1 Introduction

My primary research area is extremal graph theory, which is the art of finding ranges of parameters for common families of graphs. The most well-known example of this is the Four Color Theorem, which states that every planar graph has chromatic number at most four. My research focuses on the relationship between graphs with large substructures and the number of colors used in coloring the graph. In one result, we prove that graphs that require many colors in any proper coloring contain a subgraph with a high edge-to-vertex ratio. In a second result, we prove that graphs whose edges are colored with many colors contain a large substructure in which no color is repeated. In the next section I will discuss these results in greater detail; in Section 3 I will list several projects to continue working on, including solving Ore’s conjecture and supersaturation of rainbow matchings given a local property on edge-colorings.

2 Research Summary

2.1 Definitions

A \( k \)-coloring, or a \( k \)-vertex-coloring, is a mapping \( c : V(G) \to \{1, \ldots, k\} \). A coloring is proper if for every edge \( uv \in E(G) \), \( c(u) \neq c(v) \). The chromatic number of a graph, \( \chi(G) \), is the smallest integer \( k \) such that \( G \) has a proper \( k \)-coloring. A graph with \( \chi(G) = k \) is called \( k \)-chromatic. A minimal \( k \)-chromatic graph is called \( k \)-critical.

A \( k \)-edge-coloring is a mapping \( c : E(G) \to \{1, \ldots, k\} \). An edge-coloring is proper if for every pair of edges \( e_1, e_2 \in E(G) \) such that \( e_1 \cap e_2 \neq \emptyset \), then \( c(e_1) \neq c(e_2) \). A rainbow subgraph of an edge-colored graph is a subgraph \( H \subseteq G \) such that for all pairs of edges \( e_1, e_2 \in E(H) \), \( c(e_1) \neq c(e_2) \). A matching is a subgraph \( H \subseteq G \) such that every vertex of \( H \) is in exactly one edge. For \( v \in V(G) \) and an edge-coloring \( c \) on \( E(G) \), \( \hat{\delta}(v) \) is the number of distinct colors on the edges incident to \( v \). This is called the color degree of \( v \). The smallest color degree of all vertices in \( G \) is the minimum color degree of \( G \), or \( \hat{\delta}(G,c) \). The size of a graph \( G \) is \( |E(G)| \).

2.2 \( k \)-Critical Graphs

The motivation for studying \( k \)-critical graphs comes from the fact that most graph coloring problems can be reduced through induction to critical graphs, which have more structure. For example, \( k \)-critical graphs are 2-connected and have minimum degree at least \( k - 1 \). Our goal was simple: to prove that \( k \)-critical graphs have many edges.

The set of \( k \)-critical graphs is well known for small \( k \). The only 1-critical graph is \( K_1 \), the only 2-critical graph is \( K_2 \), and the only 3-critical graphs are the odd cycles. However, much is left open about \( k \)-critical graphs when \( k \geq 4 \). For the rest of this discussion, assume \( k \geq 4 \).

Because of the application to so many coloring problems, our goal has been at the center of attention for more than 50 years. Dirac defined \( k \)-critical graphs in 1952, and in 1957 he asked for the minimum number of edges in an \( n \)-vertex \( k \)-critical graph. Gallai reiterated this question in 1963, and Ore brought it to the forefront again in 1967. Since then, this question has appeared...
as Problem 5.3 in Jensen and Toft’s book [6] on open graph theory problems and the first half of Problem P1 (page 347) in the *Handbook on Graph Theory* [18].

Let \( f_k(n) \) be the minimum number of edges in a \( k \)-critical graph with \( n \) vertices. It is known that this function is well-defined (e.g., there exists a \( k \)-critical graph on \( n \) vertices) if \( n = k \) or \( n \geq k + 2 \). Gallai found exact values of \( f_k(n) \) for small \( n \) [5]: if \( k + 2 \leq n \leq 2k - 1 \), then \( f_k(n) = \frac{1}{2}((k-1)n + (n-k)(2k-n)) - 1 \).

Since the minimum degree of a \( k \)-critical graph is at least \( k - 1 \), \( f_k(n) \geq \frac{k-1}{2}n \), which is tight when \( n = k \). Dirac [4] proved that for \( n \neq k \), \( f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2} \), which is tight for \( n = 2k - 1 \). This was improved by Kostochka and Stiebitz [9] to \( f_k(n) \geq \frac{k-1}{2}n + k - 3 \) when \( n \notin \{2k-1,k\} \), which is tight for \( n \in \{2k,3k-2\} \). Proving a constant lower bound on the value of \( (f_k(n) - \frac{k-1}{2}n) \) is referred to as a *Dirac-type bound*.

Gallai [5] also proved that \( f_k(n) \geq \left(\frac{k-1}{2} + \frac{k-3}{2(k^2-3)}\right)n \) for all \( n \geq k + 2 \). This was improved by Krivelevich [13] to \( f_k(n) \geq \left(\frac{k-1}{2} + \frac{k-3}{2(k^2-2k-1)}\right)n \). Later Kostochka and Stiebitz [9] proved that for \( k \geq 6 \) and \( n \geq k + 2 \), \( f_k(n) \geq \left(\frac{k-1}{2} + \frac{k-3}{k^2+6k-11-6/(k-2)}\right)n \). Giving a lower bound on \( \frac{f(n)}{n} \) for large \( n \) is referred to as a *Gallai-type bound*.

Ore observed that a construction by Hajós implies

\[
    f_k(n + k - 1) \leq f_k(n) + \frac{(k-2)(k+1)}{2} = f_k(n) + (k-1)\left(\frac{k-1}{2} + \frac{k-3}{2(k-1)}\right),
\]

(giving an upper bound on the best possible Gallai-type bound. Ore [17] also conjectured that in (1) equality holds. In [12] Kostochka and I proved that the best possible Gallai-type bound is true.

**Theorem 2.1 (Kostochka and Yancey)** If \( k \geq 4 \), \( n \geq k \), and \( n \neq k + 1 \), then \( f_k(n) \geq F(k,n) = \left(\frac{k-1}{2} + \frac{k-3}{2(k-1)}\right)n - \frac{k(k-3)}{2(k-1)} \). Moreover, if \( |E(G[W])| < F(k,|W|) \) for all \( W \subseteq V(G) \), then there exists a polynomial-time algorithm to \((k-1)\)-color \( G \).

We have already begun applying this result to what made \( k \)-critical graphs so popular: solving coloring problems. Kostochka and I have written a very short proof of the celebrated Grötzsch’s...
Theorem, which states that every triangle-free planar graph is 3-colorable; at least one prominent author [21] intends to use it in the next edition of his textbook. A second team, Borodin, Kostochka, Lidicky, and I, are currently preparing a paper [1] devoted exclusively to other applications of Theorem 2.1.

Although Theorem 2.1 settles the result asymptotically, the full version of Ore’s Conjecture remains open. For each \( k \), there is a family of \( k \)-critical graphs, called \( k \)-Ore graphs, where each graph \( G \) has \( F(k, |V(G)|) \) edges. Kostochka and I [10] can give an explicit characterization of \( k \)-Ore graphs, and we can prove that if \( G \) is \( k \)-critical but not \( k \)-Ore then \( G \) has at least \( F(k, |V(G)|) + y_k \) edges, where \( y_1 = \frac{1}{7}, y_5 = \frac{3}{4}, \) and \( y_k = 1 \) for \( k \geq 6 \). We can also give a construction of graphs for which this new result is tight. These two results combine to confirm Ore’s Conjecture from 1967 as true if \( k \leq 5 \) or \( n \equiv 1 \) (mod \( k - 1 \)). Furthermore, Ore’s Conjecture can be false at most \( k^3/12 \) times.

2.3 Rainbow Matchings

The topic of rainbow matchings has been well studied, along with a more general topic of rainbow subgraphs (see [7] for a survey). Let \( r(G, \phi) \) be the size of a largest rainbow matching in a graph \( G \) with edge coloring \( \phi \). In 2008, Wang and Li [20] showed that \( r(G, \phi) \geq \left\lceil \frac{5\delta(G, \phi) - 3}{12} \right\rceil \) for every graph \( G \) and conjectured that if \( \delta(G, \phi) \geq k \geq 4 \) then \( r(G, \phi) \geq \left\lceil \frac{k}{2} \right\rceil \). The conjecture is known to be tight for properly colored complete graphs. LeSaulnier et al. [14] proved that \( r(G, \phi) \geq \left\lceil \frac{k}{2} \right\rceil \) for general graphs, and gave several conditions sufficient for a rainbow matching of size \( \left\lceil \frac{k}{2} \right\rceil \). In [11], Kostochka and I proved the conjecture in full.

**Theorem 2.2 (Kostochka and Yancey)** If \( \delta(G, \phi) \geq k \geq 4 \) then \( r(G, \phi) \geq \left\lceil \frac{k}{2} \right\rceil \).

The only known extremal examples for the bound have at most \( k + 2 \) vertices.

Wang [19] proved that every properly edge-colored graph \( (G, \phi) \) with \( \delta(G, \phi) = k \) and \( |V(G)| \geq 1.6k \) has a rainbow matching of size at least \( \frac{3k}{5} \) and that every such triangle-free graph has a rainbow matching of size at least \( \lceil 2k/3 \rceil \). He also asked if there is a function, \( f(k) \), such that for every graph \( G \) and proper edge coloring \( \phi \) of \( G \) with \( \delta(G, \phi) \geq k \) and \( |V(G)| \geq f(k) \), we have \( r(G, \phi) \geq k \). The bound on \( r(G, \phi) \) is sharp for any properly \( k \)-edge-colored \( k \)-regular graph.

Diemunsch et al. [2] answered the question in the positive and proved that \( f(k) \leq 6.5k \). Shortly thereafter, Lo [15] improved the bound to \( f(k) \leq 4.5k \), and finally Diemunsch et al. [3] combined the two manuscripts and improved the bound to \( f(k) \leq \frac{9k}{13} \). The largest matching in a graph with \( n \) vertices contains at most \( n/2 \) edges, which implies \( f(k) \geq 2k \). By considering the relationship of Latin squares to edge-colored \( K_{n,n} \), the lower bound can be improved to \( f(k) \geq 2k + 1 \) for even \( k \). This is the best known lower bound on the number of vertices required for both the properly edge-colored and general cases.

Independently from the above results, Kostochka, Pfender, and I proved a similar result in [8].

**Theorem 2.3 (Kostochka and Yancey)** Let \( G \) be an \( n \)-vertex graph and \( \phi \) be an edge-coloring of \( G \) with \( n > 4.25\delta(G, \phi)^2 \). Then \( (G, \phi) \) contains a rainbow matching with at least \( \delta(G, \phi) \) edges.

Our result gives a significantly weaker bound on the order of \( G \) than the bound in [3] but for a significantly wider class of edge-colorings. Lo and Tan [16] later gave a single result which improved both theorems.
3 Research Proposal

3.1 Ore’s Conjecture in Full

Question 3.1 What is $f_k(n)$?

The quadratic polynomial in Gallai’s formula for small values of $n$ in combination with the recursive nature of Ore’s Conjecture makes solving for exact values of $f_k(n)$ a delightful challenge. The technique used in Theorem 2.1 uses a “potential function” to evaluate the density of subgraphs of a minimal counterexample. Kostochka and I have begun to attack the full version of Ore’s Conjecture by manipulating the potential function to account for the periodic-parabolic shape that Ore conjectures for $f_k(n)$. We have made some progress, but there are still many gaps to fill.

3.2 Sparse $k$-critical Hypergraphs and $K_s$-free graphs

Stronger bounds than Theorem 2.1 have been found for classes of graphs. Specifically, $r$-uniform hypergraphs and $K_s$-free graphs appear in the literature.

An analogue of Theorem 2.1 can be proven for hypergraphs. However, every hypergraph for which Theorem 2.1 is sharp contains at least 3 edges that contain only two vertices.

Question 3.2 What is the fewest number of edges a $k$-critical $n$-vertex $r$-uniform hypergraph can have?

I intend to investigate the problem of minimal $K_s$-free $k$-critical graphs in the future, as I believe taking my research in this direction will lead to the most applications. At least half of the applications in [1] involve 3-coloring planar graphs without specific subgraphs. One large open problem is Steinberg’s Conjecture that every planar graph that is $C_4$ and $C_5$-free is 3-colorable. However, it seems more natural to start with $K_3$-free graphs, given the presence of such results in the literature.

Question 3.3 What is the fewest number of edges a $k$-critical $n$-vertex $K_s$-free graph can have?

3.3 Rainbow Subgraphs

Given the size of the field of rainbow subgraphs, one direction for my research would be to investigate subgraphs other than matchings. It is not likely that positive results can be obtained for all graphs.

Question 3.4 For what graphs $H$ does there exist a value $f'(H)$ such that if $G$ is edge-colored with $\hat{\delta}(G,\phi) \geq f'(H)$ then $(G,\phi)$ contains a rainbow copy of $H$?

Lo and Tan achieved their result by finding two vertex-disjoint rainbow matchings simultaneously. There have recently been results in supersaturation: the idea that once a graph meets a threshold to contain a specific substructure, it then contains many such substructures.

Question 3.5 Does there exist a function $g(k,n)$ such that if $|V(G)| = n$ and $\hat{\delta}(G,\phi) \geq k$, then $(G,\phi)$ would contain at least $g(k,n)$ distinct rainbow matchings?

Lo and Tan’s result implies that $g(k,n)$ should be at least exponential. More importantly, trying to solve this question may lead us to sharper results about $f(k)$. 

4
References


[21] D. B. West, personal communication