Large Rainbow Matchings in Edge-Colored Graphs

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Rainbow Matchings in Graphs

A *rainbow subgraph* of an edge-colored graph is a subgraph whose edges have distinct colors.

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\[ r = 3 \]
Color Degree

For \( v \in V(G) \), \( \hat{d}(v) \) is the number of distinct colors on the edges incident to \( v \).

\[
k = \delta(G) = \min \hat{d}(v)
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d(A) = 3, \quad \hat{d}(A) = 2.
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$d(A) = 3$, $\hat{d}(A) = 2$. $d(B) = 4$, $\hat{d}(B) = 3$. 
For $v \in V(G)$, $\hat{d}(v)$ is the number of distinct colors on the edges incident to $v$.

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- $d(A) = 3$, $\hat{d}(A) = 2$.
- $d(B) = 4$, $\hat{d}(B) = 3$.
- $d(C) = 4$, $\hat{d}(C) = 2$.
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\[ k = 2 \]
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  d(A) &= 3, \quad \hat{d}(A) = 2. \\
  d(B) &= 4, \quad \hat{d}(B) = 3. \\
  d(C) &= 4, \quad \hat{d}(C) = 2
\end{align*} \]

\[ k = 2 \]
## Early Results

### Theorem (Li and Wang, 2008)

\[ r \geq \left\lceil \frac{5k-3}{12} \right\rceil. \]

### Theorem (Li and Wang, 2008)

For \( k \geq 3 \) and \( G \) bipartite,
\[ r \geq \left\lceil \frac{2k}{3} \right\rceil. \]

### Conjecture (Li and Wang, 2008)

For \( k \geq 4 \),
\[ r \geq \left\lceil \frac{k}{2} \right\rceil. \]
Tight Examples

1. \( k = 3 \), \( r = 1 \)
   - Bipartite

2. \( k = 2 \), \( r = 1 \)
   - \( K_n \)
     - \( k = n - 1 \)
     - \( r = n/2 \) or \((n-1)/2\)
Further Results

Theorem (Li and Xu, 2007)

If $G$ is a properly colored complete graph other than $K_4$, then

$$r \geq \left\lceil \frac{k}{2} \right\rceil.$$ 

Theorem (LeSaulnier, Stocker, Wegner, and West, 2010)

$r \geq \left\lfloor \frac{k}{2} \right\rfloor$ and $r \geq \left\lceil \frac{k}{2} \right\rceil$ if any of the following are true

- $G$ is triangle-free
- $G$ is properly colored and $n \neq k + 2$
- $n \geq \frac{3(k-1)}{2}$
Our Results

**Theorem (Kostochka and Y, 2011+)**

If \( k \geq 4 \), then \( r \geq \left\lceil \frac{k}{2} \right\rceil \)

**Theorem (Kostochka and Y, 2011+)**

If \( G \) is triangle-free, then \( r \geq \left\lceil \frac{2k}{3} \right\rceil \).
Notation

\[ r = \frac{k - 1}{2} \]

\[ p = n - k + 1 \]

Edge \( e_i \) has color \( i \)
Let $\phi$ be an ordering on the vertices such that

$$
\phi(u_1) < \phi(v_1) < \phi(u_2) < \phi(v_2) < \cdots < \phi(u_r) < \phi(v_r)
$$

An important edge, $e = wz$, is an edge with color not in $M$ and with $w \in H$ and $z \in V(M)$ such that $\phi(z)$ is the minimum of all edges incident to $w$ with the same color as $wz$. 
The Inequality

\[
\frac{\text{number of important edges out of } M}{\text{number of important edges out of } H} \geq p(k - r) = (k + 1) \frac{p}{2}
\]

On average the vertices in \( M \) are incident to more than \( \frac{p}{2} \) important edges.
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Lemma (LeSaulnier, Stocker, Wegner, and West, 2010)

Each \( e_i \) is incident to at most \( p + 1 \) important edges. Furthermore, if \( e_i \) is incident to \( p + 1 \) important edges, then \( E_i \) has Configuration A or Configuration B.
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$p + 1$ Configurations

Configuration A

Configuration B

$p = 3$
A special vertex is a vertex $v$ with $d(v) = n - 1$, $\hat{d}(v) = k$, and one color is incident to it $n - k$ times (all other colors are incident to it once).

If $E_i$ has Configuration A then $v_i$ is special.
Special Vertices

A *special vertex* is a vertex $v$ with $d(v) = n - 1$, $\hat{d}(v) = k$, and one color is incident to it $n - k$ times (all other colors are incident to it once).

If $E_i$ has Configuration A then $v_i$ is special.

Let $v$ be a special vertex. The *main color* of $v$ is the color that is repeated on the edges incident to $v$. A *main edge* of $v$ is an edge incident to $v$ colored the main color of $v$. 
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Main Edges

We will assume that $G$ is a minimal counter-example to the theorem, and that $M$ contains the most main edges out of all maximum rainbow matchings in $G$.

Lemma

If $E_i$ has Configuration A, then $u_i$ is special with main color $i$. 
Main Edges

We will assume that $G$ is a minimal counter-example to the theorem, and that $M$ contains the most main edges out of all maximum rainbow matchings in $G$.

Lemma

*If $E_i$ has Configuration A, then $u_i$ is special with main color $i$.***

Proof.

Suppose not. $v_i$ is special with a main color that is important, so it can not be $i$. Since $u_i$ is not special with main color $i$, edge $e_i$ is not a main edge.

Replace $e_i$ with one of the main edges of $v_i$. This is still a rainbow matching of same size, and now has more main edges, which contradicts our choice of $M$. 
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The Two Cases

Let $a$ be the number of times Configuration A occurs and $b$ be the number of times Configuration B occurs.

**Case 1:** $a > 0$ We will assume that $E_1$ has Configuration A. Consider edges $u_1u_i$ for $i$ such that $E_i$ has Configuration A or B.

We will attempt to replace $e_1$ and $e_i$ with $u_1u_i$ and important edges of $v_1$ and $v_i$ to prove that $M$ was not a maximum rainbow matching.
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**Case 1:** $a > 0$ We will assume that $E_1$ has Configuration A. Consider edges $u_1 u_i$ for $i$ such that $E_i$ has Configuration A or B.

We will attempt to replace $e_1$ and $e_i$ with $u_1 u_i$ and important edges of $v_1$ and $v_i$ to prove that $M$ was not a maximum rainbow matching.
If edge $u_1 u_i$ has color 1, i, or a color not in $M$, then we generate a contradiction to either the maximality of $M$ or minimality of $G$. 
5-Alternating Path Plus a 2-Alternating Path

If the color of edge $u_1u_i$ matches the color of edge $e_h$ in $M$.

![Graph diagram]
5-Alternating Path Plus a 2-Alternating Path

If the color of edge $u_1u_i$ matches the color of edge $e_h$ in $M$.

Try to replace $e_1$, $e_i$, and $e_h$ with $u_1u_i$ and important edges adjacent to $v_1$, $v_i$, and $v_h$. 
If the color of edge $u_1u_i$ matches the color of edge $e_h$ in $M$.

Try to replace $e_1$, $e_i$, and $e_h$ with $u_1u_i$ and important edges adjacent to $v_1$, $v_i$, and $v_h$. 

Conclusion of Case 1

\[
\frac{p \cdot (k + 1)}{2} \leq \text{number of important edges out of } H
\]
\[
= \text{number of important edges out of } M
\]
\[
\leq (a + b)(p + 1) + (a + b - 1)2 + \left(\frac{k - 1}{2} - 2a - 2b + 1\right)p
\]

And Case 1 is done!
We no longer have special vertices to use.

However, we know that $p = 3$.

\[(k - 1) + p = n\]

\[n = k + 2\]
Case 2

We no longer have special vertices to use. However, we know that \( p = 3 \).

\[
(k - 1) + p = n
\]

\[
n = k + 2
\]

Every vertex is adjacent to \( n - 2 \) distinct colors. Every vertex is "like" a special vertex!
We no longer have special vertices to use. However, we know that $p = 3$.

\[(k - 1) + p = n\]
\[n = k + 2\]

Every vertex is adjacent to $n - 2$ distinct colors. Every vertex is "like" a special vertex!
Open Problem

**Conjecture** (Li and Wang, 2008)

*For bipartite graphs, \( r \geq k - 1 \) if \( k \) is even and \( r \geq k \) if \( k \) is odd.*

If true, this would be a generalization of H. J. Ryser’s conjecture (1967) for the maximum size of a transversal in a latin square.


