On Vertex-Disjoint Chorded Cycles

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Slides available at
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Overview

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Notation

- The number of edges from $S$ to $T$ regardless of whether they are vertex sets or subgraphs
  \[ ||S, T|| = \sum_{s \in V(S)} N_G(s) \cap V(T) \]

- The minimum degree
  \[ \delta(G) = \min\{d_G(x) : x \in V(G)\} \]

- The minimum Ore-degree is
  \[ \sigma_2(G) = \min\{d_G(x) + d_G(y) : xy \not\in E(G)\} \]
Theorem (Corrádi, Hajnal 1963)

Every simple graph $G$ on $|G| \geq 3k$ vertices with $\delta(G) \geq 2k$ contains $k$ vertex-disjoint cycles.


Every simple graph $G$ on $|G| \geq 3k$ vertices with $\sigma_2(G) \geq 4k - 1$ contains $k$ vertex-disjoint cycles.
Theorem (Corrádi, Hajnal 1963)

Every simple graph $G$ on $|G| \geq 3k$ vertices with $\delta(G) \geq 2k$ contains $k$ vertex-disjoint cycles.
Background

**Theorem (Finkel 2008)**

Every graph $G$ on $n = |G| \geq 4k$ vertices with $\delta(G) \geq 3k$ contains $k$ disjoint chorded cycles.

**Theorem (Chiba, Fujita, Gao, Li 2010)**

Every graph $G$ on $n = |G| \geq 4k$ vertices with $\sigma_2(G) \geq 6k - 1$ contains a collection of $k$ disjoint chorded cycles.
Background

**Theorem (Molla, Santana, Yeager 2017)**

For $k \geq 2$, let $G$ be a graph with $n = |G| \geq 4k$ and $\sigma_2(G) \geq 6k - 2$. Then $G$ does not contain $k$ disjoint chorded cycles if and only if $G \in \{G_1(n, k), G_2(k)\}$, where $G_1(n, k) = K_{3k-1, n-3k+1}$ for $n \geq 6k - 2$ and $G_2(k) = K_{3k-2, 3k-2, 1}$ for $k \geq 2$. \\

$G_1(n, 2)$

$G_2(2)$
The Main Theorem

Theorem (Kostochka, Y, Yu 2018+++)

Let $k \geq 2$, $G$ be an $n$-vertex graph with $n \geq 4k$ and $\sigma_2(G) \geq 6k - 3$. Then $G$ does not contain $k$ disjoint chorded cycles if and only if $G$ is not an exceptional graph.
The (Simplified) Rules

Choose a collection $\mathcal{C}$ of chorded cycles such that

1. the number of chorded 4-cycles is maximum,
2. subject to the preceding, the number of $K_4$ is maximum,
3. subject to the preceding, the cycles are small
4. subject to the preceding, the cycles have the maximum number of edges
5. subject to the preceding, the length of a longest path $P$ in $R := V(G) - V(\mathcal{C})$ is maximum. If $|P| = |R|$, then the number of Hamiltonian cycles in $G[R]$ is maximum.
6. subject to the preceding, $|E(G[R])|$ is maximum, and
7. subject to the preceding, $\sum_{v \in R} d_G(v)$ is maximum.
The Rules

1. the number of chorded $4$-cycles is maximum,

2. subject to the preceding, the number of $K_4$ is maximum,

3. subject to the preceding, the $k$-tuple $(C_1, \ldots, C_k)$ has 
   \((|V(C_1)|, \ldots, |V(C_k)|)\) least lexicographically,

4. subject to the preceding tuple, 
   \((|E(C_1)|, \ldots, |E(C_k)|)\) is greatest lexicographically,

5. subject to the preceding, the length of a longest path $P$ in
   $R := V(G) - V(C)$ is maximum. If $|P| = |R|$, then the number of
   Hamiltonian cycles in $G[R]$ is maximum, unless $|R| = 4$ in which case
   we prefer the paw $G[R] = K_{1,3}^+$ over $G[R] = C_4$.

6. subject to the preceding, $|E(G[R])|$ is maximum, and

7. subject to the preceding, $\sum_{v \in R} d_G(v)$ is maximum.
Proof Outline

1. \( G[R] \) does not have a Hamiltonian path
2. \( G[R] \) has a Hamiltonian path and \( k \geq 3 \)
3. \( G[R] \) has a Hamiltonian path and \( k = 2 \)
Given a collection of chorded cycles chosen by our rules, if $G[R]$ does not have a Hamiltonian path, then $G[V(P)]$ cannot be a cycle.

- First, we can utilize the fact that for all $z \in R$, $d_R(z) \geq 2$.
- Moreover, for a maximal path $P' \subseteq G[R - P]$, one of the endpoints must have degree 2 in $G[R]$. 
A Proof

Claim

*Given a collection of chorded cycles chosen by our rules, if $G[R]$ does not have a Hamiltonian path, then $G[V(P)]$ cannot be a cycle.*
A Proof

Claim

Given a collection of chorded cycles chosen by our rules, if $G[R]$ does not have a Hamiltonian path, then $G[V(P)]$ cannot be a cycle.

Since $\|W, V(C)\| \geq 2(6k - 3) - 4(2) = 12(k - 1) - 2$, if $|C| < k - 1$, then there exists $C \in C$ such that $\|W, C\| \geq 13$. 
A Proof

**Claim**

*Given a collection of chorded cycles chosen by our rules, if $G[R]$ does not have a Hamiltonian path, then $G[V(P)]$ cannot be a cycle.*

A contradiction! Therefore, $|C| = k - 1$ and we need only trade one cycle for two in order to reach a contradiction. Also, there exists $C \in C$ such that $\|W, C\| \geq 10$. 
Given a collection of chorded cycles chosen by our rules, if $G[R]$ does not have a Hamiltonian path, then $G[V(P)]$ cannot be a cycle.

Since $\|\{w_1, w_q\}, C\| \geq 4$, they must have neighborhoods in different partite sets. In what remains, $v_1$ and $v_2$ still have neighbors which are also in different partite sets. Therefore, $C \neq K_{3,3}$. 
Claim

Given a collection of chorded cycles chosen by our rules, if $G[R]$ does not have a Hamiltonian path, then $G[V(P)]$ cannot be a cycle.

There exists $x \in V(C)$ such that $N(x) \supseteq \{v_1, v_2\}$. Otherwise, ...
A Proof

Claim

Given a collection of chorded cycles chosen by our rules, if $G[R]$ does not have a Hamiltonian path, then $G[V(P)]$ cannot be a cycle.

Otherwise, there exists a $K_4^- \cong K_4 - e$ contradicting our rules.
A Proof

Claim

Given a collection of chorded cycles chosen by our rules, if $G[R]$ does not have a Hamiltonian path, then $G[V(P)]$ cannot be a cycle.

If $w_1$ and $w_q$ each have a neighbor in $C - x$, we can construct 2 disjoint chorded cycles.
A Proof

Claim

Given a collection of chorded cycles chosen by our rules, if \( G[R] \) does not have a Hamiltonian path, then \( G[V(P)] \) cannot be a cycle.

\[
||\{v_1, v_2\}, C|| = 6 \\
||\{w_1, w_q\}, C|| = 4
\]

Now, there exists a vertex \( x' \in V(C) - x \) with \( x' \in N(v_1) \cap N(v_2) \) so we can again get 2 disjoint chorded cycles.
Claim

Given a collection of chorded cycles chosen by our rules, if $G[R]$ does not have a Hamiltonian path, then $G[V(P)]$ cannot be a cycle.

This proves the claim.
• Mixed Cycles, currently done for
  • $\delta(G) \geq 2k_{cycle} + 3k_{chorded} - 1$ and
  • $\sigma_2(G) \geq 4k_{cycle} + 6k_{chorded} - 1$
Thank You
Shuya Chiba, Shinya Fujita, Yunshu Gao, and Guojun Li.
On a sharp degree sum condition for disjoint chorded cycles in graphs.

Keresztély Corradi and András Hajnal.
On the maximal number of independent circuits in a graph.

Hikoe Enomoto.
On the existence of disjoint cycles in a graph.

Daniel Finkel.
On the number of independent chorded cycles in a graph.

Theodore Molla, Michael Santana, and Elyse Yeager.
A refinement of theorems on vertex-disjoint chorded cycles.