A List Version of Graph Packing

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The Königsberg Bridge Problem

Can you start somewhere, traverse every bridge exactly once, and return to the starting point?
Introduction

The Königsberg Bridge Problem

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A graph $G$ is comprised of a set of vertices $V$ and a set of edges $E$, where the edges are 2-element subsets of $V$. 
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$V(G) = \{u, v, x, y\}$

$E(G) = \{xu, uv, vx, xy\}$
Definition

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$d(x) = 3,$
$d(y) = 1,$
$d(u) = d(v) = 2$

$\Delta(G) = 3$
Definitions

Definition

A packing of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, both on the same number of vertices, is a bijection $f : V_1 \to V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$.

Example: $G_1 \cong K_{1,n-2} + K_1$ and $G_2 \cong C_{n-1} + K_1$ pack.
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For any graph $G$ with vertex set $V$ and edge set $E$, we call $\overline{G}$ the complement of $G$ where $V(\overline{G}) = V(G)$ and $E(\overline{G})$ consists of all edges that were not in $G$.

Figure: A graph $G$ and its complement $\overline{G}$
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Figure: A packing of $G$ and $\overline{G}$
Figure: Two graphs $G_1$ and $G_2$ that pack
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Figure: Two new graphs $G_1$ and $G_2$ that do not pack
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Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs of order $n$. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.
**Previous Results and Extensions**

**Theorem (Sauer, Spencer 1978)**

Let $G_1$ and $G_2$ be two graphs of order $n$. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.

**Theorem (Kaul, Kostochka 2007)**

Let $\Delta(G_1)\Delta(G_2) \leq \frac{n}{2}$. $G_1$ and $G_2$ do not pack if and only if one of $G_1$ and $G_2$ is a perfect matching and the other is either $K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$. 
Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.

1) Suppose we consider a best mapping that minimizes the number of conflicts, and suppose for contradiction that this number is not zero.
Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1) \Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.

2) Then, we want to reposition the blue graph on top of the red graph so that there are no new conflicts and no conflicts at our focal point.
Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.

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- $a$ with $b_1$; conflict edge remains
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- $a$ with $b_1$; conflict edge remains
- $a$ with $b_2$ or $b_3$; blue edge moves onto red edge thus creating a new conflict

Are these the only problems?
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4) How many bad "swaps"?

<table>
<thead>
<tr>
<th>Type of Swap</th>
<th>Maximum Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ with $b_1$</td>
<td>$t$</td>
</tr>
<tr>
<td>$a$ with $b_2$</td>
<td>$\Delta(G_1)\Delta(G_2) - t$</td>
</tr>
<tr>
<td>$a$ with $b_3$</td>
<td>$\Delta(G_2)\Delta(G_1) - t$</td>
</tr>
<tr>
<td>any bad</td>
<td>$t + 2[\Delta(G_1)\Delta(G_2) - t]$</td>
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Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.

5) Thus, our desired vertex $b$ is available as long as

$$t + 2[\Delta(G_1)\Delta(G_2) - t] < n - 1$$

or equivalently

$$2\Delta(G_1)\Delta(G_2) - t < n - 1$$
Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1) \Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.
Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs of order $n$. If $|E_1| + |E_2| \leq \frac{3}{2} n - 2$, then $G_1$ and $G_2$ pack.

Sharpness Example:
Definition

A graph triple $G = (G_1, G_2, G_3)$, of size $n$, consists of a pair of $n$-vertex graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ together with a bipartite graph $G_3 = (V_1 \cup V_2, E_3)$.

Example: $G_1 \cong K_{1,n-2}$, $G_2 \cong C_{n-1} \cup K_1$, $G_3 \cong 4K_2$
Definition

A list packing of the graph triple $G$ is a bijection $f : V_1 \rightarrow V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and $v \in V_1$ implies $vf(v) \notin E_3$. Observe that the newly introduced set of edges basically represent forbidden mappings.

Example 1: This graph triple can pack.
A *list packing* of the graph triple $G$ is a bijection $f : V_1 \to V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and $v \in V_1$ implies $vf(v) \notin E_3$. Observe that the newly introduced set of edges basically represent forbidden mappings.

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Example 2: This graph triple does not pack.
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A list packing of the graph triple $G$ is a bijection $f : V_1 \rightarrow V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and $v \in V_1$ implies $vf(v) \notin E_3$.
Observation

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Now, we can ask if this packs. But how did the number of edges change? And the maximum degrees?
A fixed-point free embedding is a packing of $G$ and $\overline{G}$ such that for each $v \in V$, $f(v) \neq v$.

Fixed-point free embedding:
Theorem (Győri, Kostochka, McConvey, Y 2015+)

Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$ with $\Delta_1 \Delta_2 + \Delta_3 \leq \frac{n}{2}$. Then $G$ does not pack if and only if $\Delta_3 = 0$ and one of $G_1$ and $G_2$ is a perfect matching and the other is either $K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

Theorem (Győri, Kostochka, McConvey, Y 2015+)

Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$. If $|E_1| + |E_2| + |E_3| \leq \frac{3}{2} n - 2$, then $G$ packs.
Theorem (Bollobás, Eldridge 1978)

If $\Delta_1, \Delta_2 \leq n - 2, |E_1| + |E_2| \leq 2n - 3,$ and $\{G_1, G_2\}$ is not one of the 7 pairs shown below, then $G_1$ and $G_2$ pack.
Theorem (Győri, Kostochka, McConvey, Y 2015+)

Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$. If $\Delta_1, \Delta_2 \leq n - 2$, $\Delta_3 \leq n - 1$, and $|E_1| + |E_2| + |E_3| \leq 2n - 3$, and $\{G_1, G_2\}$ is not one of the Bollobás-Eldridge pairs. Then either $G$ packs or is one of the same 7 examples.

The result is sharp:
A Conjecture of Zâk

Theorem (Zâk 2014)

Let $G_1$ and $G_2$ be graphs on $n$ vertices with $\Delta_1, \Delta_2 \leq n - 2$. If $|E_1| + |E_2| + \Delta_1 + \Delta_2 \leq 3n - 68n^{3/4} - 62$, then $G_1$ and $G_2$ pack.

Conjecture (Zâk 2014)

Let $G_1$ and $G_2$ be graphs on $n$ vertices with $\Delta_1, \Delta_2 \leq n - 2$. If $|E_1| + |E_2| + \Delta_1 + \Delta_2 \leq 3n - 3$, then $G_1$ and $G_2$ pack.
Zák's conjecture false for small $n$:

$$|E_1| + |E_2| + \Delta_1 + \Delta_2 = 3n - 5$$

For large $n$, conjecture is best possible:

$$|E_1| + |E_2| + \Delta_1 + \Delta_2 = 3n - 2$$
Main Result

Theorem (Győri, Kostochka, McConvey, Y 2015+)

Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$ with $\Delta_1, \Delta_2 \leq n - 2$ and $\Delta_3 \leq n - 1$. There is an absolute constant $C$ such that if $|E_1| + |E_2| + |E_3| + D_1 + D_2 \leq 3n - C$, then $G$ packs.

Our current proof gives $C = 418275$.

Corollary

Let $G_1$ and $G_2$ be a graphs of order $n$ with $\Delta_1, \Delta_2 \leq n - 2$. There is an absolute constant $C$ such that if $|E_1| + |E_2| + \Delta_1 + \Delta_2 \leq 3n - C$, then $G_1$ and $G_2$ pack.
Thank You