A List Version of Graph Packing

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Joint work with Ervin Győri, Alexandr Kostochka, and Andrew McConvey
Overview

1. Graph Packing Definitions
2. Previous Results
3. List Packing Definitions
4. List Packing Results
A packing of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, both on the same number of vertices, is a bijection $f : V_1 \rightarrow V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$.

Example: $G_1 \cong K_{1,n-2} + K_1$ and $G_2 \cong C_{n-1} + K_1$ pack.
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Example: The two graphs $G_1$ and $G_2$ do not pack.
Definitions

Definition

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Example: The two graphs $G_1$ and $G_2$ do not pack.
Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs of order $n$. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.
Previous Results

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Theorem (Kaul, Kostochka 2007)

Let $\Delta(G_1)\Delta(G_2) \leq \frac{n}{2}$. $G_1$ and $G_2$ do not pack if and only if one of $G_1$ and $G_2$ is a perfect matching and the other is either $K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.
Theorem (Sauer, Spencer 1978)

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Sharpness Example:
A **graph triple** $G = (G_1, G_2, G_3)$, of size $n$, consists of a pair of $n$-vertex graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ together with a bipartite graph $G_3 = (V_1 \cup V_2, E_3)$.

**Example:** $G_1 \cong K_{1,n-2} \cup K_1$, $G_2 \cong C_{n-1} \cup K_1$, $G_3 \cong 4K_2 \cup \overline{K_{2n-8}}$
Definition

A list packing of the graph triple $G$ is a bijection $f : V_1 \rightarrow V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and $v \in V_1$ implies $vf(v) \notin E_3$. Observe that the newly introduced set of edges basically represent forbidden mappings.

Example 1: This graph triple can pack.
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Example 2: This graph triple does not pack.
Observation

With list packing, we still have the same problems with induction, but we can fix them with a minor adjustment.
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Now, we can ask if this packs. But how did the number of edges change? And the maximum degrees?
Theorem (Győri, Kostochka, McConvey, Y 2014+)

Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$ with $\Delta_1 \Delta_2 + \Delta_3 \leq \frac{n}{2}$. Then $G$ does not pack if and only if $\Delta_3 = 0$ and one of $G_1$ and $G_2$ is a perfect matching and the other is either $K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$. 
Extensions of Previous Results

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\[
(n - 1) - [(\Delta(G_3) - 1) + (\Delta(G_3) - 1) + 2\Delta(G_1)\Delta(G_2)] \leq 0
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Theorem (Győri, Kostochka, McConvey, Y 2015+)

Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$. If $|E_1| + |E_2| + |E_3| \leq \frac{3}{2}n - 2$, then $G$ packs.
Extensions of Previous Results

Theorem (Bollobás, Eldridge 1978)

If $\Delta_1, \Delta_2 \leq n - 2$, $|E_1| + |E_2| \leq 2n - 3$, and $\{G_1, G_2\}$ is not one of the 7 pairs shown below, then $G_1$ and $G_2$ pack.

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The result is sharp:
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## Applications of List Packing

<table>
<thead>
<tr>
<th>Conjecture (Ţak 2014)</th>
</tr>
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<tbody>
<tr>
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Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$. If $|E_1| + |E_2| + |E_3| \leq \frac{3}{2} n - 2$, then $G$ packs.

**Case 1:** There exists a vertex $x \in V_i$ with $d_i(x) = d_3(x) = 0$. Assume $x \in V_1$.

If $d_2(y) + d_3(y) \geq 2$ for some $y \in V_2$, done.
Proof of Sauer-Spencer Extension

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**Case 2:** There exists a vertex \( x \in V_i \) with \( d_i(x) = 0, d_3(x) > 0 \). Assume \( x \in V_1 \).
If \( d_3(x) \geq 2 \), done.
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If $d_3(x) = 1$, 

![Diagram](image)
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**Case 3:** $\delta_1, \delta_2 > 0$.
Based on edge and degree conditions, there exists $x \in V_1$ with
$d_1(x) = 1, d_3(x) = 0$. 

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Thank You