A List Version of Graph Packing

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Slides available at
http://math.illinois.edu/~yager2/research.html
Overview

1. Introduction to Graphs
2. Packing Definitions
3. Previous Results
4. Proof of Sauer and Spencer Result
5. List Packing Definitions
6. List Packing Results
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The Königsberg Bridge Problem

Can you start somewhere, traverse every bridge exactly once, and return to the starting point?
Introduction

The Königsberg Bridge Problem

Can you start somewhere, traverse every bridge exactly once, and return to the starting point?
The Königsberg Bridge Problem

What if I don’t require the starting and ending points to be the same?
A graph $G$ is comprised of a set of vertices $V$ and a set of edges $E$, where the edges are 2-element subsets of $V$. 
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$V(G) = \{u, v, x, y\}$

$E(G) = \{xu, uv, vx, xy\}$
A graph $G$ is comprised of a set of vertices $V$ and a set of edges $E$, where the edges are 2-element subsets of $V$.

- $d(x) = 3$
- $d(y) = 1$
- $d(u) = d(v) = 2$
- $\Delta(G) = 3$
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A packing of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, both on the same number of vertices, is a bijection $f : V_1 \rightarrow V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$.

Example: $G_1 \cong K_{1,n-2} + K_1$ and $G_2 \cong C_{n-1} + K_1$ pack.
Definitions

Definition

A packing of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, both on the same number of vertices, is a bijection $f : V_1 \to V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$.

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For any graph $G$ with vertex set $V$ and edge set $E$, we call $\overline{G}$ the \textit{complement of $G$} where $V(\overline{G}) = V(G)$ and $E(\overline{G})$ consists of all edges that were not in $G$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{graph_and_complement}
\caption{A graph $G$ and its complement $\overline{G}$}
\end{figure}
Definitions

Definition

For any graph $G$ with vertex set $V$ and edge set $E$, we call $\overline{G}$ the complement of $G$ where $V(\overline{G}) = V(G)$ and $E(\overline{G})$ consists of all edges that were not in $G$.

Figure: A packing of $G$ and $\overline{G}$
Induction fails?

**Figure:** Two graphs $G_1$ and $G_2$ that pack
Induction fails?

Figure: Two graphs $G_1$ and $G_2$ that pack
Induction fails?

Add vertices and edge(s)
Add one edge

Figure: Two new graphs $G_1$ and $G_2$ that do not pack
Induction fails?

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Add vertices and edge(s)

Figure: Two new graphs $G_1$ and $G_2$ that do not pack
Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.
Previous Results and Extensions

Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1) \Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.

Theorem (Kaul, Kostochka 2007)

Let $\Delta(G_1) \Delta(G_2) \leq \frac{n}{2}$. $G_1$ and $G_2$ do not pack if and only if one of $G_1$ and $G_2$ is a perfect matching and the other is either $K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$. 

\[
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3} \\
\end{array}
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1) Suppose we consider a best mapping that minimizes the number of conflicts, and suppose for contradiction that this number is not zero.
Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.

2) Then, we want to reposition the blue graph on top of the red graph so that there are no new conflicts and no conflicts at our focal point.
Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1) \Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.

3) Bad “swaps”
- $a$ with $b_1$; conflict edge remains
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- $a$ with $b_1$; conflict edge remains
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Are these the only problems?
**Sketch of Proof**

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Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1) \Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.

4) How many bad “swaps”?

<table>
<thead>
<tr>
<th>Type of Swap</th>
<th>Maximum Quantity</th>
</tr>
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<tbody>
<tr>
<td>$a$ with $b_1$</td>
<td>$t$</td>
</tr>
<tr>
<td>$a$ with $b_2$</td>
<td>$\Delta(G_1) \Delta(G_2) - t$</td>
</tr>
<tr>
<td>$a$ with $b_3$</td>
<td>$\Delta(G_2) \Delta(G_1) - t$</td>
</tr>
<tr>
<td>total bad</td>
<td>$t + 2[\Delta(G_1) \Delta(G_2) - t]$</td>
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Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.

5) Thus, our desired vertex $b$ is available as long as

$$t + 2[\Delta(G_1)\Delta(G_2) - t] < n - 1$$

or equivalently

$$2\Delta(G_1)\Delta(G_2) - t < n - 1$$
Theorem (Sauer, Spencer 1978)

Let $G_1$ and $G_2$ be two graphs on $n$ vertices. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then $G_1$ and $G_2$ pack.
Theorem (Sauer, Spencer 1978 and Bollobás, Eldridge 1978)

Let $G_1$ and $G_2$ be two graphs of order $n$. If $|E_1| + |E_2| \leq \frac{3}{2} n - 2$, then $G_1$ and $G_2$ pack.

Sharpness Example:

Figure: There are $n - 1 + \frac{n}{2} = \frac{3}{2} n - 1 > \frac{3}{2} n - 2$ total edges
Definition

A graph triple $G = (G_1, G_2, G_3)$, of size $n$, consists of a pair of $n$-vertex graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ together with a bipartite graph $G_3 = (V_1 \cup V_2, E_3)$.

Example: $G_1 \cong K_{1, n-2} + K_1$, $G_2 \cong C_{n-1} + K_1$, $G_3 \cong 4K_2 + (2n - 8)K_1$
Definition

A list packing of the graph triple $G$ is a bijection $f : V_1 \rightarrow V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and $v \in V_1$ implies $vf(v) \notin E_3$. Observe that the newly introduced set of edges basically represent forbidden mappings.

Example 1: This graph triple can pack.
More Definitions

**Definition**

A *list packing* of the graph triple $G$ is a bijection $f : V_1 \rightarrow V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and $v \in V_1$ implies $vf(v) \notin E_3$. Observe that the newly introduced set of edges basically represent forbidden mappings.

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Example 2: This graph triple does not pack.
A list packing of the graph triple $G$ is a bijection $f : V_1 \rightarrow V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and $v \in V_1$ implies $vf(v) \notin E_3$. Observe that the newly introduced set of edges basically represent forbidden mappings.

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Does the above configuration pack?
Observation

With list packing, we still have the same problems with induction, but we can fix them with a minor adjustment.
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Induction Revisited

Observation

With list packing, we still have the same problems with induction, but we can fix them with a minor adjustment.

Now, we can ask if this packs. But how did the number of edges change? And the maximum degrees?
Theorem (Győri, Kostochka, McConvey, Y 2016)

Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$ with $\Delta_1 \Delta_2 + \Delta_3 \leq \frac{n}{2}$. Then $G$ does not pack if and only if $\Delta_3 = 0$ and one of $G_1$ and $G_2$ is a perfect matching and the other is either $K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

Theorem (Győri, Kostochka, McConvey, Y 2016)

Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$. If $|E_1| + |E_2| + |E_3| \leq \frac{3}{2} n - 2$, then $G$ packs.
Theorem (Bollobás, Eldridge 1978)

If $\Delta_1, \Delta_2 \leq n - 2, |E_1| + |E_2| \leq 2n - 3$, and \{G_1, G_2\} is not one of the 7 pairs shown below, then $G_1$ and $G_2$ pack.
Extensions of Previous Results

Theorem (Győri, Kostochka, McConvey, Y 2016)

Let \( G = (G_1, G_2, G_3) \) be a graph triple of size \( n \). If \( \Delta_1, \Delta_2 \leq n - 2 \), \( \Delta_3 \leq n - 1 \), and \( |E_1| + |E_2| + |E_3| \leq 2n - 3 \), and \( \{G_1, G_2\} \) is not one of the Bollobás-Eldridge pairs. Then either \( G \) packs or is one of the same 7 examples.

The result is sharp:
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The result is sharp:

$$|E_1| + |E_2| + |E_3| = (n - 1) + n = 2n - 2$$
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Let \( G = (G_1, G_2, G_3) \) be a graph triple of size \( n \). If \( \Delta_1, \Delta_2 \leq n - 2 \), \( \Delta_3 \leq n - 1 \), and \( |E_1| + |E_2| + |E_3| \leq 2n - 3 \), and \( \{G_1, G_2\} \) is not one of the Bollobás-Eldridge pairs. Then either \( G \) packs or is one of the same 7 examples.

The result is sharp:

\[
|E_1| + |E_2| + |E_3| = 2(n - 1) = 2n - 2
\]
Theorem (Győri, Kostochka, McConvey, Y 2016)

Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$. If $\Delta_1, \Delta_2 \leq n - 2$, $\Delta_3 \leq n - 1$, and $|E_1| + |E_2| + |E_3| \leq 2n - 3$, and \{ $G_1, G_2$ \} is not one of the Bollobás-Eldridge pairs. Then either $G$ packs or is one of the same 7 examples.

The result is sharp:

$$|E_1| + |E_2| + |E_3| = 1 + 1 + 2(n - 2) = 2n - 2$$
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**Theorem (Győri, Kostochka, McConvey, Y 2016)**

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The result is sharp:
\[ |E_1| + |E_2| + \max\{\Delta_1, \Delta_2\} \leq 3n - 68n^{3/4} - 62, \]

then \( G_1 \) and \( G_2 \) pack.

Conjecture (Zak 2014)

Let \( G_1 \) and \( G_2 \) be graphs on \( n \) vertices with \( \Delta_1, \Delta_2 \leq n - 2 \). If

\[ |E_1| + |E_2| + \max\{\Delta_1, \Delta_2\} \leq 3n - 7, \]

then \( G_1 \) and \( G_2 \) pack.
\[ |E_1| + |E_2| + \max\{\Delta_1, \Delta_2\} = 10 + 12 + 4 = 3n - 10 \leq 3n - 7 \]

For large \( n \), conjecture is close to best possible:

\[ |E_1| + |E_2| + \max\{\Delta_1, \Delta_2\} = (n - 2) + n + (n - 2) = 3n - 4 \geq 3n - 7 \]
Main Result

**Theorem (Győri, Kostochka, McConvey, Y 2016)**

Let $G = (G_1, G_2, G_3)$ be a graph triple of size $n$ with $\Delta_1, \Delta_2 \leq n - 2$ and $\Delta_3 \leq n - 1$. There is an absolute constant $C$ such that if $|E_1| + |E_2| + |E_3| + D \leq 3n - C$, then $G$ packs.

Our current proof gives $C = 418275$.

**Corollary**

Let $G_1$ and $G_2$ be a graphs of order $n$ with $\Delta_1, \Delta_2 \leq n - 2$. There is an absolute constant $C$ such that if $|E_1| + |E_2| + \max\{\Delta_1, \Delta_2\} \leq 3n - C$, then $G_1$ and $G_2$ pack.
Thank You
References

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Extremal graphs for a graph packing theorem of sauer and spencer.  

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On packing two graphs with bounded sum of size and maximum degree.  
We assign an initial charge to each vertex, edge, and face of a graph. This yields a total charge of the entire graph. We then apply rules that move charge around locally while conserving the total charge of the graph. By choosing appropriate initial charges and applying clever rules, we hope to either obtain the desired structure OR reach a contradiction to our conserved total charge.
Discharging

We assign an initial charge to each vertex, edge, and face of a graph. This yields a total charge of the entire graph. We then apply rules that move charge around locally while conserving the total charge of the graph. By choosing appropriate initial charges and applying clever rules, we hope to either obtain the desired structure OR reach a contradiction to our conserved total charge.

Figure: Every vertex receives charge equal to its degree.
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Figure: Rule - Every vertex of degree 4 gives 1 to each of its neighbors.
We assign an initial charge to each vertex, edge, and face of a graph. This yields a total charge of the entire graph. We then apply rules that move charge around locally while conserving the total charge of the graph. By choosing appropriate initial charges and applying clever rules, we hope to either obtain the desired structure OR reach a contradiction to our conserved total charge.
Discharging Overview

General steps:
1. Place a set amount of charge on each of the vertices
2. Rearrange the charge using a set of rules
3. Recount the total charge on vertices to arrive at contradiction

Overview of our set-up:
- Each vertex \( v \) gets charge equal to \( d(v) \)
- Total Charge should be about twice the edge sum, which is less than \( 6n \)
- After discharging, each vertex has charge roughly 3
Completion of the Proof

The discharging argument guarantees the following structure, and the proof of the theorem follows.