MATH 241 :: Calculus III

Mock Exam II

October 20, 2018

Name: _______________________________________________________

- Show ALL your work.
- There are NO calculators allowed on this exam.
- This exam has 13 questions.
1. Mark the following as True or False

<table>
<thead>
<tr>
<th>Equation</th>
<th>T or F</th>
</tr>
</thead>
<tbody>
<tr>
<td>1  The integral of $f(x, y) = x^2 + y^2 - 2$ along $x^2 + y^2 = 1$ is the surface area of the “curtain” formed by the function along the curve.</td>
<td></td>
</tr>
<tr>
<td>2  If a given vector field $F = (P, Q)$ on $\mathbb{R}^2 \setminus {0}$ is conservative, then $P_y = Q_x$</td>
<td></td>
</tr>
<tr>
<td>3  The gradient vector $\nabla f(p)$ is tangent to the level set of $f$ at $p$</td>
<td></td>
</tr>
<tr>
<td>4  The integral of the vector field $F$ along a curve $C$ is independent of the orientation.</td>
<td></td>
</tr>
<tr>
<td>5  A given vector field $F = (P, Q)$ on the domain $\mathbb{R}^2 \setminus {0}$ is conservative if $P_y = Q_x$.</td>
<td></td>
</tr>
</tbody>
</table>

2. Consider the curve:

$$\gamma : [-1, 1] \to \mathbb{R}^2, \quad \gamma(t) = (t^2, t^3)$$

Identify which of the following corresponds to the image of this curve.

![Figure 1](image1.png)

![Figure 2](image2.png)

![Figure 3](image3.png)

![Figure 4](image4.png)
3. Determine whether the following sets are (a) open, (b) closed, (c) connected, (d) simply connected, (e) bounded, and/or (f) none of the above. Put all that apply.

A. \( \{(x, y) \in \mathbb{R}^2 \mid |x| > 1\} \)

B. \( \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\} \)

C. \( \{(x, y) \in \mathbb{R}^2 \mid xy = 1\} \)

D. \( \{(x, y) \in \mathbb{R}^2 \mid |x| < 1\} \)

4. Consider the curve:
\[ \gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = (t^2, \cos(t), t^3) \]

(a) Find the velocity and acceleration vectors of this curve.

(b) Find a parametric equation for the tangent line at \((\pi^2, -1, \pi^3)\).
5. Compute:

(a) \( \int_C xy^5 \, ds \) where \( C \) is composed of the line segment from \((0, -1)\) to \((0, 1)\) and the right half of the unit circle.

(b) \( \int_C F \cdot dr \) where \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) is defined by \( F(x, y, z) = (x, z, x + y) \) and \( C \) is the line segment starting at \((1, 0, 0)\) and ending at \((1, 2, 1)\).

(c) \( \int_C y^3 \, dx + x^2 \, dy \) where \( C \) is part of the parabola \( x = 1 - y^2 \) starting at \((0, -1)\) and ending at \((0, 1)\).
6. Find the maxima and minima of $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = 2x + y^2 - x^2$ on the region bounded by and including the triangle with vertices (0, 2), (0, -2), and (2, 0).
7. The following table gives the values of an infinitely differentiable function \( f : \mathbb{R}^2 \to \mathbb{R} \) and its derivatives.

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(f(x, y))</th>
<th>(f_x(x, y))</th>
<th>(f_y(x, y))</th>
<th>(f_{xx}(x, y))</th>
<th>(f_{yy}(x, y))</th>
<th>(f_{xy}(x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>20</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>(\sqrt{2})</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>18</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>32</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>(1, −2)</td>
<td>32</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>10</td>
</tr>
</tbody>
</table>

(a) Identify the critical points from the above choices.

(b) If possible from the given data, classify the critical points you identified and state the corresponding value. Indicate if a critical point can’t be classified from the given data.
8. Consider the vector field:

\[ F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F(x, y) = (2xy, x^2 - y) \]

Show that \( F \) is conservative by finding a potential function. Verify it.

9. Consider the curve:

\[ \gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = 3t\mathbf{i} + (2 - t)\mathbf{j} - (3 - 4t)\mathbf{k} \]

(a) Calculate the arc-length function \( s(t) \) for \( t \geq 0 \).

(b) Find a reparametrization \( \tilde{\gamma}(s) = \gamma(\phi(s)) \) with \( \tilde{\gamma}(0) = \gamma(0) \) and the length of the segment of the curve \( \tilde{\gamma} \) between \( \tilde{\gamma}(0) \) and \( \tilde{\gamma}(s) \) is \( s \).
10. Consider the above level sets of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and answer the following questions:

(a) Is $D_u f(p)$ positive, negative, or 0?
(b) $\int_C f \, ds$ positive, negative, or 0?
(c) What is the value of $\int_E \nabla f \cdot dr$?
11. Find the work done by the force \( \mathbf{F}(x,y) = -yi + xj \) on a particle moving clockwise around the ellipse \( 4x^2 + 9y^2 = 36. \)

12. Let \( D \) be the domain shown below and suppose that \( \mathbf{F} \) is a vector field that is continuous on \( D \). Let \( C_1 \) and \( C_2 \) be the paths shown that travel from \( B \) to \( A \), and let \( C_3 \) and \( C_4 \) be the paths shown that travel from \( A \) to \( B \).

Suppose that \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2 \), \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2 \), \( \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 4 \), and \( \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = 4 \). Is \( \mathbf{F} \) conservative?
A. Yes  B. No  C. Not enough information
13. Consider the following three vector fields.

(a) Which is the vector field $\langle -2x, y \rangle$?  
A. Figure 5  
B. Figure 6  
C. Figure 7

(b) Exactly one of these vector fields is not conservative. Which one is it?  
A. Figure 5  
B. Figure 6  
C. Figure 7