MATH 241 :: Calculus III

Mock Exam I

September 22, 2018

Name: ________________________________

• Show **ALL** your work.
• There are **NO** calculators allowed on this exam.
• This exam has 9 questions.
1. Mark the following as True or False

<table>
<thead>
<tr>
<th>Equation</th>
<th>T or F</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) If the planes $ax + by + cz = d$ and $Ax + By + Cz = D$ are parallel then $a = A, b = B, c = C$.</td>
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<tr>
<td>b) If $i, j,$ and $k$ are the standard unit vectors, then $(k \times i) \times i = j \times (i \times k)$</td>
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<tr>
<td>c) Two non-parallel planes with normal vectors $v$ and $w$ intersect in a plane with normal vector $v \times w$.</td>
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<tr>
<td>d) The volume of the parallelepiped spanned by the vectors $(1,0,0),(0,2,0)$ and $(0,0,3)$ is 6.</td>
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<tr>
<td>e) For all vectors $v$ and $w$, the vector $w \times (w \times v)$ is perpendicular to $v$.</td>
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<tr>
<td>f) The number $</td>
<td>u \times (v \times w)</td>
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<tr>
<td>g) The equation $(u + v) \cdot (u - v) = 0$ implies $</td>
<td></td>
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<tr>
<td>h) If $a \cdot b = a \cdot c$ then $b = c$.</td>
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<tr>
<td>i) Two lines in $\mathbb{R}^3$ are either parallel or intersect.</td>
<td></td>
</tr>
<tr>
<td>j) Two planes in $\mathbb{R}^3$ are either parallel or intersect.</td>
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</table>

**Solution:** (a)F  (b)F  (c)F  (d)T  (e)F  (f)F  (g)T  (h)F  (i)F  (j)T

2. Consider the partial differential equation $u_t = u_{xx}$. Determine which of the following are solutions of this equation:

A. $u(x, t) = e^{-t} \sin(x)$
B. $u(x, t) = \sin(x - t)$
C. $u(x, t) = e^{-t} \cos(x)$
D. $u(x, t) = \cos(x - t)$
E. $u(x, t) = e^{x+t}$

**Solution:**


Take the relevant partial derivatives and see if they satisfy the equation. For example, for A:

$u_t = -e^{-t} \sin(x)$
$u_x = e^{-t} \cos(x)$
$u_{xx} = -e^{-t} \sin(x)$

and hence:

$u_t - u_{xx} = -e^{-t} \sin(x) - (-e^{-t} \sin(x)) = 0$
3. Find the limit, if it exists, or show that the limit does not exist. Show all your work.

(a) \[ \lim_{(x,y) \to (0,0)} \frac{x^4y}{x^8 + y^2} \]

Solution:
Along \( y = 0 \):
\[ \lim_{x \to 0} \frac{x^40}{x^8 + 0^2} = \lim_{x \to 0} \frac{0}{x^8} = \lim_{x \to 0} 0 = 0 \]
For \( y = x^k \) note:
\[ \frac{x^4x^k}{x^8 + x^{2k}} = \frac{x^{4+k}}{x^8 + x^{2k}} = \frac{1}{(x^8 + x^{2k})x^{-4-k}} = \frac{1}{x^{4-k} + x^{k-4}} \]
Hence, along \( y = x^4 \) (that is, for \( k = 4 \)):
\[ \lim_{x \to 0} \frac{x^4x^4}{x^8 + x^{2(4)}} = \lim_{x \to 0} \frac{1}{x^0 + x^0} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2} \]
Thus, the limit does not exist.

(b) \[ \lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} \]

Solution: Use polar coordinates.
\[ \lim_{(x,y) \to 0} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r \to 0} \frac{\sin(r^2)}{r^2} = \lim_{u \to 0} \frac{\sin(u)}{u} = 1 \]

(c) \[ \lim_{(x,y) \to (1,-1)} e^{-xy} \cos(x + y) \]

Solution: The function is continuous at \((1, -1)\) so:
\[ \lim_{(x,y) \to (1,-1)} e^{-xy} \cos(x + y) = e^{-(1)(-1)} \cos(1 - 1) = e \]
4. Consider the differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ whose level curves (or contours) are shown in the figure.

(a) What is the sign of $f_x(P)$? Negative Zero Positive.

(b) What is the sign of $f_{yy}(P)$? Negative Zero Positive.

(c) What is the sign of $f_{xy}(P)$? Negative Zero Positive.

(d) Circle the best estimate for $h'(0)$, where $h(t) = f(t^2 + 4t + 2, \cos(t) + 1)$.

-4 -2 0 2 4

**Solution:** (a) We fix $y$ at $-2$. As $x$ increases, $f$ increases, and so $f_x(P)$ is positive.

(b) Fix $x$. Now, we look at how $f_y$ changes as $y$ increases. Below $P$, $f_y$ is negative. Above $P$, $f_y$ is also negative, but it is less negative than below $P$ because the level curves are further apart in the vertical direction. Hence, $f_{yy}(P)$ is positive.

(c) Fix $x$. Now, we look at how $f_x$ changes as $y$ increases. Below $P$, $f_x$ is positive. After $P$, $f_x$ is also positive. As the level curves below $P$ are closer together in the horizontal direction than above $P$, this means that $f_x$ is larger below $P$ than above $P$. Hence, as $y$ increases, $f_x$ decreases, making $f_{xx}(P)$ negative.

(d) *** There was a TYPO in the problem. It should have said $h(t) = f(t^2 + 4t + 2, \cos(t) + 1)$.***
By the chain rule, we have that \( h'(0) = \frac{\partial f}{\partial x}(x(0), y(0)) \frac{dx}{dt}(0) + \frac{\partial f}{\partial y}(x(0), y(0)) \frac{dy}{dt}(0). \)

Since \( y'(0) = -\sin(0) = 0 \), this means we do not have to go about estimating \( \frac{\partial f}{\partial y}(x(0), y(0)) \). So, we only need to estimate \( \frac{\partial f}{\partial x}(x(0), y(0)) = \frac{\partial f}{\partial x}(2, 2) \). To estimate this quantity, we fix \( y = 2 \) and look at how the function changes around \( x = 2 \). We see that \( f(1, 2) = 0 \) and \( f(3, 2) = 2 \). Hence, \( \frac{\partial f}{\partial x}(x(0), y(0)) \approx \frac{f(3, 2) - f(2, 2)}{1} = 1 \). As \( x'(0) = 2(0) + 4 = 4 \), we see by the chain rule that \( h'(0) = 4 \).
5. Consider the following cross-sections:

Figure 1: $x$-fixed cross sections
Figure 2: $y$-fixed cross sections
Figure 3: $z$-fixed cross sections

And consider the following quadric surfaces:

Figure I
Figure II
Figure III

(a) Choose which of the above quadric surfaces (I, II, or III) has the above cross sections.

Solution: Figure II.

(b) Match EACH of the following equations to the corresponding surface (I, II, III) shown above:

A. $z^2 = 1 + 4x^2 + 4y^2$

Solution: Figure I
B. \( x^2 + y^2 = z^2 \)

**Solution:** Figure II

C. \( z = x^2 - y^2 \)

**Solution:** Figure III
6. Consider the plane $P$ defined by $x + y + z = 1$ and a line $L$ defined by

$$\frac{x - 3}{a} = \frac{2 - y}{a + 1} = \frac{z + 1}{3 - 2a}.$$ 

(a) For what value of $a$ is the line $L$ parallel to the plane $P$?

**Solution:** We first find the parametric equation of the line by setting each component of our symmetric equations equal to the parameter $t$. Then, we solve for $x$, $y$, and $z$. So, we get that $L$ is parametrized by

$$L = l(t) = (3 + at, 2 - (a + 1)t, -1 + (3 - 2a)t).$$

Now, this line is parallel to the plane $P$ exactly when the direction vector of the line is perpendicular to the normal vector of the plane. So, we set

$$(1, 1, 1) \cdot (a, -(a + 1), 3 - 2a) = 0$$

and solve for $a$:

$$a - a - 1 + 3 - 2a = 0$$

$$2a = 2$$

Therefore, when $a = 1$, the line is parallel to the plane.

(b) Find the distance from the line $L$ to the plane $P$ when $L$ is parallel to $P$.

**Solution:** Since $L$ is parallel to $P$, the distance from $L$ to $P$ is the same as the distance from ANY point that lies on the line $L$ to the plane $P$. So, we will find the distance from the point $(3, 2, -1)$ to the plane $P$. A point that lives on the plane is $(1, 0, 0)$, and the vector $v = (2, 2, -1)$ goes from the point $(1, 0, 0)$ on the plane to the point $(3, 2, -1)$. Then, the distance from the line to the plane is the length of the projection of $v$ onto the normal vector of the plane $n = (1, 1, 1)$:

$$\text{proj}_n v = \left(\frac{v \cdot n}{n \cdot n}\right) n = \left(\frac{2 + 2 - 1}{1 + 1 + 1}\right) n = (1, 1, 1).$$

Thus, the distance between $L$ and $P$ is $|(1, 1, 1)| = \sqrt{3}$.

7. Match the functions with their contour plots.
<table>
<thead>
<tr>
<th>function</th>
<th>Enter I,II,III, IV,V, VI here</th>
</tr>
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<tbody>
<tr>
<td>1 $f(x, y) = \sin(x)$</td>
<td></td>
</tr>
<tr>
<td>2 $f(x, y) = x^2 + 2y^2$</td>
<td></td>
</tr>
<tr>
<td>3 $f(x, y) =</td>
<td>x</td>
</tr>
<tr>
<td>4 $f(x, y) = \sin(x)\cos(y)$</td>
<td></td>
</tr>
<tr>
<td>5 $f(x, y) = xe^{-x^2-y^2}$</td>
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</tr>
<tr>
<td>6 $f(x, y) = x^2/(x^2 + y^2)$</td>
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**Solution:** (1)III  (2)II  (3)V  (4)IV  (5)I  (6)VI
8. Find the equation of the plane containing the two lines:

\[ l_1(t) = (-1 + t, 3t, -2 + 2t) \quad l_2(t) = (-3 - 2t, -1 - t, 1 + 3t). \]

**Solution:** Find the velocity vectors by looking at \(t\)-coefficients. For \(l_1\):

\[ v_1 = (1, 3, 2), \]

while for \(l_2\):

\[ v_2 = (-2, -1, 3). \]

Find a normal vector for the plane by taking the cross product of the velocity vectors (sketch a picture to visualize):

\[ n := v_1 \times v_2 = (11, -7, 5). \]

Pick a point on the plane:

\[ l_1(0) = (-1, 0, -2) \]

To obtain the equation of the plane simplify:

\[
0 = n \cdot \left( (x, y, z) - l_1(0) \right)
= (11, -7, 5) \cdot (x + 1, y, z + 2)
= 11x + 11 - 7y + 5z + 10
= 11x - 7y + 5z + 21,
\]

which yields the equation:

\[ 11x - 7y + 5z = -21. \]

9. Consider the vectors \(u\), \(v\), \(w\), and \(s\) shown below.

For each of the following, circle the best answer.

(a) A. \(u = 2(w + v)\)  B. \(u = 2w - s\)  C. \(u = s - 2w\)  D. \(u = 2(s + v)\)

(b) A. \(|u \times v| > |w \times u|\)  B. \(|u \times v| = |w \times u|\)  C. \(|u \times v| < |w \times u|\)
(c) A. \( u \cdot v > u \cdot w \)  
B. \( u \cdot v = u \cdot w \)  
C. \( u \cdot v < u \cdot w \)

(d) A. \( \text{proj}_s u = \frac{1}{4}u \)  
B. \( \text{proj}_s u = 2s \)  
C. \( \text{proj}_s u = s \)  
D. \( \text{proj}_s u = \frac{1}{2}u \)