THE GAP DISTRIBUTION OF DIRECTIONS IN SOME SCHOTTKY GROUPS

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Abstract. We prove the existence and some properties of the limiting gap distribution for the directions of some Schottky group orbits in the Poincaré disk. A key feature is that the fundamental domains for these groups have infinite area.

1. Introduction

In this paper, we investigate the gap distribution of directions of orbits of a class of discrete groups of hyperbolic isometries. Specifically, we let $\Gamma_0$ be the Schottky group generated by three reflections $\rho_1, \rho_2, \rho_3$, with non-intersecting isometry half-circles $C_1, C_2, C_3$ in the Poincaré disk $\mathbb{D}$ (see Figure 1). We give coordinates to $\mathbb{D}$ by identifying $\mathbb{D}$ with the unit disk centered at the origin in the complex plain $\mathbb{C}$. The region $\mathcal{F}$ bounded by $C_1, C_2, C_3$ is a fundamental domain for $\Gamma_0$ and has infinite area. We assume $1 \in \mathcal{F}$ and we want to study the distribution of $B_T = B_T(1)$ on $\partial\mathbb{D}$ as $T$ goes to infinity, where $B_T$ is the collection of all $\gamma \in \Gamma_0$ such that $\|\gamma\| \leq T$ for some norm $\|\cdot\|$ on $\text{Isom}(\mathbb{D})$. Note that $\partial\mathbb{D}$ can be identified with directions of geodesic rays emitted from the origin, which explains the title.

Let $C(z, r)$ be the circle on $\mathbb{C}$ having center $z$ and radius $r$. The norm $\|\cdot\|$ is defined as follows: we place a circle $C_I = C(1-r_0, r_0) \subset \mathcal{F}$ tangent to $\partial\mathbb{D}$ at 1 (see Figure 2), where $I$ denotes the identity element of $\text{Isom}(\mathbb{D})$ and $r_0$ is restricted to the condition given at (1) in the following. Let $C_\gamma = \gamma(C_I) = C((1-r_\gamma)e^{i\theta}, r_\gamma)$. We define

$$\|\gamma\| := \sqrt{\kappa(C_\gamma)} = \sqrt{1/r_\gamma},$$

where $\kappa(C_\gamma)$ is the curvature, or the reciprocal of the radius of $C_\gamma$. In other words, we can view $B_T$ as tangencies of circles in $\Gamma_0(C_I)$ with curvatures bounded by $T^2$.

The group orbit $\Gamma_0(1)$ under our investigation converges to a limit set $\Lambda_{\Gamma_0} \subset \partial\mathbb{D}$, where $\Lambda_{\Gamma_0}$ is a disconnected, perfect set with Hausdorff dimension $< 1$. This is in contrast with a lattice orbit on $\partial\mathbb{D}$, which is a dense subset of $\partial\mathbb{D}$. Let $\delta$ be the critical exponent for $\Gamma_0$ (roughly speaking, $\delta$ measures the asymptotic growth rate of $\Gamma$). In our setting, $\Gamma_0$ is convex co-compact, so $\delta$ also agrees with the Hausdorff dimension of the limit set $\Lambda_{\Gamma_0}$ (see Theorem 7 of [18], and see Page 12 and Page 32 of [13] for the computation of $\delta$ for some of such groups). There is a canonical probability measure...
\( \mu \) supported on \( \Lambda_{\Gamma_0} \), which is called the Patterson-Sullivan measure. The measure \( \mu \) is a \( \Gamma_0 \) invariant, \( \delta \)-dimensional geometric measure without atom [15][18][19].

We choose \( r_0 \) in the following way: for any \( x, y \in \mathbb{C} \), let \( d_E(x, y) \) be the Euclidean distance between \( x \) and \( y \). We assume none of \( C_i (i = 1, 2, 3) \) is a diameter, then each \( C_i \) divides the circle \( \partial \mathbb{D} \) into a major arc \( \mathfrak{M}_i \) and a minor arc \( \mathfrak{m}_i \). We further assume that the minor arcs \( \mathfrak{m}_i (i = 1, 2, 3) \) are disjoint. Let \( D_i \) be the region in \( \mathbb{D} \) bounded by \( C_i \) and \( \mathfrak{m}_i \), we then define the Euclidean distance between the two sets \( D_i \) and \( D_j \) (we still use the notation \( d_E \)) by

\[
d_E(D_i, D_j) = \inf_{x, y} \{d_E(x, y) | x \in D_i, y \in D_j \},
\]

We choose \( r_0 \) so that

\[
r_0 < \frac{1}{3} \min_{i \neq j} \{d_E(D_i, D_j)\} \quad (1)
\]

Now we introduce our problem. Let \( \mathcal{I} \) be an interval of \( \partial \mathbb{D} \) with \( \mu(\mathcal{I}) > 0 \). Let \( \mathcal{B}_{\mathcal{I}} = \{\gamma \in \mathcal{B}_\gamma, \gamma(1) \in \mathcal{I} \} \), and let \( \mathcal{B}_{\mathcal{I}} = \{x_i\} \) be the counter-clockwise oriented sequence of the points in \( \mathcal{B}_{\mathcal{I}}(1) \). The gap between \( x_i \) and \( x_{i+1} \), denoted by \( d(x_i, x_{i+1}) \), is just the arc length distance between \( x_i \) and \( x_{i+1} \). The size of \( B_{\mathcal{I}} \) is asymptotically \( c_0 \mu(\mathcal{I})T^{2\delta} \) for some positive constant \( c_0 \) independent of \( \mathcal{I} \) (Theorem 2.3). Therefore, when defining the gap distribution function, the first thought is to normalize the gaps in \( B_{\mathcal{I}} \) by dividing them by \( 1/T^{2\delta} \). However, it turns out most gaps are not of the order of the average gap \( 1/T^{2\delta} \), but of the order \( 1/T^2 \), which is the scale for a lattice. Moreover, unlike a typical lattice orbit (for example, Farey sequences), any gap in \( B_{\mathcal{I}} \) has two end points \( \gamma_1(1), \gamma_2(1) \) with \( \|\gamma_1\|, \|\gamma_2\| \) comparable (see Theorem 4.2).
We define our gap distribution function to be
\[ F_{T,I}(s) = \frac{1}{\mu(T)} \sum_{x_i \in B_{T,I}} 1\{d(x_i, x_{i+1}) \leq s\}. \]  
\( (2) \)

Our main theorem is the following:

**Theorem 1.1.** Let \( F_{T,I} \) be as above. There exists a monotone, continuous function \( F \) on \([0, \infty)\), independent of \( I \), such that
\[
\lim_{T \to \infty} F_{T,I}(s) = F(s).
\]

Moreover, \( F \) is supported away from 0, and
\[
\lim_{s \to \infty} F(s) = 1.
\]

Figure 3 and Figure 4 are both empirical plots for the case illustrated at Figure 2, where \( m_1, m_2, m_3 \) are evenly spaced and each of them has arc length \( \frac{7\pi}{12} \). From [13], the critical exponent in this case is \( \delta \approx 0.62627635 \). The circle \( C_I \) in Figure 2 is \( C(0.92, 0.08) \). Figure 3 is the plot for the gap distribution function \( F_{T,I} \), where \( T = 10^4 \) and \( I \) is the full boundary \( \partial \mathbb{D} \). Figure 4 are normalized histograms for the densities of the gap distributions, or “\( F'_{T,I} \)”, with step taken to be 1, for various \( T \)’s and \( I \)’s.

![Figure 3. The plot for the gap distribution function \( F_{10^4, \partial \mathbb{D}} \), for the example illustrated in Figure 2](image_url)

The scale \( 1/T^2 \) agrees with what a random point process on \( \Lambda_{\Gamma_0} \) predicts (See Section 6). This is not surprising, as \( B_T \) tends to accumulate at those tiny clusters which make up the limit set \( \Lambda_{\Gamma_0} \), instead of the whole boundary.

We will work with the group \( \Gamma = \langle \rho_1 \rho_2, \rho_1 \rho_3 \rangle \), which is the index-2, orientation-preserving and free subgroup of \( \Gamma_0 \). As a finite-index subgroup of an infinite co-volume group \( \Gamma_0 \), \( \Gamma \) is also of infinite co-volume, and the limit set \( \Lambda_\Gamma = \Lambda_{\Gamma_0} \). If \( \Gamma' \) is of finite co-volume, or a lattice in \( \text{Isom}(\mathbb{H}) \), Kelmer and Kontorovich [9] proved the limiting pair correlation of the directions of a single \( \Gamma' \)-orbit in \( \mathbb{H} \). This generalizes the work of Boca, Popa, and Zaharescu [3] which deals with the case when \( \Gamma' \) is
SL(2, Z) and the observer is at an elliptic point. Later, the work [9] was further generalized by Risager and Södergren [16] to cover the cases SO(n, 1) with explicit convergence rate, and by Marklof and Vinogradov [12] which determines a large class of local statistics, including gap distribution.

Each of [9], [16] and [12] used some tools either from homogeneous dynamics or from spectral theory. Applying such tools to spacing statistics studies dates back at least to [10]. See also [6], [11], [17], [1], [2] for some interesting works of this flavor.

In fact, all these works have the underlying groups lattices. The novelty of this paper is that it deals with some infinite co-volume orbits. In doing so, we exhibit both similarities and dissimilarities with lattice cases. Furthermore, our work serves to give a general flavor for this type of problems.

Compared to [9], [12], we use the norm \(\|\cdot\|\) instead of the Frobenius norm, given by

\[
\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_F = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}.
\]

The advantage is that the gaps between these tangencies can be explicitly parametrized by \(\Gamma\). Indeed, we manage to reduce counting gaps in \(B_T(1)\) with normalized length \(s\) to counting elements in sets of the form \(\{\gamma \in \Gamma : \gamma \in \Omega_T\}\), where \(\Omega_T\) are a family of homogeneously expanding sets (see the statement of Corollary 2.2). Then we can apply a bisector counting theorem from Oh-Shah [14] to achieve the counting. The idea of reduction to orbital counting grows out of the work [17]. However, the combinatorics here is more intricate. Unlike [17] where we can classify gaps by finitely many cases (corresponding to finitely many \(\Omega_T\)'s), in this paper we need to consider finitely many families (depending on the ending of \(\gamma\)), each of which contains infinite many cases. Nevertheless, a limiting argument shows that the limiting gap distribution is indeed the sum of the limiting contribution of gaps from each of

\[
(A) \quad T = 10^4, I = \partial \mathbb{D} \\
(b) \quad T = 10^4, I = \partial \mathbb{D} \\
(c) \quad T = 10^4, I = (-3.080528, -0.613722)
\]

FIGURE 4. The histograms of \(F'_{T,I}\) of different \(T\), for the example illustrated in Figure 2.
these infinitely many cases.

**Notation:** For convenience we adopt the following standard notations throughout the paper: our main growing parameter is $T$. By $f \ll g$ ($f \gg g$, $f \asymp g$) we mean there exists a positive constant $B$ such that $|f| \leq B|g|$ ($|f| \geq B|g|$, $\frac{1}{B}|g| \leq |f| \leq B|g|$). The expression $f = O(g)$ is synonymous with $f \ll g$, and $f = o(g)$ is synonymous with $\frac{f}{g} \to 0$. Without further explanation all the implied constants depend at most on $\Gamma_0$ and the norm $\|\cdot\|$.

**Plan for the paper:** In Sec. 2 we state several counting theorems about $\Gamma$. In Sec. 3 we establish several statements on the relationships between the norm $\|\cdot\|$ and the word length of an group element in $\Gamma_0$. An auxiliary tool is to compare the norm $\|\cdot\|$ with the Frobenius norm $\|\cdot\|_F$. In Sec. 4 we divide $\Gamma$ into finite families of infinitely many cases according to the ending letters of $\gamma$, where we use the results in Sec. 3 which guarantees that in each case we only need to check bounded many conditions when determining the two neighbors of a given point in $B_{T,I}$. In Sec. 5 we prove our main theorem, where the results in Sec. 3 allow us to pass the limit when assembling the contributions from these infinite many cases. We conclude this paper by analyzing some limiting behaviors of a random process on $\Lambda_{\Gamma_0}$ in Sec. 6.

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### 2. Several Counting Statements

Let $\mathbb{D}$ be the Poincaré disc $\{z = x + yi \in \mathbb{C} : x^2 + y^2 < 1\}$, with the metric

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2}.$$

The orientation-preserving symmetry group of $\mathbb{D}$ is

$$G = PSU(1, 1) = \left\{ \begin{pmatrix} \xi & \eta \\ \bar{\eta} & \bar{\xi} \end{pmatrix} \bigg| |\xi|^2 - |\eta|^2 = 1, \xi, \eta \in \mathbb{C} \right\} \cong PSL_2(\mathbb{R}).$$

Let

$$K = \left\{ \begin{pmatrix} e^{\phi i} & 0 \\ 0 & e^{-\phi i} \end{pmatrix} \bigg| \phi \in [0, 2\pi) \right\},$$

$$A = \left\{ \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} \bigg| t \in (-\infty, \infty) \right\}.$$
and

$$A^+ = \left\{ \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} \middle| t \in (0, \infty) \right\}. $$

Recall the Cartan decomposition $G = KA^+K$ that each $g \in G - K$ can be written in a unique way as

$$g = k_{\phi_1(g)}a_{t(g)}k_{\pi - \phi_2(g)} = \left( e^{-\frac{\phi_1(g)}{2}} \right) \begin{pmatrix} \cosh \frac{t(g)}{2} & \sinh \frac{t(g)}{2} \\ \sinh \frac{t(g)}{2} & \cosh \frac{t(g)}{2} \end{pmatrix} \left( e^{-\frac{\pi - \phi_2(g)}{2}} \right) \begin{pmatrix} \sinh \frac{t(g)}{2} \\ \cosh \frac{t(g)}{2} \end{pmatrix}$$

with $\phi_1(g), \phi_2(g) \in [0, 2\pi)$ and $t(g) > 0$, so this determines $\phi_1, \phi_2, t$ as functions of $g$. The Haar measure is given by $dg = e^t d\phi_1 d\phi_2 dt$.

Our crucial tool is the following bisector counting theorem:

**Theorem 2.1** (Good, Bourgain-Kontorovich-Sarnak, Oh-Shah). Let $\Gamma$ be a non-elementary, geometrically finite discrete subgroup in $PSU(1, 1)$. Let $I, J$ be two intervals in $[0, 2\pi)$ with $\mu(I), \mu(J) > 0$, then

$$\sum_{\gamma \in \Gamma} 1 \begin{cases} \phi_1(\gamma) \in I \\ \phi_2(\gamma) \in J \\ t(\gamma) \leq T \end{cases} \sim c_\Gamma \mu(I)\mu(J)e^{\delta T}$$

for some positive constant $c_\Gamma$ which only depends on $\Gamma$.

The case when $\Gamma$ is a lattice, or equivalently $\delta = 1$ was proved by Good [8]. Later it was generalized by Bourgain, Kontorovich and Sarnak to cover the cases when $\delta > \frac{1}{2}$ using the unitary representation theory of $SL(2, \mathbb{R})$ [4]. Oh and Shah [14] further extended this counting result to cover the cases when $0 < \delta \leq \frac{1}{2}$ using mixing of the geodesic flow under the Bowen-Margulis-Sullivan measure [14].

Theorem 2.1 has the following corollary which is applied to our analysis:

**Corollary 2.2.** Let $I$ be as above. Define a measure $dS = e^{\delta t} dt \times d\mu$ on $[0, \infty) \times [0, 2\pi)$. Let $\Omega = \Omega_0$ be a bounded set with piecewise smooth boundary in $[0, \infty) \times [0, 2\pi)$ and $S(\Omega_0) > 0$, and let $\Omega_T = \{(t, \theta) \in [0, \infty) \times [0, 2\pi) | (t - T, \theta) \in \Omega\}$. Then as $T$ goes to $\infty$,

$$\sum_{\gamma \in \Gamma} 1 \begin{cases} \phi_1(\gamma) \in I \\ (t(\gamma), \phi_2(\gamma)) \in \Omega_T \end{cases} \sim c_\Gamma \mu(I)S(\Omega_0)e^{\delta T},$$

where $c_\Gamma > 0$ depends only on $\Gamma$. 
Theorem 2.3. There exists a positive constant \( c_0 \) such that, as \( T \) goes to \( \infty \),
\[
#B_{T,I} \sim c_0 \mu(I) T^{2\delta}.
\]
The constant \( c_0 \) depends only on \( \Gamma_0 \) and \( r_0 \).

Theorem 2.3 also follows from a simple modification of the method in [17].

3. Compare Norms

Each \( \gamma \in \Gamma_0 \) can be expressed in a unique way as \( S_1 S_2 \cdots S_l(\gamma) \), where each \( S_i \in \{ \rho_1, \rho_2, \rho_3 \} \) and \( S_i \neq S_{i+1} \). We call \( S_1 S_2 \cdots S_l(\gamma) \) the reduced word for \( \gamma \) and \( l(\gamma) \) the word length of \( \gamma \). Clearly we have \( \gamma \in \Gamma \) if and only if \( l(\gamma) \) is even. We denote the initial letter \( S_1 \) for \( \gamma \) by \( \text{Int}(\gamma) \), and the ending letter \( S_l(\gamma) \) for \( \gamma \) by \( \text{End}(\gamma) \). By convention, all the words in this paper are reduced. In this section we prove several relationships between \( \| \cdot \| \) and \( l(\cdot) \).

Using the decomposition (3), any \( \gamma \in \Gamma \) maps the circle \( C((1 - r)e^{\theta i}, r) \) to a circle of curvature
\[
\kappa(\gamma(C((1 - r)e^{\theta i}, r))) = e^{t(\gamma)} \frac{1 - r}{2r} (1 + \cos(\pi - \phi_2(\gamma) + \theta))
\]
\[
+ e^{-t(\gamma)} \frac{1 - r}{2r} (1 - \cos(\pi - \phi_2(\gamma) + \theta)) + 1
\]
and tangent to \( \partial \mathbb{D} \) at
\[
ex_p \left( \left( \phi_1(\gamma) + \arcsin \frac{2 \sin(\pi - \phi_2(\gamma) + \theta)}{e^{t(\gamma)}(1 + \cos(\pi - \phi_2(\gamma) + \theta)) + e^{-t(\gamma)}(1 - \cos(\pi - \phi_2(\gamma) + \theta))} \right) i \right).
\]

Lemma 3.1. The norms \( \| \cdot \| \) and \( \| \cdot \|_F \) are equivalent in \( \Gamma \).

Proof. First we write \( \gamma = k_0(\gamma) a_t(\gamma) k_{\pi - \phi_2(\gamma)} \). Then \( \gamma(0) = e^{\phi_1(\gamma) i} \sec t(\gamma) \) since \( 1 \not \in \Lambda_\Gamma \), \( \phi_1(\gamma) \) stays uniformly away from 0 for \( \gamma \in \Gamma - \{ I \} \).

Since \( \gamma = (k_0(\gamma) a_t(\gamma) k_{\pi - \phi_2(\gamma)})^{-1} = k_0(\gamma) a_t(\gamma) k_{\pi - \phi_1(\gamma)} \), we have \( \phi_1(\gamma^{-1}) = \phi_2(\gamma) \), \( \phi_2(\gamma^{-1}) = \phi_1(\gamma) \) and \( t(\gamma^{-1}) = t(\gamma) \). By similar consideration as above, we have \( \phi_2(\gamma) \) stays uniformly away from 0.

Now Lemma 3.1 can be seen directly. First we have \( \| \gamma \|_F^2 = \| a_t(\gamma) \|_F^2 = \frac{e^{t(\gamma)} + e^{-t(\gamma)}}{2} \approx e^{t(\gamma)} \), since \( \| \cdot \|_F \) is invariant under \( K \).
From (4) we have
\[
\|\gamma\|^2 = \kappa(\gamma(C)) = e^{t(\gamma)} \frac{1-r_0}{2r_0} (1 + \cos(\pi - \phi_2(\gamma))) + e^{-t(\gamma)} \frac{1-r_0}{2r_0} (1 - \cos(\pi - \phi_2(\gamma))) + 1.
\]
Since \(\phi_2(\gamma)\) stays uniformly away from 0, we have \(\|\gamma\|^2 \asymp e^{t(\gamma)}\). \(\square\)

**Figure 5.** Reflecting a circle

Recall \(\gamma = S_1S_2 \cdots S_{l(\gamma)}\) is the reduced word for \(\gamma\). The following lemma gives a relationship between \(\|\gamma\|\) and \(l(\gamma)\):

**Lemma 3.2.** There exists \(b > a > 1\), such that
\[
a^{t(\gamma)} \ll \|\gamma\| \ll b^{l(\gamma)}.\]

**Proof.** The isometry circles of \(\rho_1, \rho_2, \rho_3\) are \(C_1, C_2, C_3\), and each of them divides \(\partial \mathbb{D}\) into a major arc \(M_1, M_2, \text{ or } M_3\) and a minor arc \(m_1, m_2, \text{ or } m_3\). By induction one can show that for each \(i\), \(S_i \cdots S_n(1) \in m_i \subset M_{i+1}\). Each \(\rho_i\) maps a circle tangent on \(M_i\) to a circle tangent on \(m_i\) with smaller radius. This can be quantified as follows:

Let \(R_i\) be the radius of \(C_i\). Let \(C'\) be any circle tangent to \(m_j(j \neq i)\) with radius \(r' < r_0\) so that \(C'\) does not intersect \(C_i\) (See Figure 5 where we set \(i = 1\) and \(j = 2\)). Let \(L\) be the distance between the centers of \(C_i\) and \(C'\). Then \(\rho_i\) will map \(C'\) tangent to \(m_j\) with curvature
\[
\frac{L^2 - r'^2}{R_i^2} \frac{1}{r'}.
\]

By elementary geometric consideration, we have the following crude estimate:
\[
\frac{(R_i + 2)^2}{R_i^2} > \frac{L^2 - r'^2}{R_i^2} > 1 + \frac{r_0}{3R_i}.
\]
The first inequality holds because
\[ \frac{L^2 - r^2}{R_i^2} < \frac{(L + r')^2}{R_i^2}, \]
and \( L + r' \) is the Euclidean length of \( O_iP \), the segment connecting \( O_i \) and \( P \). Here \( O_i \) is the center of \( C_i \), and \( P \) is the point on \( C' \) such that the center of \( C' \) lies on the segment \( O_iP \). Suppose \( C_i \) intersects \( \partial D \) at a point \( Q \), then by the triangle inequality,
\[ L + r' = d_E(O_i, P) < d_E(O_i, Q) + d_E(Q, P) < R_i + 2. \]

The second inequality is obtained in a similar fashion.

Let \( a = \min_i \left\{ 1 + \frac{r_0}{3R_i} \right\} \) and \( b = \max_i \left\{ \frac{(R_i + 2)^2}{R_i^2} \right\} \). Then by induction we can prove that
\[ a^n \frac{1}{r_0} < \kappa(\gamma(C)) < b^n \frac{1}{r_0}. \]

The above lemma has the following generalization:

Lemma 3.3. For any \( w, \gamma \in \Gamma_0 \) with \( l(w) = n \), if \( \text{End}(\gamma) \neq \text{Int}(w) \), then
\[ a^n \| \gamma \| \ll \| w \cdot \gamma \| \ll b^n \| \gamma \|. \]
Similarly, if \( \text{Int}(\gamma) \neq \text{End}(w) \), then
\[ a^n \| \gamma \| \ll \| \gamma \cdot w \| \ll b^n \| \gamma \|. \]
The implied constants are independent of \( w \) and \( \gamma \).

Proof. If \( \text{Int}(\gamma) = \rho_i \) and \( \text{End}(w) = \rho_j \) with \( j \neq i \), then \( \gamma(C_i) \) is a circle tangent to \( m_i \). Then a same argument as in Lemma 3.2 shows that
\[ a^n \kappa(\gamma(C)) \ll \kappa(\gamma(w(C))) \ll b^n \kappa(\gamma(C)), \]
or
\[ a^n \| \gamma \| \ll \| w \cdot \gamma \| \ll b^n \| \gamma \|. \]
The second case is reduced to the first case immediately, using the fact that \( \| \gamma \cdot w \| \asymp \| \gamma \cdot w \|_F \) and \( \| \gamma \cdot w \|_F = \| (\gamma \cdot w)^{-1} \|_F \). \( \square \)

We will also need the following lemma:

Lemma 3.4. Let \( \rho_i, \rho_j, \rho_k \) be three distinct reflections from \( \{ \rho_1, \rho_2, \rho_3 \} \). For any \( \gamma \in \Gamma_0 \) with \( \text{End}(\gamma) = \rho_i \), we have
\[ \| \gamma(\rho_j \rho_k)^n \| \asymp \| \gamma(\rho_k \rho_j)^n \|, \]
and the implied constants independent of \( n \).
Proof. We first assume \( \gamma \in \Gamma \). Applying (5), we see that \( \gamma \) maps 1 to the point
\[
\exp\left(\left(\phi_1(\gamma) + \arcsin \frac{2 \sin(\pi - \phi_2(\gamma))}{e^{t(\gamma)}(1 + \cos(\pi - \phi_2(\gamma))) + e^{-t(\gamma)}(1 - \cos(\pi - \phi_2(\gamma))}\right)i\right).
\]
(8)

Since 1 is not a limit point for \( \Gamma \), \( \phi_1(\gamma) \), \( \phi_2(\gamma) \) stays uniformly away from 0, so that when \( t(\gamma) \) is large, the second summand in the exponent of (8) becomes small, and the position of \( \gamma(1) \) and the position of \( e^{\phi_1(\gamma)i} \) are close. On the other hand we know that \( \gamma(1) \) must lie in \( m_i \) if \( \text{Int}(\gamma) = \rho_i \). In summary we have: there exists a positive integer \( N_0 \) depending \( \Gamma_0 \) such that whenever \( l(\gamma) > N_0 \), the condition
\[
\text{Int}(\gamma) = \rho_i
\]
is equivalent to
\[
e^{\phi_1(\gamma)i} \in m_i.
\]
As a corollary, whenever \( l(\gamma) > N_0 \), the condition
\[
\text{End}(\gamma) = \rho_i
\]
is equivalent to
\[
e^{\phi_2(\gamma)i} \in m_i.
\]

From Lemma 3.3, if \( l(\gamma) \leq N_0 \), then
\[
\|\gamma(\rho_j \rho_k)^n\| \asymp \|\rho_j \rho_k\|^n = \|\rho_j (\rho_k \rho_j)^{n-1} \rho_k\| \asymp \|\rho_k \rho_j\|^n \asymp \|\gamma(\rho_k \rho_j)^n\|.
\]

Now we deal with the case \( l(\gamma) > N_0 \). Applying Lemma 3.3 successively, we have
\[
(\rho_j \rho_k)^n \asymp (\rho_k \rho_j)^{n-1} \rho_k \asymp (\rho_k \rho_j)^n,
\]
so that
\[
\kappa((\rho_j \rho_k)^n(C_1)) \asymp \kappa((\rho_k \rho_j)^n(C_1)).
\]

Since \( (\rho_j \rho_k)^n(C_1), (\rho_k \rho_j)^n(C_1) \) has tangencies \( e^{\alpha_1i}, e^{\alpha_2i} \) in \( m_j, m_k \), and \( \phi_2(\gamma) \in m_i \), we have both \( \phi_2(\gamma) + \pi - \alpha_1 \) and \( \phi_2(\gamma) + \pi - \alpha_2 \) are uniformly away from \( \pi \). Then applying (4) to the circles \( (\rho_j \rho_k)^n(C_1) \) and \( (\rho_k \rho_j)^n(C_1) \) we can see that
\[
\kappa(\gamma(\rho_j \rho_k)^n(C_1)) \asymp \kappa(\gamma(\rho_k \rho_j)^n(C_1)),
\]
or
\[
\|\gamma(\rho_j \rho_k)^n\| \asymp \|\gamma(\rho_k \rho_j)^n\|.
\]

If \( \gamma \notin \Gamma \), then write \( \gamma = \rho \gamma' \), where \( \rho \in \{\rho_1, \rho_2, \rho_3\} \), \( \gamma' \in \Gamma \) and \( \rho \neq \text{Int}(\gamma') \). Then
\[
\kappa(\gamma'(\rho_j \rho_k)^n(C_1)) \asymp \kappa(\gamma'(\rho_j \rho_k)^n(C_1)).
\]
Using the argument in Lemma 3.2, we see that
\[
a \cdot \kappa(\gamma'(\rho_j \rho_k)^n(C_1)) < \kappa(\gamma(\rho_j \rho_k)^n(C_1)) < b \cdot \kappa(\gamma'(\rho_j \rho_k)^n(C_1))
\]
and
\[
a \cdot \kappa(\gamma'(\rho_k \rho_j)^n(C_1)) < \kappa(\gamma(\rho_k \rho_j)^n(C_1)) < b \cdot \kappa(\gamma'(\rho_k \rho_j)^n(C_1)).
\]

(9)
Combining (9) and (10), we also obtain
\[ \|\gamma (\rho_j \rho_k)^n\| \asymp \|\gamma (\rho_k \rho_j)^n\| \]
when \( \gamma \in \Gamma_0 - \Gamma \). \(\square\)

4. Parametrization of gaps

Each gap in \( B_{T,\mathcal{I}} \) has two end points \( \gamma_1(1), \gamma_2(1) \) with \( \|\gamma_1\|, \|\gamma_2\| < T \). We will first partition \( \Gamma_0 \) into infinitely many classes \( \mathcal{C}_{i,j,k}^n \), where \( n \in \mathbb{N}^+ \) and \( i, j, k \) are three distinct numbers from \( \{1, 2, 3\} \), corresponding to the three reflections \( \rho_i, \rho_j, \rho_k \), and \( n \in \mathbb{N}^+ \). We have this convention for the indices \( i, j, k \) throughout this paper. Given \( \gamma_1 \in \mathcal{C}_{i,j,k}^n \cap \mathcal{B}_T \), we will completely describe the two neighbors \( \gamma_2^\pm \) of \( \gamma_1 \).

Figure 2 illustrates that in some parts of \( \mathbb{D} \) a circle \( \gamma_1(C_i) \) in \( \mathcal{B}_T(C_i) \) needs only to cross one (or a few) fundamental domain(s) to meets its neighbors; in this case typically the tail of the word of \( \gamma_1 \) contains a string \( \rho_1 \rho_2 \rho_1 \rho_3 \cdots \), because \( C_i \) lies in between the two isometry circles \( C_1, C_3 \) of the two reflections \( \rho_1, \rho_3 \). In some other parts of \( \mathbb{D} \), a circle \( \gamma_1(C_i) \) needs to cross many fundamental domains to meet its neighbors, and typically the tail of \( \gamma_1 \) contains a string \( \rho_1 \rho_2 \rho_1 \rho_3 \cdots \) or \( \rho_2 \rho_3 \rho_2 \rho_3 \cdots \).

The neighbor determinations in these two cases are different, which indeed suggests the necessity to divide \( \Gamma_0 \) according to the ending of the words.

First, by Lemma 3.3 and Lemma 3.4, we can choose a positive integer \( N_1 \) satisfying the following conditions:

1. For any \( \gamma \in \Gamma_0 \) with \( \text{End}(\gamma) = \rho_i \), we have \( \|\gamma \rho_j\| \leq \|\gamma \rho_k w\| \) for any \( w \in \Gamma_0 \) with \( l(w) \geq N_1 \) and \( \text{Int}(w) \neq \rho_k \).

2. For any \( \gamma \in \Gamma_0 \) with \( \text{End}(\gamma) = \rho_i \), we have \( \|\gamma (\rho_j \rho_k)^n w\| \leq \|\gamma (\rho_k \rho_j)^n w\| \) for any \( w \in \Gamma_0 \) with \( l(w) \geq N_1 \) and \( \text{Int}(w) \neq \rho_j \).

Let \( \mathcal{W}_{N_1} \) be the collection of \( w \in \Gamma_0 \) with \( l(w) = N_1 \). Now we define
\[
\mathcal{C}_{i,j,k}^n = \{ \gamma_1 = (\rho_j \rho_k)^n w | \gamma \in \Gamma, w \in \mathcal{W}_{N_1}, \text{End}(\gamma) = \rho_i, \text{Int}(w) \neq \rho_k \}. 
\]
We let \( \mathcal{E} \) be the set of sporadic elements, defined by
\[
\mathcal{E} = \{ \gamma_1 \in \Gamma_0 | l(\gamma_1) \leq N_1 + 2 \}
\cup \{ \gamma_1|_{\gamma_1 = (\rho_i \rho_j)^n w, w \in \mathcal{W}_{N_1}, \text{Int}(w) \neq \rho_j, i, j \in \{1, 2, 3\} \text{ distinct}, n \geq 2 \}
\cup \{ \gamma_1|_{\gamma_1 = (\rho_i \rho_j)^n w, w \in \mathcal{W}_{N_1}, \text{Int}(w) \neq \rho_j, i, j \in \{1, 2, 3\} \text{ distinct}, n \geq 1 \}
\]
It can be checked that
\[
\Gamma_0 = \left( \bigcup_{i,j,k \in \{1,2,3\} \text{ distinct}} \bigcup_{n=1}^{\infty} \mathcal{C}_{i,j,k}^n \right) \cup \mathcal{E}
\]
gives a partition of \( \Gamma_0 \). In our analysis we will remove \( \mathcal{E}(1) \) from \( B_{T,\mathcal{I}} \). As \( \#(\mathcal{E} \cap \mathcal{B}_T) \ll \log T \), removing these points only affects \( O(\log T) \) gaps, which is negligible.
compared to the number of total gaps ($\gg T^{2\delta}$), so it doesn’t affect the limiting gap distribution.

Recall that the minor arcs $\{m_1, m_2, m_3\}$ are the shorter arcs cut by the isometry circles $C_1, C_2, C_3$ from reflections $\rho_1, \rho_2, \rho_3$. For $\gamma = S_1 S_2 \cdots S_{l(\gamma)}$, we define $m_\gamma$ to be the arc $S_1 S_2 \cdots S_{l(\gamma)-1}(m)$ where $m \in \{m_1, m_2, m_3\}$ corresponding to the reflection $S_{l(\gamma)}$. We have the following simple but important observation:

**Lemma 4.1.** The following is true:

(1) If $\gamma' = \gamma \beta_1$ with $\text{End}(\gamma) \neq \text{Int}(\beta_1)$, then $m_{\gamma'} \subset m_\gamma$.

(2) If $\gamma'(1) \in m_\gamma$, then $\gamma'$ is of the form $\gamma' = \gamma \beta_2$ with $\text{End}(\gamma) \neq \text{Int}(\beta_2)$.

(3) Suppose $\gamma(1), \gamma \beta(1) \in B_{T, I}$, then the neighbor of $\gamma \beta(1)$ in $B_{T, I}$ lying on the same side with $\gamma(1)$ is of the form $\gamma \beta'(1)$.

**Proof.** (1) and (2) follow from simple induction. To prove (3), we observe that both $\gamma(1)$ and $\gamma \beta(1)$ lie in the arc $m_\gamma$, so the neighbor of $\gamma \beta(1)$ on the same side with $\gamma(1)$ is either $\gamma(1)$ itself, or some other point lying in between $\gamma(1)$ and $\gamma \beta(1)$, which must be in the arc $m_\gamma$. In any case, from (2) this neighbor is of the form $\gamma \beta'(1)$. \qed

Our goal is to determine the two neighbors of $\gamma_1(1)$ for each $\gamma_1 = (\rho_1 \rho_3)^n w \in C_{i,j,k}^{n} \cap \mathcal{B}_T$. First we observe that $\gamma_1(1), \gamma (\rho_1 \rho_3)^{-1} \rho_2 \rho_1(1) \in m_{\gamma (\rho_1 \rho_3)^{-1} \rho_2 \rho_1}$, and $|\gamma_1| \leq |\gamma (\rho_1 \rho_3)|^{-1} \rho_2 \rho_1|$. Therefore, one neighbor of $\gamma_1(1)$ in $B_T$, which we denote by $\gamma^{-}_2(1)$ for some $\gamma^{-}_2 \in \Gamma_0$, must lie in between $\gamma_1(1)$ and $\gamma (\rho_1 \rho_3)^{-1} \rho_2 \rho_1(1)$. We use $\gamma^{+}_2(1)$, for some $\gamma^{+}_2 \in \Gamma_0$, to denote the other neighbor of $\gamma_1(1)$ in $B_T$. For convenience, we say that points in a small neighborhood of $\gamma_1(1)$ are in the $-$ direction, resp., $+$ direction, to $\gamma_1(1)$ if they lie in the same direction as $\gamma^{-}_2(1)$, resp., $\gamma^{+}_2(1)$, to $\gamma_1(1)$.

We analyze two typical families: (a) $\gamma_1 = (\rho_1 \rho_3)^n w$ with $\text{End}(\gamma) = \rho_2, w \in W_{N_1}$ and $\text{Int}(w) \neq \rho_3$; (b) $\gamma_1 = (\rho_2 \rho_3)^n w(1)$ with $\text{End}(\gamma) = \rho_1, w \in W_{N_1}$ and $\text{Int}(w) \neq \rho_3$. The analysis for the other four cases are similar.

To visualize our analysis, we label the arcs from the orbits $\Gamma_0(C_1), \Gamma_0(C_2), \Gamma_0(C_3)$ by the digits 1,2,3 respectively. Then one can read the word for $\gamma_1$ by reversing the order of the digits of the arcs when $\gamma_1(1)$ is moving to the principal fundamental domain $F$.

**Case (a):** Suppose $\gamma_1(1) = (\rho_1 \rho_3)^n w(1)$ with $\text{End}(\gamma) = \rho_2, w \in W_{N_1}$ and $\text{Int}(w) \neq \rho_3$ (see Figure 6), then in the $-$ direction lies the point $\gamma (\rho_1 \rho_3)^{-1} \rho_1 \rho_2(1)$ by the definition of $-$. Since we have $|\gamma (\rho_1 \rho_3)^{-1} \rho_1 \rho_2(1)| \leq |\gamma (\rho_1 \rho_3)^n w| < T$, in the $-$ direction $\gamma^{-}_2(1)$ either lies in $m_{\gamma (\rho_1 \rho_3)^n}$ or lies in $m_{\gamma (\rho_1 \rho_3)^{-1} \rho_1 \rho_2}$, so $\gamma^{-}_2(1)$ lies in $m_{\gamma (\rho_1 \rho_3)^{n-1} \rho_1}$ in any case. Therefore, $\gamma^{-}_2(1)$ is of the form $\gamma^{-}_2(1) = (\rho_1 \rho_3)^{n-1} \rho_1 w'(1)$, and from Lemma 3.3 we have $|\gamma^{-}_2| \gg |\gamma (\rho_1 \rho_3)^{n-1} | \gg |\gamma_1|$. 
In the + direction lies \( \gamma(r_1 r_3)^{n-1} r_1(1) \). Since \( \gamma_1(1), \gamma(r_1 r_3)^{n-1} r_1(1) \in m_{\gamma(r_1 r_3)^{n-1} r_1} \) and \( \|\gamma(r_1 r_3)^{n-1} r_1\| \leq \|\gamma_1\| < T \), \( \gamma_2^+(1) \) must lie in \( m_{\gamma(r_1 r_3)^{n-1} r_1} \), thus be of the form \( \gamma_2^+(1) = \gamma(r_1 r_3)^{n-1} r_1 w'(1) \), and similarly \( \|\gamma_2^+\| \gg \|\gamma_1\| \).

Case (b): Suppose \( \gamma_1(1) = \gamma(r_2 r_3)^n w(1) \) with \( \text{End}(\gamma) = r_1 \), \( w \in W_{N_1} \) and \( \text{Int}(w) \neq r_3 \) (see Figure 7). Then in the – direction lies the point \( \gamma(r_2 r_3)^{n-1} r_2 r_1(1) \). We have \( \|\gamma_1\| \leq \|\gamma(r_2 r_3)^{n-1} r_2 r_1(1)\| < T \), so in the – direction \( \gamma_2^- \) is contained either in the arc \( m_{\gamma(r_2 r_3)^n} \) or in the arc \( m_{\gamma(r_2 r_3)^{n-1} r_2 r_1} \), so \( \gamma_2^- \) is of the form \( \gamma(r_2 r_3)^{n-1} r_2 w' \).

In the + direction, either \( \gamma_2^+(1) \) lies in the arc \( m_{\gamma(r_2 r_3)^n} \), or we have to cross \( 4n \) fundamental domains from \( \gamma_1(1) \) to find \( \gamma_2^+(1) \) in the arc \( m_{\gamma(r_2 r_3)^n} \), this is because there’s no point from \( \Gamma_0(1) \) lying in between the two arcs \( m_{\gamma(r_2 r_3)^n} \) and \( m_{\gamma(r_2 r_3)^{n-1} r_2 r_1} \), yet we manage to find a point \( \gamma(r_2 r_3)^n \) in the family \( B_T \), as \( \|\gamma(r_2 r_3)^n w\| \leq \|\gamma(r_2 r_3)^n w\| < T \).

In summary we must have \( \gamma_2^+ = \gamma(r_2 r_3)^n w' \) or \( \gamma(r_2 r_3)^n w' \) for some \( w' \in \Gamma_0 \). Similarly we also have \( \|\gamma_2^+\| \gg \|\gamma_1\| \) from Lemma 3.3 and Lemma 3.4.

The above argument shows that, for any \( \gamma_1(1) \in B_T \) with \( \gamma_1 \not\in \mathcal{E} \), if \( \gamma_2(1) \) is a neighbor of \( \gamma_1(1) \), then we must have

\[
\|\gamma_2\| \gg \|\gamma_1\|,
\]

using Lemma 3.3 and Lemma 3.4.

In fact a similar analysis shows that the relation (11) can be extended to cover the case \( \gamma_1 \in \mathcal{E} \). Since being neighbors is a reflexiv relation, we also have \( \|\gamma_1\| \gg \|\gamma_2\| \).

Thus we have the following theorem:

**Theorem 4.2.** If \( \gamma_1(1) \) and \( \gamma_2(1) \) are neighbors in \( B_T \), then \( \|\gamma_1\| \gg \|\gamma_2\| \), and the implied constant depends only on the group \( \Gamma_0 \) and \( r_0 \).
Theorem 4.2, together with Lemma 3.3 and Lemma 3.4 also implies that there exists a positive constant $N_2$ which only depends on $\Gamma_0$ and $r_0$, such that for any $w'$ in the above settings, we have $l(w') \leq N_2$. Otherwise the ratio $\|\gamma_2\|/\|\gamma_1\|$ can get arbitrarily large.

We summarize what we have obtained in this section:

**Theorem 4.3.** For any $j,k \in \{1,2,3\}$ distinct, any $w \in W_{N_1}$ with $\text{Int}(w) \neq \rho_k$, and $\square \in \{+,-\}$, there exists a sequence in $\Gamma_0$ of length at most $N_3$: $v_1^\square, v_2^\square, v_3^\square, \cdots$ satisfying the following property:

Fix $\square \in \{+,-\}$. For each $\gamma_1 = \gamma(\rho_j \rho_k)^n w \in C_{i,j,k}^n$, the neighbor of $\gamma_1(1)$ in $B_T$ in the $\square$ direction is of the form $\gamma_2^\square(1) = \gamma w^\square$, where $w^\square = v_l^\square$ for some $v_1^\square$ from the above sequence. The ordering is such that $\gamma v_1^\square(1), \gamma v_2^\square(1), \gamma v_3^\square(1) \cdots$ is from near to far to $\gamma_1(1)$ in the $\square$ direction.

The sequence $v_1^\square, v_2^\square, v_3^\square \cdots$ depends only on the word $w$ and the indexes $i,j,k,n$, but is independent of $\gamma$ (this is the key) because each $\gamma \in \Gamma_0$ acts as a diffeomorphism on $\partial \mathbb{D}$. As a result, two points $\gamma_1(1)$ and $\gamma v_1^\square(1)$ form a gap in the family $B_T$ if and only if

$$\|\gamma_1\|, \|\gamma v_1^\square\| < T \quad \text{and} \quad \|\gamma v_i^\square\| \geq T \quad \text{for} \quad i = 1, \cdots l - 1. \quad (12)$$

5. **Analyzing the gap distribution function**

In this section we are going to prove Theorem 1.1.

Recall that

$$\Gamma_0 = \bigcup_{i,j,k \in \{1,2,3\} \text{ distinct}}^\infty C_{i,j,k}^n,$$

neglecting the sporadic elements from $\mathcal{E}$. So we have

$$B_T = \bigcup_{i,j,k \in \{1,2,3\} \text{ distinct}}^\infty (B_T \cap C_{i,j,k}^n).$$

We further write

$$B_T \cap C_{i,j,k}^n = \bigcup_{w \in W_{N_1}} \bigcup_{\square \in \{+,-\}}^{\infty} \bigcup_{i=1}^{N_3} \tilde{\Omega}_{i,j,k}^n(w, \square, l, \infty; T),$$

where

$$\tilde{\Omega}_{i,j,k}^n(w, \square, l, \infty; T) = \left\{ \gamma_1 \in \Gamma_0 \mid \gamma_1 = \gamma(\rho_j \rho_k)^n w, \text{End}(\gamma) = \rho_i, w \in W_{N_1}, \text{Int}(w) \neq \rho_k, \gamma v_1^\square(1) \text{ is the neighbor of } \gamma_1(1) \text{ in } B_T \text{ in the } \square \text{ direction} \right\}.$$
We add an extra condition regarding the relative length to define $\tilde{\Omega}_{i,j,k}^n(w, \Box, l, s; T)$:

$$\tilde{\Omega}_{i,j,k}^n(w, \Box, l, s; T) = \left\{ \gamma_1 \in \Gamma_0 \left| \begin{array}{l}
\gamma_1 = (\rho_j \rho_k)^n w, \ \text{End}(\gamma) = \rho_i, w \in W_{N_1}, \ \text{Int}(w) \neq \rho_k, \\
\gamma_{v(1)}^l \text{ is the neighbor of } \gamma_1(1) \text{ in } B_T \text{ in the } \Box \text{ direction,} \\
\#(\tilde{\Omega}_{i,j,k}^n(w, \Box, l, s; T) \cap B_T) \leq \frac{s}{T^2}
\end{array} \right. \right\}. \quad (13)$$

Then the gap distribution function $F_{T,I}(s)$ can be rewritten as

$$F_{T,I}(s) = \frac{1}{2} \sum_{i,j,k \in \{1,2,3\} \text{ distinct}} \sum_{w \in W_{N_1}} \sum_{l \leq N_3} \sum_{n=1}^{\infty} \frac{\#(\tilde{\Omega}_{i,j,k}^n(w, \Box, l, s; T) \cap B_{T,I})}{c_0 \mu(I) T^{2\delta}} + O \left( \frac{\log T}{T^2} \right). \quad (14)$$

The factor $1/2$ stands for the fact that each gap is counted twice. The symbol $*$ means that the sum is restricted to those terms $w$ with $\text{Int}(w) \neq \rho_k$. The error term is a bound for the contribution from the sporadic gaps formed by $\mathcal{E}(1)$. We will show that each summand in (14) converges to a limit, and then show that the limit of $F_{T,I}$ is indeed the sum of limit from each summand.

We further split $\tilde{\Omega}_{i,j,k}^n(w, \Box, l, s; T)$ into two sets $\tilde{\Omega}_{i,j,k,+}^n(w, \Box, l, s; T)$ and $\tilde{\Omega}_{i,j,k,-}^n(w, \Box, l, s; T)$ according to whether $\gamma \in \Gamma$ or not in definition (13). We first analyze $\tilde{\Omega}_{i,j,k,+}^n(w, \Box, l, s; T) \cap B_{T,I}$.

Recall the notation $C_{v'} = v'C_1 = C((1-r_{v'})e^{\theta_{v'}}, r_{v'})$ for $v' \in \Gamma_0$. We observe that since $e^{\theta_{v'}} = v'(1)$ and 1 is not a limit point, $e^{\theta_{v'}}$ is not a limit point as well. As a result, excluding finitely elements in $\Gamma$, $\phi_2(\gamma)$ is uniformly away from $\theta_{v'}$, thus the term $1 + \cos(\pi - \phi_2(\gamma) + \theta_{v'})$ has a uniform positive lower bound.

The condition $\gamma_1 = (\rho_j \rho_k)^n w \in B_{T,I}$ is the same as

$$\gamma(\rho_j \rho_k)^n w(1) \in I \quad (15)$$

We use the notation $\tilde{I}$ to denote the interval in $[0,2\pi)$ such that $e^{\tilde{I}} = I$.

We define

$$\tilde{I}^+ = \{ x \in \mathbb{D} : d(x, \tilde{I}) < \epsilon \}$$

and

$$\tilde{I}^- = \{ x \in \mathbb{D} : d(x, \tilde{I}) < \epsilon \}.$$

We apply these notations to other arcs of $\partial \mathbb{D}$.

From (5), when $t(\gamma)$ is large (depending on $\epsilon$), the condition (15) implies

$$\phi_1(\gamma) \in \tilde{I}^+, \quad (16)$$
The condition $\gamma_1 \in \tilde{\Omega}_{i,j,k,+}^{n}(w, \Box, l, s; T)$ is equivalent to the following conditions:

\begin{align}
\text{End}(\gamma) &= \rho_i, \quad \text{(17)} \\
\|\gamma(\rho_j \rho_k)^n w\| &< T, \quad \text{(18)} \\
\|\gamma v_m^{\Box}\| &\geq T \text{ for } 1 \leq m < l, \quad \text{(19)} \\
\|\gamma v_l^{\Box}\| &< T, \quad \text{(20)} \\
\frac{d(\gamma v(1), \gamma v_l^{\Box}(1))}{T^2} &\leq s. \quad \text{(21)}
\end{align}

For (17), we have $\text{End}(\gamma) = \rho_i \iff \text{Int}(\gamma^{-1}) = \rho_i \iff \gamma^{-1}(1) \in m_i$. Again from (5), the condition (17) implies $\phi_1(\gamma^{-1}) \in m_i^{+}$, or equivalently,

$$\phi_2(\gamma) \in m_i^{+} \quad \text{(22)}$$

From (4), when $t(\gamma)$ is large, the conditions (18), (19), (20) imply

\begin{align}
&\quad e^{t(\gamma)} \frac{1 - r_v}{2r_v} (1 + \cos(\pi - \phi_2(\gamma) + \theta_v)) < (T(1 + \epsilon))^2, \quad \text{(23)} \\
&\quad e^{t(\gamma)} \frac{1 - r_v}{2r_v}^{\Box} (1 + \cos(\pi - \phi_2(\gamma) + \theta_v^{\Box})) \geq (T(1 - \epsilon))^2, \quad \text{(24)} \\
&\quad e^{t(\gamma)} \frac{1 - r_v}{2r_v} l (1 + \cos(\pi - \phi_2(\gamma) + \theta_v^{\Box})) < (T(1 + \epsilon))^2. \quad \text{(25)}
\end{align}

Now we deal with the last condition (21). We can use (5) to obtain

$$\phi_1(\gamma) + 2 \tan\left(\frac{\pi - \phi_2(\gamma) + \theta_v^{\Box}}{2} e^{-t(\gamma)}\right) + O(e^{-2t(\gamma)}) \quad \text{(26)}$$

Therefore,

\begin{align}
&d(\gamma(\rho_j \rho_k)^n(1), \gamma v_m^{\Box}(1)) = 2e^{-t(\gamma)} \left|\tan\left(\frac{\pi - \phi_2(\gamma) + \theta_v}{2}\right) - \tan\left(\frac{\pi - \phi_2(\gamma) + \theta_v^{\Box}}{2}\right)\right| \\
&\quad + O(e^{-2t(\gamma)}), \quad \text{(27)}
\end{align}

so the condition (21) implies when $t(\gamma)$ is large,

$$\frac{e^{t(\gamma)}}{2} \left|\tan\left(\frac{\pi - \phi_2(\gamma) + \theta_v}{2}\right) - \tan\left(\frac{\pi - \phi_2(\gamma) + \theta_v^{\Box}}{2}\right)\right| \geq \frac{(T(1 - \epsilon))^2}{s}. \quad \text{(28)}$$

We observe that $e^{\theta(\rho_j \rho_k)^n}$ and those $e^{\theta v_l^{\Box}}$'s are not in the limit set. Thus, we can choose a set $J_0 \subset [0, 2\pi)$ which is a union of small neighborhoods of these points such that $\phi_2(\gamma) \in J_0$ for at most finitely many $\gamma \in \Gamma$, and necessarily we must have
\(\mu(\mathcal{J}_0) = 0\). The set \(\mathcal{J}_0\) depends on \(w, i, j, k, n\). The reason we choose this set \(\mathcal{J}_0\) is that we want the term \(1 + \cos(\pi - \theta + \theta_{\square l})\) to have a uniform positive lower bound for \(\theta \in [0, 2\pi) - \mathcal{J}_0\), to ensure the compactness of the set \(\Omega_{i,j,k,+}^n(w, \square, l, s; T)\), which we define below.

In light of the conditions (22), (23), (24), (25) and (28), we define \(\Omega_{i,j,k,+}^n(w, \square, l, s; T)\) to be the set of \((t, \theta)\) such that

\[
\begin{align*}
\theta &\in m_i - \mathcal{J}_0 \quad (29) \\
e^t \frac{1 - r_v}{2r_v} (1 + \cos(\pi - \theta + \theta_v)) &< T^2, \quad (30) \\
e^t \frac{1 - r_m}{2r_m} (1 + \cos(\pi - \theta + \theta_{\square m})) &\geq T^2 \quad \text{for } 1 \leq m < l \quad (31) \\
e^t \frac{1 - r_{\square l}}{2r_{\square l}} (1 + \cos(\pi - \theta + \theta_{\square l})) &< T^2, \quad (32) \\
e^t \frac{e^{t(\gamma)}}{2} \left| \tan \left( \frac{\pi - \theta + \theta_v}{2} \right) - \tan \left( \frac{\pi - \theta + \theta_{\square m}}{2} \right) \right| &\geq \frac{T^2}{s}. \quad (33)
\end{align*}
\]

In an analogous way, we define \(\Omega_{i,j,k,+}^{n,\epsilon}(w, \square, l, s; T)\) to be the set of \((t, \theta)\) satisfying the following conditions:

\[
\begin{align*}
\theta &\in m_i^{+} - \mathcal{J}_0 \quad (34) \\
e^t \frac{1 - r_v}{2r_v} (1 + \cos(\pi - \theta + \theta_v)) &< (T(1 + \epsilon))^2, \quad (35) \\
e^t \frac{1 - r_m}{2r_m} (1 + \cos(\pi - \theta + \theta_{\square m})) &\geq (T(1 - \epsilon))^2 \quad \text{for } 1 \leq m < l \quad (36) \\
e^t \frac{1 - r_{\square l}}{2r_{\square l}} (1 + \cos(\pi - \theta + \theta_{\square l})) &< (T(1 + \epsilon))^2. \quad (37) \\
e^t \frac{e^{t(\gamma)}}{2} \left| \tan \left( \frac{\pi - \theta + \theta_v}{2} \right) - \tan \left( \frac{\pi - \theta + \theta_{\square m}}{2} \right) \right| &\geq \frac{(T(1 - \epsilon))^2}{s}. \quad (38)
\end{align*}
\]

It is clear that both \(\Omega_{i,j,k,+}^n(w, \square, l, s; T)\) and \(\Omega_{i,j,k,+}^{n,\epsilon}(w, \square, l, s; T)\) are compact sets bounded by finitely many analytic curves. Moreover, both sets are homogeneous with respect \(T\):

\[(t, \theta) \in \Omega_{i,j,k,+}^n(w, \square, l, s; T) \iff (t - 2 \log T, \theta) \in \Omega_{i,j,k,+}^n(w, \square, l, s; 1)\]
and
\[(t, \theta) \in \Omega_{i,j,k,+}^{n,\epsilon}(w, \square, l, s; T) \iff (t - 2 \log T, \theta) \in \Omega_{i,j,k,+}^{n,\epsilon}(w, \square, l, s; T).\]

Therefore, we can apply Corollary 2.2 to count group elements in these sets.

From our discussion above, ignoring finitely many \(\gamma\)'s such that \(\phi_2(\gamma) \in J_0\), and finitely many \(\gamma\)'s such that the conditions (35), (36), (37), (38) fail, we have
\[
\{\gamma \in \Gamma | \gamma(\rho_j \rho_k)^n w(1) \in \mathcal{I}, \gamma(\rho_j \rho_k)^n w \in \tilde{\Omega}_{i,j,k,+}^{n}(w, \square, l, s; T)\} 
\subset \{\gamma \in \Gamma | \phi_2(\gamma) \in \tilde{T}^+, (t(\gamma), \phi_2(\gamma)) \in \Omega_{i,j,k,+}^{n,\epsilon}(w, \square, l, s; T)\} \quad (39)
\]

Applying Corollary 2.2 to the right hand side of (39), we thus obtained
\[
\limsup_{T \to \infty} \frac{\#(\Omega_{i,j,k,+}^n(w, \square, l, s; T) \cap B_{T,\mathcal{I}})}{c_0 \mu(\mathcal{I}) T^{2\delta}} \leq \frac{c_\tau \mu(\tilde{T}^+)}{c_0} \mu(\Omega_{i,j,k,+}^n(w, \square, l, s; 1)).
\]

We have the clear inclusion \(\mathcal{I} \subset \tilde{T}^+\) and \(\Omega_{i,j,k,+}^n(w, \square, l, s; 1) \subset \Omega_{i,j,k,+}^{n,\epsilon}(w, \square, l, s; 1)\).

Since \(\mu\) has no atom, we also have
\[
\lim_{\epsilon \to 0} \mu(\tilde{T}^+) = \mu(\mathcal{I})
\]
and
\[
\lim_{\epsilon \to 0} S(\Omega_{i,j,k,+}^{n,\epsilon}(w, \square, l, s; 1)) = S(\Omega_{i,j,k,+}^n(w, \square, l, s; 1)) = S(\Omega_{i,j,k,+}^n(w, \square, l, s; 1)).
\]

Therefore, we have
\[
\limsup_{T \to \infty} \frac{\#(\Omega_{i,j,k,+}^n(w, \square, l, s; T) \cap B_{T,\mathcal{I}})}{c_0 \mu(\mathcal{I}) T^{2\delta}} \leq \frac{c_\tau S(\Omega_{i,j,k,+}^n(w, \square, l, s; 1))}{c_0}.
\]

Similarly, we can prove
\[
\liminf_{T \to \infty} \frac{\#(\Omega_{i,j,k,+}^n(w, \square, l, s; T) \cap B_{T,\mathcal{I}})}{c_0 \mu(\mathcal{I}) T^{2\delta}} \geq \frac{c_\tau S(\Omega_{i,j,k,+}^n(w, \square, l, s; 1))}{c_0}.
\]

Therefore, we have
\[
\lim_{T \to \infty} \frac{\#(\Omega_{i,j,k,+}^n(w, \square, l, s; T) \cap B_{T,\mathcal{I}})}{c_0 \mu(\mathcal{I}) T^{2\delta}} = \frac{c_\tau S(\Omega_{i,j,k,+}^n(w, \square, l, s; 1))}{c_0}.
\]

We can analyze \(\Omega_{i,j,k,-}^n(w, \square, l, s; T) \cap B_{T,\mathcal{I}}\) in a completely analogous way as \(\Omega_{i,j,k,+}^n(w, \square, l, s; T) \cap B_{T,\mathcal{I}}\), and obtain a family of compact, homogeneously expanding sets \(\Omega_{i,j,k,+}^n(w, \square, l, s; T)\), bounded by finitely many analytic curves such that
\[
\lim_{T \to \infty} \frac{\#(\Omega_{i,j,k,-}^n(w, \square, l, s; T) \cap B_{T,\mathcal{I}})}{c_0 \mu(\mathcal{I}) T^{2\delta}} = \frac{c_\tau S(\Omega_{i,j,k,-}^n(w, \square, l, s; 1))}{c_0}.
\]
Writing $\Omega_{i,j,k}^n(w, \Box, l, s; 1) = \Omega_{i,j,k,+}^n(w, \Box, l, s; 1) \cup \Omega_{i,j,k,-}^n(w, \Box, l, s; 1)$, we thus have obtained

**Lemma 5.1.** Let the notations be as above. We have

$$\lim_{T \to \infty} \frac{\#(\tilde{\Omega}^{n}_{i,j,k}(w, \Box, l, s; T) \cap \mathcal{B}_{T,I})}{c_0 \mu(I) T^{2\delta}} = \frac{c_T}{c_0} S(\Omega_{i,j,k}^n(w, \Box, l, s; 1)).$$

Now we can analyze the gap distribution function $F_{T,I}(s)$ defined in (2).

**Proof of Theorem 1.1.** Return to (14). First we drop all the terms with $n > M$ in the innermost sum of the above, for some big number $M$. The contribution from these terms to (14) is

$$\leq \frac{1}{c_0 \mu(I) T^{2\delta}} \sum_{i,j,k \in \{1,2,3\}} \sum_{n > M} \#(C_{i,j,k}^n \cap \mathcal{B}_T)$$

(40)

If $\gamma(\rho_j \rho_k)^n w \in C_{i,j,k}^n$, then from Lemma 3.3 we have $\|\gamma\| \ll \frac{\|\gamma(\rho_j \rho_k)^n w\|}{a^n} \leq \frac{T}{a^n}$. So we only have $\ll \frac{T^{2\delta}}{a^n}$ choices for $\gamma$ for each fixed $w$. Since the number of choices for $w$ is also bounded (recall $l(w) < N_2$), we have

$$\#(C_{i,j,k}^n \cap \mathcal{B}_T) \ll \frac{T^{2\delta}}{a^{2n}}$$

Therefore,

$$\text{(40)} \ll \frac{1}{a^{2M} (a^2 - 1)}$$

(41)

and we can rewrite $F_{T,I}(s)$ as

$$F_{T,I}(s) = \frac{1}{2} \sum_{i,j,k \in \{1,2,3\}} \sum_{w \in \mathcal{W}_{N_1}} \sum_{\Box \in \{+, -\}} \sum_{l \leq N_3} \sum_{n = 1}^{M} \frac{\#(\tilde{\Omega}_{i,j,k}^n(w, \Box, l, s; T) \cap \mathcal{B}_{T,I})}{c_0 \mu(I) T^{2\delta}}$$

$$+ O \left( \frac{\log T}{T^{2\delta}} + \frac{1}{a^{2M}} \right).$$
Now we have a finite sum. Applying Lemma 5.1, we obtain

\[
\frac{1}{2} \sum_{i,j,k \in \{1,2,3\}} \sum_{w \in \mathcal{W}} \sum_{\square \in \{+, -\}} \sum_{l \leq N_3} \sum_{n=1}^{M} \frac{c_T}{c_0} S(\Omega_{i,j,k}^n(w, \square, l, s; 1)) - \frac{c_1}{a^2M} \leq \liminf_{T \to \infty} F_{T,I}(s)
\]

\[
\leq \limsup_{T \to \infty} F_{T,I}(s) \leq \frac{1}{2} \sum_{i,j,k \in \{1,2,3\}} \sum_{w \in \mathcal{W}} \sum_{\square \in \{+, -\}} \sum_{l \leq N_3} \sum_{n=1}^{M} \frac{c_T}{c_0} S(\Omega_{i,j,k}^n(w, \square, l, s; 1)) + \frac{c_2}{a^2M},
\]

for some positive absolute constant \(c_1, c_2\). Letting \(M\) go to \(\infty\), we obtain

\[
\lim_{T \to \infty} F_{T,I}(s) = \frac{1}{2} \sum_{i,j,k \in \{1,2,3\}} \sum_{w \in \mathcal{W}} \sum_{\square \in \{+, -\}} \sum_{l \leq N_3} \sum_{n=1}^{\infty} \frac{c_T}{c_0} S(\Omega_{i,j,k}^n(w, \square, l, s; 1)) := F(s)
\]

Thus the existence of the limiting distribution is proved, and clearly \(F(s)\) is independent of \(I\).

Next we show that \(F(s)\) is continuous and \(\lim_{s \to \infty} F(s) = 1\).

We can define \(\tilde{\Omega}_{i,j,k}^n(w, \square, l, \infty; T)\), \(\Omega_{i,j,k}^n(w, \square, l, \infty; T)\) in the same way as \(\Omega_{i,j,k}^n(w, \square, l, s; T)\), \(\Omega_{i,j,k}^n(w, \square, l, s; T)\), respectively, with the conditions regarding relative length (see (21),(28) ) removed. An identical argument as above shows that

\[
\lim_{T \to \infty} \frac{1}{2} \sum_{i,j,k \in \{1,2,3\}} \sum_{w \in \mathcal{W}_{N_1}} \sum_{\square \in \{+, -\}} \sum_{l \leq N_3} \sum_{n=1}^{\infty} \frac{c_T}{c_0} S(\tilde{\Omega}_{i,j,k}^n(w, \square, l, s; 1))
\]

\[
= \frac{1}{2} \sum_{i,j,k \in \{1,2,3\}} \sum_{w \in \mathcal{W}_{N_1}} \sum_{\square \in \{+, -\}} \sum_{l \leq N_3} \sum_{n=1}^{\infty} \frac{c_T}{c_0} S(\Omega_{i,j,k}^n(w, \square, l, 1)) = 1
\]

But the left hand side of (43) is \(\frac{\# \text{ (gaps in } B_{T,I}) + O(\log T)}{c_0 \mu(I) T^{23}}\), which approaches 1 as \(T\) approaches infinity. Here \(O(\log T)\) comes from the gaps with at least one point in \(\mathcal{E}\). So in fact we have an identity:

\[
\frac{1}{2} \sum_{i,j,k \in \{1,2,3\}} \sum_{w \in \mathcal{W}_{N_1}} \sum_{\square \in \{+, -\}} \sum_{l \leq N_3} \sum_{n=1}^{\infty} \frac{c_T}{c_0} S(\Omega_{i,j,k}^n(w, \square, l, 1)) = 1
\]
It is clear that each $S(\Omega_{i,j,k}^n(w,\Box,l,s;1))$ is a continuous, monotone increasing function of $s$ and

$$\lim_{s \to \infty} S(\Omega_{i,j,k}^n(w,\Box,l,s;1)) = S(\Omega_{i,j,k}^n(w,\Box,l,\infty;1))$$

From this we obtain $F(s)$ is continuous and

$$\lim_{s \to \infty} F(s) = 1.$$

Finally, it is elementary to see that $F$ is supported away from 0: if two points $\gamma_1(1), \gamma_2(1)$ are adjacent in $B_T$, then the radius $r_{\gamma_1}, r_{\gamma_2}$ of the corresponding two disjoint circles $\geq 1/T^2$. This forces the distance between $\gamma_1(1), \gamma_2(1) \gg \frac{1}{T^2}$. □

6. A RANDOM POINT PROCESS ON THE LIMIT SET

We prove some properties of the asymptotic behavior of a random point process on the limit set $\Lambda_{\Gamma_0}$. We closely follow the argument from the appendix of [5] where it is shown that the limiting nearest spacing of a random point process on $\mathbb{S}^n$ is Poisson.

Let $\tilde{d}(x,y)$ be the counterclockwise distance from $x$ to $y$ on $\partial \mathbb{D}$ (so $\tilde{d}(\cdot,\cdot)$ is not symmetric, and $\tilde{d}(x,y) = 2\pi - \tilde{d}(y,x)$ if $x \neq y$). Let $P_1, P_2, \ldots, P_N$ be $N$ independent random variables distributed according to the Patterson-Sullivan measure $\mu$ on $\partial \mathbb{D}$ and $d_i$ be the arc length distance of $P_i$ and its neighbor in the counterclockwise direction. Note that these $d_1, d_2, \ldots, d_N$ are not independent. The $d_1, d_2, \ldots, d_N$ are typically of the scale $N^{-\frac{1}{2}}$, this is due to the following fact: if we let $L(x,\eta)$ be the interval $\{y \in \partial \mathbb{D} | \tilde{d}(x,y) < \eta\}$ and $p(x,\eta) = \mu(L(x,\eta))/\eta^\delta$, then according to Proposition 3 and the comment in Section 3 from [18], for any $x \in \Lambda_{\Gamma_0}$, we have

$$c < p(x,\eta) < C$$  \hspace{1cm} (46)

If $\mu$ is the Lebesgue measure, then $p(x,\eta) \equiv \frac{1}{2\pi}$. However, if $\delta < 1$, (46) can not be improved (i.e, the limit of $p(x,\eta)$ when $\eta \to 0$ does not exist for most points) according to Corollary 4.10 from [7].

We define the scaled gap distribution measure $\nu_N$ on $[0,\infty)$ in the following way: for any Borel set $A \subset [0,\infty)$, let

$$\nu_N(A) = \nu(P_1,\ldots,P_N)(A) := \sum_{i=1}^{N} \frac{1}{N} 1\{d_i N^{\frac{1}{2}} \in A\}$$  \hspace{1cm} (47)
Let $s \geq 0$, we examine the expectation
\[
E(\nu_N([0, s])) = E\left(\frac{1}{N} \sum_{i=1}^{N} 1\{d_i N^{\frac{2}{p}} \leq s\}\right)
\] (48)
\[
= E\left(\frac{1}{N} \sum_{i=1}^{N} 1\{\min_{j \neq i} \{\tilde{d}(P_i, P_j)\} \leq \frac{s}{N^{\frac{2}{p}}}\}\right) \tag{49}
\]

Let $\nu_N$ be a probability measure on $[0, \infty)$ such that
\[
\bar{\nu}_N = 1 - \int_{\partial D} e^{-s \frac{p(x, s \cdot N^{-\frac{1}{2}})}{2}} d\mu(x).
\]

We will show that

**Theorem 6.1.** As $N$ goes to $\infty$, $\nu_N([0, s]) - \bar{\nu}_N([0, s])$ converges to 0 in probability.

When $\delta < 1$, we cannot show that $\bar{\nu}_N$ converges to a limit measure as $N$ goes to $\infty$, because we don’t know if the random variable $p(P, \eta)$ converges in distribution as $\eta$ goes to 0. Nevertheless, Theorem 6.1 still indicates that the gap distribution of a random point process on $\Lambda_{\Gamma_0}$ has exponential decay.

We define for $k \geq 1$,
\[
A_k(P_1, P_2, \cdots, P_N) := \frac{1}{N^k} \sum_{i=1}^{N} \sum_{j_1, \cdots, j_k \neq i} \prod_{l=1}^{k} 1\{\tilde{d}(P_i, P_{j_l}) \leq \frac{s}{N^{\frac{2}{p}}}\}, \tag{50}
\]
and for $n \geq 1$, we let $B_n = \sum_{k=1}^{n} (-1)^{k+1} A_k$.

Using inclusion-exclusion, we have
\[
B_{2n} < \nu_N([0, s]) < B_{2n+1},
\]
so that
\[
E(\nu_N([0, s])) \leq E(B_{2n+1}) = E(A_1 - A_2 + \cdots + A_{2n+1}) \tag{51}
\]
and
\[
E(\nu_N([0, s])) \geq E(B_{2n}) = E(A_1 - A_2 + \cdots - A_{2n}) \tag{52}
\]
For each $k$,

$$
\mathbb{E}(A_k) = \mathbb{E} \left( \frac{1}{N^k} \sum_{i=1}^{N} \sum_{j_i, \ldots, j_k \neq i}^{j_1, \ldots, j_k \text{ distinct}} \prod_{l=1}^{k} 1\{d(P_i, P_{j_l}) \leq \frac{s}{N^{\frac{1}{2}}} \} \right)
$$

$$
= \frac{(N-1)(N-2) \cdots (N-k)}{k!} \int_{\partial D} \mu \left( L \left( x, \frac{s}{N^{\frac{1}{2}}} \right) \right)^k \mu(x)
$$

$$
= \frac{(N-1)(N-2) \cdots (N-k) s^k}{N^k k!} \int_{\partial D} \left( p \left( x, sN^{-\frac{1}{2}} \right) \right)^k \mu(x) \quad (53)
$$

Therefore,

$$
\mathbb{E}(A_k) = \left( \frac{s^k}{k!} + O \left( \frac{k + 1}{(k-1)!N} \right) \right) \int_{\partial D} \left( p \left( x, sN^{-\frac{1}{2}} \right) \right)^k \mu(x) \quad (54)
$$

From (54) we can work out

$$
\mathbb{E}(B_n) = 1 - \int_{\partial D} e^{-s^k p \left( x, sN^{\frac{1}{2}} \right)} \mu(x) + O \left( \frac{(s^k C)^n}{n!} + \frac{ne^{s^k C}}{N} \right) \quad (55)
$$

From (51) and (52), and setting $n = [\log N]$ in (55), we can see that

$$
\lim_{N \to \infty} \mathbb{E}(\nu_N([0, s])) - \left( 1 - \int_{\partial D} e^{-s^k p \left( x, sN^{-\frac{1}{2}} \right)} \mu(x) \right) = 0 \quad (56)
$$

We can also give an upper bound for $\text{Var}(A_k)$, the variance of $A_k$. Using Cauchy-Schwarz inequality,

$$
\text{Var}(A_k) = \mathbb{E} \left( (A_k - \mathbb{E}(A_k))^2 \right)
$$

$$
\leq \frac{1}{N^2 k!} \sum_{i=1}^{N} \sum_{j_i, \ldots, j_k \neq i}^{j_1, \ldots, j_k \text{ distinct}} \text{Var} \left( \prod_{l=1}^{k} 1\{d(P_i, P_{j_l}) \leq \frac{s}{N^{\frac{1}{2}}} \} \right)
$$

$$
\leq \frac{s^k}{N^k k!} \int_{\partial D} \left( p \left( x, sN^{-\frac{1}{2}} \right) \right)^k \mu(x) \left( 1 - \frac{s^k}{N^k} \int_{\partial D} \left( p \left( x, sN^{-\frac{1}{2}} \right) \right)^k \mu(x) \right)
$$

$$
\leq \frac{s^k}{N^k k!} \int_{\partial D} \left( p \left( x, sN^{-\frac{1}{2}} \right) \right)^k \mu(x) \leq \frac{(s^k C)^k}{N^k k!} \quad (57)
$$
Therefore we have
\[
\sum_{k=1}^{\infty} \text{Var}(A_k) \leq \frac{e^{s\delta} C}{N} \tag{58}
\]

Finally, using (58) and (54) we can give an upper bound for \( E \left( (\nu_N([0, s]) - \bar{\nu}_N([0, s]))^2 \right) \) in the following way: since \( B_{2n} < \nu_N([0, s]) < B_{2n+1} \), we have
\[
E \left( (\nu_N([0, s]) - \bar{\nu}_N([0, s]))^2 \right) \\
\leq E \left( (B_{2n+1} - \nu_N([0, s]))^2 \right) + E \left( (B_{2n} - \bar{\nu}_N([0, s]))^2 \right) \\
\ll E \left( (B_{2n+1} - E(B_{2n+1}))^2 \right) + E \left( (B_{2n} - E(B_{2n}))^2 \right) \\
+ (E(B_{2n+1}) - \bar{\nu}_N([0, s]))^2 + (E(B_{2n}) - \bar{\nu}_N([0, s]))^2 \\
\ll \frac{e^{Cs^4}}{N^2} + \frac{n^2 e^{2s\delta} C}{N^2} + \frac{(s\delta C)^4 n}{(2n!)^2} \tag{59}
\]

Setting \( n = \lfloor \log N \rfloor \), we see that as \( N \) goes to infinity, \( E \left( (\nu_N([0, s]) - \bar{\nu}_N([0, s]))^2 \right) \) goes to 0. A direct application of Chebyshev’s inequality shows that \( \nu_N([0, s]) - \bar{\nu}_N([0, s]) \) converges to 0 in probability.

**References**


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