Local-global principles in circle packings

Xin Zhang, joint with Elena Fuchs and Kate Stange

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Integral Apollonian circle packings

Figure: Integral Apollonian circle packings
Figure: Guettler and Mallows’ Apollonian 3-circle packing and Stange’s $\mathbb{Q}[\sqrt{-2}]$-Apollonian packing
Other integral circle packings

Figure: Kontorovich-Nakamura’s integral crystallographic packing

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Question

What integers arise as curvatures from an integral circle packing?

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Local-global principles in circle packings
A brief overview of the work on Apollonian packings

Theorem (Descartes)

The curvatures $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ from any four mutually tangent circles in an Apollonian packing satisfy the following relation:

$$Q(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = 2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) - (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2 = 0$$
A brief overview of the work on Apollonian packings

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**Corollary**

Fix $\kappa_2, \kappa_3, \kappa_4$. The two solutions $\kappa_1^{(1)}, \kappa_1^{(2)}$ for $\kappa_1$ in the quadratic equation

$$Q(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = 0$$

satisfies

$$\kappa_1^{(1)} + \kappa_1^{(2)} = 2(\kappa_2 + \kappa_3 + \kappa_4)$$

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Corollary

Fix a quadruple of curvatures of four mutually tangent circles \( r = (\kappa_1, \kappa_2, \kappa_3, \kappa_4) \). The set of curvatures \( K \) is given by

\[
\bigcup_{i=1}^{k} \langle A \cdot r, e_i \rangle,
\]

where \( A = \langle S_1, S_2, S_3, S_4 \rangle \prec O_Q \), and

\[
S_1 = \begin{pmatrix}
-1 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
S_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
2 & -1 & 2 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
S_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
2 & 2 & -1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix},
S_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
2 & 2 & 2 & -1
\end{pmatrix}.
\]
To understand $K$, a first step is to understand $K \pmod{q}$ for each $q$. For this purpose, it is not very convenient to work with $\mathcal{A}$, because $\mathcal{A}$ does not have strong approximation property.
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Choose a spin homomorphism $\rho : PSL(2, \mathbb{C}) \to SO_Q$, and work with $\Lambda = \rho^{-1}(A \cap SO_Q) \subset PSL(2, \mathbb{Z}[i])$, where

$$\Lambda = \langle \pm \begin{pmatrix} 1 & 4i \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} -2 & i \\ i & 0 \end{pmatrix}, \pm \begin{pmatrix} 2 + 2i & 4 + 3i \\ -i & -2i \end{pmatrix} \rangle$$
Definition

Let \( \mathcal{P} \) be any integral Apollonian circle packings with \( \gcd\{\text{curvatures}\} = 1 \). A congruence class \( \text{mod} \ 24 \) is admissible if it contains at least one curvature from \( \mathcal{P} \).
Definition

Let $\mathcal{P}$ be any integral Apollonian circle packings with $\gcd\{\text{curvatures}\} = 1$. A congruence class mod 24 is admissible if it contains at least one curvature from $\mathcal{P}$.

Figure: Admissible congruence classes mod 24 are $\{0, 8, 9, 12, 17, 20\}$ (mod 24)
Theorem (Bourgain-Kontorovich)

Let $\mathcal{P}$ be any integral Apollonian circle packings with $\gcd\{\text{curvatures}\} = 1$. Almost every positive integer in admissible congruence classes mod 24 is a curvature.
Theorem (Bourgain-Kontorovich)

Let \( \mathcal{P} \) be any integral Apollonian circle packings with \( \gcd\{\text{curvatures}\} = 1 \). Almost every positive integer in admissible congruence classes mod 24 is a curvature.

Why 24?
Let \( \Lambda(m) \) be the principle congruence subgroup of \( \Lambda \) of level \( m \):
\[
\left\{ \lambda \in \Lambda : \lambda \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{m} \right\},
\]
then for any \( q \in \mathbb{Z}^+ \),
\[
\Lambda(24)(\text{mod } q) = SL(2, \mathbb{Z}[i])(24)(\text{mod } q).
\]
The Schmidt arrangement

Figure: The orbit $SL(2, \mathbb{Z}[i]) \cdot \hat{\mathbb{R}}$
Consider a union of circles $\mathcal{P}$ from the Schmidt arrangement maximal with respect to

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Then $\mathcal{P}$ is an Apollonian packing, and any other integral Apollonian packing is a Möbius transform of $\mathcal{P}$ by a matrix $M \in SL(2, \mathbb{Z}[i])$. 
Consider a union of circles $\mathcal{P}$ from the Schmidt arrangement maximal with respect to

- $\mathcal{P}$ is connected.

Then $\mathcal{P}$ is an Apollonian packing, and any other integral Apollonian packing is a Möbius transform of $\mathcal{P}$ by a matrix $M \in SL(2, \mathbb{Z})$. 

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Local-global principles in circle packings
Consider a union of circles $\mathcal{P}$ from the Schmidt arrangement \textit{maximal} with respect to

- $\mathcal{P}$ is connected.
- Circles from $\mathcal{P}$ do not traverse.

There does not exist a triple of circles $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$, such that $\mathcal{C}_1$ lies in $\mathcal{C}_3$ and $\mathcal{C}_2$ lies outside $\mathcal{C}_3$. Then $\mathcal{P}$ is an Apollonian packing, and any other integral Apollonian packing is a Möbius transform of $\mathcal{P}$ by a matrix $M \in \text{SL}(2, \mathbb{Z}[i])$. 

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Then $\mathcal{P}$ is an Apollonian packing, and any other integral Apollonian packing is a Möbius transform of $\mathcal{P}$ by a matrix $M \in SL(2, \mathbb{Z}[i])$. 
In general, one can replace $\mathbb{Z}[i]$ by any ring of integers $\mathcal{O}_K$ of an imaginary quadratic field $K$ (e.g. $K = \mathbb{Q}[\sqrt{-5}]$, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$). Consider the maximal set $\mathcal{P}$ of circles from $\text{SL}(2, \mathcal{O}_K)$ satisfying:

- $\mathcal{P}$ is connected.
- Circles from $\mathcal{P}$ do not traverse.
- There does not exist a triple of circles $(C_1, C_2, C_3)$, such that $C_1$ lies in $C_3$ and $C_2$ lies outside $C_3$. 

Then $\mathcal{P}$ is a packing of circles with integral curvatures. The stabilizer $\Lambda_{\mathcal{P}}$ of $\mathcal{P}$ is an infinite co-volume, Zariski dense subgroup of $\text{SL}(2, \mathcal{O}_K)$. $\Lambda_{\mathcal{P}}$ contains a congruence subgroup $\Gamma_{\mathcal{P}}$ of $\text{SL}(2, \mathbb{Z})$. 

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In general, one can replace $\mathbb{Z}[i]$ by any ring of integers $\mathcal{O}_K$ of an imaginary quadratic field $K$ (e.g. $K = \mathbb{Q}[\sqrt{-5}]$, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$). Consider the maximal set $\mathcal{P}$ of circles from $SL(2, \mathcal{O}_K)$ satisfying:

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In general, one can replace $\mathbb{Z}[i]$ by any ring of integers $\mathcal{O}_K$ of an imaginary quadratic field $K$ (e.g. $K = \mathbb{Q}[\sqrt{-5}]$, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$). Consider the maximal set $\mathcal{P}$ of circles from $SL(2, \mathcal{O}_K)$ satisfying:

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In general, one can replace \( \mathbb{Z}[i] \) by any ring of integers \( \mathcal{O}_K \) of an imaginary quadratic field \( K \) (e.g. \( K = \mathbb{Q}[\sqrt{-5}] \), then \( \mathcal{O}_K = \mathbb{Z}[\sqrt{-5}] \)). Consider the maximal set \( \mathcal{P} \) of circles from \( SL(2, \mathcal{O}_K) \) satisfying:

- \( \mathcal{P} \) is connected.
- Circles from \( \mathcal{P} \) do not traverse.
- There does not exist a triple of circles \( (C_1, C_2, C_3) \), such that \( C_1 \) lies in \( C_3 \) and \( C_2 \) lies outside \( C_3 \).

Then

- \( \mathcal{P} \) is a packing of circles with integral curvatures.
- The stabilizer \( \Lambda_\mathcal{P} \) of \( \mathcal{P} \) is an infinite co-volume, Zariski dense subgroup of \( SL(2, \mathcal{O}_K) \).
- \( \Lambda_\mathcal{P} \) contains a congruence subgroup \( \Gamma_\mathcal{P} \) of \( SL(2, \mathbb{Z}) \).
Let $\Delta$ be the discriminant of $O_K$. For any
\[
g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix} \in PSL(2, O_K), \ g \text{ sends the horizontal line } \hat{R} + \frac{\sqrt{\Delta}}{2} \text{ to a circle of curvature}
\]
\[
\kappa \left( g \left( \hat{R} + \frac{\sqrt{\Delta}}{2} \right) \right) = \sqrt{-\Delta} |C_g|^2 + 2\Re(C_gD_g)
\]
Statement of the problem

Let \( \Delta \) be the discriminant of \( \mathcal{O}_K \). For any
\[
g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix} \in PSL(2, \mathcal{O}_K),
g sends the horizontal line
\( \hat{\mathbb{R}} + \frac{\sqrt{\Delta}}{2} \) to a circle of curvature
\[
k \left( g \left( \hat{\mathbb{R}} + \frac{\sqrt{\Delta}}{2} \right) \right) = \sqrt{-\Delta} \lvert C_g \rvert^2 + 2 \Im(\overline{C_g D_g})
\]

The Problem

Let \( \Lambda = \langle S \rangle \) be a finitely generated subgroup of \( PSL(2, \mathcal{O}_K) \), and \( \Lambda \) contains a congruence subgroup \( \Gamma \) of \( PSL(2, \mathbb{Z}) \). Let \( M \in PSL(2, \mathcal{O}_K) \). Study the set of integers
\[
\mathcal{K} = \frac{\kappa(M \Lambda(\hat{\mathbb{R}} + \frac{\sqrt{\Delta}}{2}))}{\sqrt{-\Delta}} = \left\{ \frac{\kappa(M \lambda(\hat{\mathbb{R}} + \frac{\sqrt{\Delta}}{2}))}{\sqrt{-\Delta}} : \lambda \in \Lambda \right\},
\]

where \( \Delta \) is the discriminant of \( \mathcal{O}_K \).
The main theorem

Theorem (Fuchs-Stange-Z)

Let $\mathcal{K}(N) = \mathcal{K} \cap [0, N]$. There exists a positive integer $L$, such that

$$\#\mathcal{K}(N) = cN + O(N^{1-\eta})$$

for some $\eta > 0$, where

$$c = \frac{\# \{ \text{admissible congruence classes mod } L \}}{L}$$
The main theorem

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**Corollary**

Almost every primes in admissible congruence classes mod $L$ is a curvature.
The main theorem

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for some $\eta > 0$, where

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**Corollary**

*Almost every primes in admissible congruence classes mod $L$ is a curvature.*

**Conjecture (Local-global conjecture)**

$$\#\mathcal{K}(N) = cN + O(1)$$
The role of the congruence subgroup

1. $\Lambda$ is finitely generated $\iff$ $\Lambda$ is geometrically finite. (Tameness Theorem, Agol, Calegari-Gabai)

Comment: (1) and (2) allows us to use Lax-Phillips' Theory to count points of $\Lambda$. (3) allows us to count points of $\Lambda$ and its congruence subgroups $\Lambda(q)$ with uniform control on the error terms.
The role of the congruence subgroup

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2. The critical exponent \( \delta(\Lambda) \) of \( \Lambda \) is greater than 1
   (Bishop-Jones).

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The role of the congruence subgroup

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4. Curvatures can be produced by quadratic forms.
The role of the congruence subgroup

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2. The critical exponent $\delta(\Lambda)$ of $\Lambda$ is greater than 1 (Bishop-Jones).
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4. Curvatures can be produced by quadratic forms.

Comment: (1) and (2) allows us to use Lax-Phillips’ Theory to count points of $\Lambda$. (3) allows us to count points of $\Lambda$ and its congruence subgroups $\Lambda(q)$ with uniform control on the error terms.
Fix $\lambda \in \Lambda$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the congruence subgroup of $SL(2, \mathbb{Z})$. A computation shows that

$$\kappa \left( M\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\mathbb{R} + \frac{\sqrt{\Delta}}{2}) \right) = \frac{|C_{M\lambda}a + D_{M\lambda}c|^2 + \frac{2\Re(C_{M\lambda}D_{M\lambda})}{\sqrt{-\Delta}}}{\sqrt{-\Delta}}$$
Fix $\lambda \in \Lambda$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the congruence subgroup of $SL(2, \mathbb{Z})$. A computation shows that

$$\kappa \left( M_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\mathbb{R} + \frac{\sqrt{\Delta}}{2}) \right) \frac{\sqrt{-\Delta}}{\sqrt{-\Delta}} = |C_{M_\lambda}a + D_{M_\lambda}c|^2 + \frac{2\zeta(\overline{C_{M_\lambda}}D_{M_\lambda})}{\sqrt{-\Delta}}$$

Define $f_\lambda(a, c) = |C_{M_\lambda}a + D_{M_\lambda}c|^2 + \frac{2\zeta(\overline{C_{M_\lambda}}D_{M_\lambda})}{\sqrt{-\Delta}}$. Then $f_\lambda(a, c) \in \mathcal{K}$ if $a, c$ satisfies some congruence condition and $\gcd(a, c) = 1$. 
It is classical that

$$\# \left( \{ f_\lambda(a, c) \mid a, c \in \mathbb{Z}, \gcd(a, c) = 1 \} \cap [0, N] \right) \gg \lambda \frac{N}{(\log N)^{1/2}}$$
It is classical that

\[
\# (\{ f_\lambda(a, c) | a, c \in \mathbb{Z}, \gcd(a, c) = 1 \} \cap [0, N]) \gg \lambda \frac{N}{(\log N)^{1/2}}
\]

Idea to prove the theorem: Take many such quadratic forms, hoping to cover most of the admissible numbers from \([0, N]\).
Choose two parameters $T, X$, where $T = N^{\frac{1}{200}}, X = N^{\frac{99}{200}}$, so that $T^2 X^2 = N$. Let $B_T = \{ \lambda \in \Lambda : \|\lambda\| < T \}$, where

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$ Define

$$R(n) = \sum_{\lambda \in B_T} \sum_{x, y \leq X} 1\{f_\lambda(a, c) = n\} \text{ for } \gcd(x, y) = 1$$
Choose two parameters $T, X$, where $T = N^{\frac{1}{200}}, X = N^{\frac{99}{200}}$, so that $T^2 X^2 = N$. Let $B_T = \{ \lambda \in \Lambda : \| \lambda \| < T \}$, where

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Define

$$R(n) = \sum_{\lambda \in B_T} \sum_{x, y \leq X} \mathbf{1}\{ f_{\lambda}(a, c) = n \}$$

$R(n) > 0$ for all sufficiently large admissible numbers $\implies$ The local-global conjecture.
We have a good understanding of the sum over $R(n)$:

$$\hat{R}(0) = \sum_{n \in \mathbb{Z}} R(n) = \sum_{n \in \mathbb{Z}} \sum_{\lambda \in B_T} \sum_{\substack{x, y \leq X \\gcd(x,y)=1}} 1\{f_{\lambda}(a,c) = n\}$$

$$= \sum_{\lambda \in B_T} \sum_{x,y \leq X \\gcd(x,y)=1} 1 \sim \#B_T \cdot cX^2$$
We have a good understanding of the sum over $R(n)$:

$$\hat{R}(0) = \sum_{n \in \mathbb{Z}} R(n) = \sum_{n \in \mathbb{Z}} \sum_{\lambda \in \mathcal{B}_T} \sum_{\gcd(x, y) = 1} 1 \{ f_\lambda(a, c) = n \}$$

$$= \sum_{\lambda \in \mathcal{B}_T} \sum_{x, y \leq X} 1 \sim \# \mathcal{B}_T \cdot cX^2$$

We also have good estimate for

$$\hat{R}\left(\frac{r}{q}\right) = \sum_{n \in \mathbb{Z}} R(n)e \left( \frac{rn}{q} \right)$$

for $q$ small.
Let

\[ \hat{R}(\theta) = \sum_{n \in \mathbb{Z}} R(n)e(n\theta). \]
Let

\[ \hat{R}(\theta) = \sum_{n \in \mathbb{Z}} R(n)e(n\theta). \]

Then

\[ R(n) = \int_0^1 \hat{R}(\theta)e(-n\theta)d\theta. \]
The circle method

Let

$$\hat{R}(\theta) = \sum_{n \in \mathbb{Z}} R(n) e(n\theta).$$

Then

$$R(n) = \int_{0}^{1} \hat{R}(\theta) e(-n\theta) d\theta.$$ 

The main contribution of the above integral comes from $\mathcal{M}$, the union of small neighborhoods of $\frac{r}{q}$, with $q < Q_0$, where $Q_0$ is a small power of $T$ (recall $T$ is a small power of $N$).

Write

$$R(n) = \int_{\mathcal{M}} \hat{R}(\theta) e(-n\theta) d\theta + \int_{[0,1] - \mathcal{M}} \hat{R}(\theta) e(-n\theta) d\theta$$

$$= M(n) + E(n)$$
The total input

\[ \hat{R}(0) = \sum_{\gamma \in \mathcal{B}_T} \sum_{x, y \leq X} \sum_{(x, y) = 1} \asymp T^{2\delta} X^2, \]

where \(\delta\) is the critical exponent of \(\Lambda\).
The strategy of the proof

The total input

\[ \hat{R}(0) = \sum_{\gamma \in B_T} \sum_{x,y \leq X} T^{2\delta} X^2, \]

where \( \delta \) is the critical exponent of \( \Lambda \).
It is expected that the total mass is equidistributed among admissible numbers.
The total input

\[ \hat{R}(0) = \sum_{\gamma \in \mathcal{B}_T} \sum_{x,y \leq X, (x,y)=1} \ll T^{2\delta} X^2, \]

where \( \delta \) is the critical exponent of \( \Lambda \).

It is expected that the total mass is equidistributed among admissible numbers.

Indeed, the major arc analysis shows that for each \( n \in [N/2, N] \) admissible,

\[ M(n) \gg \frac{T^{2\delta} X^2}{T^2 X^2} = T^{2\delta-2}. \]
The strategy of the proof

We can not show $E(n) \ll T^{2\delta-2-\eta}$ for each $n \in [N/2, N]$ admissible, but we can show $E(n)$ is small on average, by establishing an $l^2$ bound:

$$\sum_{n \in \mathbb{Z}} E(n)^2 = \int_{[1,0] - \mathfrak{m}} |\hat{R}(\theta)|^2 d\theta \ll T^{4\delta-4-\eta} N.$$
The strategy of the proof

We can not show $E(n) \ll T^{2\delta-2-\eta}$ for each $n \in [N/2, N]$ admissible, but we can show $E(n)$ is small on average, by establishing an $l^2$ bound:

$$\sum_{n \in \mathbb{Z}} E(n)^2 = \int_{[1,0]} \left| \hat{R}(\theta) \right|^2 d\theta \ll T^{4\delta-4-\eta} N.$$ 

This implies that for most $n \in [N/2, N]$ admissible, $R(n) = M(n) + E(n) > 0$, with a power savings on the exceptional set.
To estimate \( \int_{\mathbb{R}} \hat{R}(\theta) e(-n\theta) d\theta \), we evaluate \( \hat{R} \left( \frac{r}{q} \right) \) when \( q \) is small. The main player is the \( \lambda \)-sum:

\[
\hat{R} \left( \frac{r}{q} \right) = \sum_{\substack{x, y \leq X \\ \gcd(x, y) = 1}} \sum_{\lambda \in B_T} e \left( f_\lambda(x, y) \frac{r}{q} \right)
\]

\[
= \sum_{\substack{x, y \leq X \\ \gcd(x, y) = 1}} \sum_{\lambda_0 \in \Lambda/\Lambda(q)} e \left( f_{\lambda_0}(x, y) \frac{r}{q} \right) \sum_{\substack{\lambda \in B_T \\ \lambda \equiv \lambda_0 \pmod{q}}} 1
\]

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Local-global principles in circle packings
Major arc analysis

To estimate \( \int_{\mathbb{R}} R(\theta) e(-n\theta) d\theta \), we evaluate \( \hat{R}(r/q) \) when \( q \) is small. The main player is the \( \lambda \)-sum:

\[
\hat{R}(r/q) = \sum_{x,y \leq X} \sum_{\lambda \in \mathcal{B}_T} e(f_\lambda(x,y) r/q)
\]

\[
= \sum_{x,y \leq X} \sum_{\lambda \in \Lambda / \Lambda(q)} e(f_{\lambda_0}(x,y) r/q) \sum_{\lambda \equiv \lambda_0 (\mod q)} 1
\]

Effective lattice point counting (Lee-Oh, Vinogradov, Mohammadi-Oh) \( \implies \)

\[
\sum_{\lambda \in \mathcal{B}_T} 1 = \frac{C}{\# \Lambda / \Lambda(q)} T^{2\delta} + O(T^{2\delta-\epsilon})
\]

It is important that there exists \( \epsilon > 0 \) independent of \( q \).
Definition

Let \( \{X_i\}_{i \in \mathbb{N}} \) be a family of \( k \)-regular, finite, connected graphs with \( |X_i| \to \infty \). Let \( M_i \) be the adjacency matrix of \( X_i \). It has eigenvalues

\[
k = \lambda_0(M_i) > \lambda_1(M_i) \geq \lambda_2(M_i) \geq \cdots \geq \lambda_s(M_i) \geq -k.
\]

\( \{X_i\}_{i \in \mathbb{N}} \) is an **expander family** if \( \exists \epsilon > 0 \) such that \( k - \lambda_1(M_i) \geq \epsilon \) for all \( i \).
Spectral gap property of $\Lambda$

**Definition**

Let $G = \langle S \rangle$ is a finitely generated, infinite subgroup of $GL_n(\mathbb{Z})$, and let $A \subset \mathbb{Z}^+$. $G$ has *spectral gap property* with respect to $A$ if \( \{ \text{Cay}(G/G(q), S) \}_{q \in A} \) is a family of expanders.

Examples

- $G$ is an arithmetic lattice, $A = \mathbb{Z}$ (Margulis, Burger-Sarnak, Clozel, etc.)
- $Z_{cl}(G) = \text{SL}_n(\mathbb{Q})$, $A = \mathbb{Z}$ (Bourgain-Varju)
- The connected component of $Z_{cl}(G)$ is perfect, $A = \{ q : q \text{ square free} \}$ (Salehi Golsefidy-Varju), or $A = \{ p^m : m \in \mathbb{Z}^+ \}$ (Salehi Golsefidy).

Xin Zhang, joint with Elena Fuchs and Kate Stange

Local-global principles in circle packings
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**Theorem (Fuchs-Stange-Z)**

$\Lambda$ has spectral gap property with respect to $\mathbb{Z}^+$. 

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**Lemma (Varju)**

Let $G$ be a finite group with a finite symmetric generating set $S$. Suppose $G_1, \cdots, G_l \leq G$ with $S \cap G_i$ generates $G_i$, and that for each $g \in G$, there exist $g_i \in G_i$ such that $g = g_1 g_2 \cdots g_l$. Then

\[
|S| - \lambda_1(G, S) \geq \min_{1 \leq i \leq l} \left\{ \frac{|S \cap G_i|}{|S|} \cdot |S| - \lambda_1(G_i, S \cap G_i) \right\}
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In our application, $G = \Lambda/\Lambda(q)$, $G_i$ are conjugates of $\Gamma/\Gamma(q)$. 

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Theorem (Salehi Golsefidy-Z)

Let $\Lambda_1$ and $\Lambda_2$ be two finitely generated subgroups of $GL_n(\mathbb{Z})$. For $i = 1, 2$, let $Zcl(\Lambda_i)^\circ$ be the Zariski-connected component of the Zariski-closure of $\Lambda_i$ in $GL_n(\mathbb{Q})$. Suppose $\Lambda_2 \leq \Lambda_1$ and the normal closure of $Zcl(\Lambda_2)^\circ$ in $Zcl(\Lambda_1)^\circ$ is $Zcl(\Lambda_1)^\circ$. Then if $\Lambda_2$ satisfies the spectral gap property with respect to some $A \subset \mathbb{Z}^+$, then $\Lambda_1$ satisfies the spectral gap property with respect to $A$. 

Corollary

Let $\Lambda \leq GL_n(\mathbb{Z})$. Assume $Zcl(\Lambda)^\circ$ is perfect: $[Zcl(\Lambda)^\circ, Zcl(\Lambda)^\circ] = Zcl(\Lambda)^\circ$. Assume further that $\Lambda$ contains a Zariski-dense subgroup of $SL_d(\mathbb{Z})$, then $\Lambda$ satisfies the spectral gap property with respect to $\mathbb{Z}^+$. 

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Xin Zhang, joint with Elena Fuchs and Kate Stange
We evaluate $\hat{R}(\frac{r}{q})$ for $q$ large. The $x, y$-sum plays the main role:

$$\hat{R}(\frac{r}{q}) = \sum_{\lambda \in \mathcal{B}_T} \sum_{x, y \leq X} e(f_\lambda(x, y) \frac{r}{q})$$

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$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{Xin Zhang, joint with Elena Fuchs and Kate Stange}
Minor arc analysis

We evaluate $\hat{R}(\frac{r}{q})$ for $q$ large. The $x, y$-sum plays the main role:

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We do not get enough cancellation from the exponential sum $\sum_{x_0, y_0 \in \mathbb{Z}/q\mathbb{Z}} e(f_\lambda(x_0, y_0) \frac{r}{q}).$
By taking norm square of $\hat{R}(r/q)$ and sum over $r \in \mathbb{Z}/q\mathbb{Z}^\times$, we encounter Kloosterman-Salié type sum

$$\sum_{x \in \mathbb{Z}/q\mathbb{Z}^\times} e\left(\frac{ax + bx^{-1}}{q}\right) \chi(x),$$

where $\chi$ is a character of $\mathbb{Z}/q\mathbb{Z}^\times$. Kloosterman’s elementary bound gives

$$\left| \sum_{x \in \mathbb{Z}/q\mathbb{Z}^\times} e\left(\frac{ax + bx^{-1}}{q}\right) \right| \ll q^{3/4}$$