Gap Distribution of Directions in Some Schottky Groups

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The study of spatial statistics originates in mathematical physics, and has received attention also in analytic number theory and probability.
The study of spatial statistics originates in mathematical physics, and has received attention also in analytic number theory and probability. In the Euclidean setting, the problem can be formulated as:

**Question**

*For a fixed vector $\vec{w}$ in $\mathbb{R}^2$, consider the following increasing sequence of finite subsets of the unit circle:*

$$A_p(N) = \left\{ \frac{\vec{v} + \vec{w}}{|\vec{v} + \vec{w}|} : \vec{v} \in \mathbb{Z}^2, |\vec{v} + \vec{w}| < N \right\}$$

*What can we say about the distribution of $A(N)$, as $N \to \infty$?*
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Using equidistribution of flows in some homogeneous space, Marklof and Strömbergsson (Ann. Math 2011) determined a class of spatial statistics. Among them is the gap distribution.
Let $d_1, d_2, \cdots, d_{\#A(N)}$ be the gaps from $A(N)$. Define the gap distribution function

$$F_N(s) = \frac{\# \{d_i : d_i/\frac{2\pi}{\#A(N)} < s \}}{\#A(N)}$$
Theorem (Marklof-Strömbergsson, 2011)

As $N \to \infty$, $F_N(s)$ pointwise converges to a continuous function $F(s)$. If $\vec{w} \notin \mathbb{Q}^2$, $F$ agrees with the limiting gap distribution of $\sqrt{n} \pmod{1}$.

Figure: Left: The distribution of gaps in the sequence $\sqrt{n} \pmod{1}$, $n = 1 \cdots 7765$, vs. the Elkies-McMullen distribution. Right: Gap distribution for the directions of the vectors $(m - \sqrt{2}, n) \in \mathbb{R}^2$ with $m \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$, $(m - \sqrt{2})^2 + n^2 < 4900$. The continuous curve is the Elkies-McMullen distribution.
The study of spatial statistics is extended to the setting of hyperbolic lattices of finite covolume by Boca-Popa-Zaharescu, Kelmer-Kontorovich and Marklof-Vinogradov:

**Figure:** Directions of lattice points observed from i. Picture by Kelmer-Kontorovich
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We consider the following group: Let $\Lambda$ be a Schottky group generated by three hyperbolic reflections, with isometric circles $C_1, C_2, C_3$

**Figure:** A hyperbolic reflection group
Let $A_{p,q}$ be the collection of tangencies from circles with curvatures $\frac{1}{\text{radius}}$. We want to study the gap distributions of $A_{p,q}$.

**Figure:** A hyperbolic reflection group
Let $A(N)$ be the collection of tangencies from circles with curvatures $(1/\text{radius}) < N$. We want to study the gap distributions of $A(N)$. 

**Figure**: A hyperbolic reflection group
Let $\delta$ be the critical exponent of $\Lambda$, which agrees with the Hausdorff dimension of the closure of the set of all tangencies. There are $\sim cN^{2\delta}$ in total, so the average gap is $\frac{1}{cN^{2\delta}}$. 
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$$F_N(s) = \frac{\# \{d_i : d_i/\frac{1}{N^2} < s \}}{\# A(N)}.$$
**Theorem (Z)**

As $N \to \infty$, $F_N(s)$ pointwise converges to $F(s)$, where $F$ is a continuous, nonnegative function which is supported away from $0$ and $\lim_{s \to \infty} F(s) = 1$. 
Histograms of $\frac{dF_N}{ds}$ for various $N$:

**Figure:** $N = 10^3$

**Figure:** $N = \sqrt{2} \times 10^3$

**Figure:** $N = 10^4$

**Figure:** $N = \sqrt{2} \times 10^3$ and tangencies are taken from $[0.695204, 2.980334]$
Ingredients of the proof:

- Reduction to a hyperbolic lattice point counting problem in $PSU(1,1)$. A typical such problem is to count lattice points asymptotically in an expanding subset of $PSU(1,1)$.
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- Tools from homogeneous dynamics (Oh-Shah’s Theorem (JAMS, 2013), mixing of the geodesic flow under Bowen-Margulis-Sullivan density)
Figure: Tangencies in an Apollonian 9-circle packing
Theorem (Rudnick-Z, 2015)

There exists a limiting gap distribution for tangencies from an Apollonian 9-circle packing.

Figure: The density $F'(s)$ of the gap distribution for Apollonian 9-circle packings.

Figure: An Apollonian 9-Circle Packing
Key ideas

- There is a finite covolume group $\Lambda$ acting on the tangent circles.
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- Each gap can be expressed uniquely as $(\gamma(\alpha_i), \gamma(\alpha_j))$, where $\alpha_i, \alpha_j$ are tangencies from $C_i, C_j$. So gaps in $A(N)$ with relative length less than $s$ can be divided into finite families $A_{i,j}(s) = \{(\gamma(\alpha_i), \gamma(\alpha_j)) : \gamma \in \Lambda, (\gamma(\alpha_i), \gamma(\alpha_j)) \text{ is a gap in } A(N) \text{ with relative length less than } s\}$
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- $\#A_{1,2}(s) = \#\{\gamma \in \Lambda : \kappa(\gamma C_1) < N, \kappa(\gamma C_2) < N, \kappa(\gamma C_3) > N, \kappa(\gamma C_4) > N, \kappa(\gamma C_5) > N, d(\gamma(\alpha_i), \gamma(\alpha_j)) < \frac{s}{N^2}\}$
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- Each gap can be expressed uniquely as \((\gamma(\alpha_i), \gamma(\alpha_j))\), where \(\alpha_i, \alpha_j\) are tangencies from \(C_i, C_j\). So gaps in \(A(N)\) with relative length less than \(s\) can be divided into finite families \(A_{i,j}(s) = \{(\gamma(\alpha_i), \gamma(\alpha_j)) : \gamma \in \Lambda, (\gamma(\alpha_i), \gamma(\alpha_j)) \text{ is a gap in } A(N) \text{ with relative length less than } s\}\).
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- Under the coordinate of Cartan decomposition, the above conditions can be rephrased as

\[
(\phi_1(\gamma), \phi_2(\gamma), t(\gamma)) \in \Omega_s(N),
\]

where

\[
\Omega_s(N) = 2 \log N \cdot \Omega_s(1) = \{(\phi_1, \phi_2, 2 \log N \cdot t) : (\phi_1, \phi_2, t) \in \Omega_s(1)\}.
\]
Cartan Decomposition

Let $\mathbb{D}$ be the Poincaré disc with the metric $ds^2 = \frac{4(dx^2 + dy^2)}{(1-(x^2+y^2))^2}$. The orientation-preserving symmetry group of $\mathbb{D}$ is

$$G = PSU(1, 1) = \left\{ \left( \begin{array}{cc} \xi & \eta \\ -\overline{\eta} & \overline{\xi} \end{array} \right) \bigg| |\xi|^2 - |\eta|^2 = 1 \right\} \cong PSL_2(\mathbb{R}).$$

Let

$$K = \left\{ k_\phi = \left( \begin{array}{cc} e^{\frac{\phi i}{2}} & 0 \\ 0 & e^{-\frac{\phi i}{2}} \end{array} \right) \bigg| \phi \in [0, 2\pi) \right\},$$

$$A = \left\{ a_t = \left( \begin{array}{cc} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{array} \right) \bigg| t \in [0, \infty) \right\}.$$  

Recall the Cartan decomposition $G = KA^+K$ that each $g \in G$ can be written in a unique way as

$$g = k_{\phi_1(g)}a_t(g)k_{\pi-\phi_2(g)}$$

with $\phi_1(g), \phi_2(g) \in [0, 2\pi)$ and $t(g) > 0$. The Haar measure is given by $dg = e^t d\phi_1 d\phi_2 dt$. 
Joint equidistribution of Lattices of Finite Covolume in Cartan Decomposition

**Theorem (Good)**

Let $\Lambda$ be a lattice of $SU(1, 1)$ of finite covolume. Let $\mathcal{I}, \mathcal{J}$ be intervals in $[0, 2\pi)$. As $N \to \infty$,

$$\#\{\gamma \in \Lambda : \phi_1(\gamma) \in \mathcal{I}, \phi_2(\gamma) \in \mathcal{J}, t(\gamma) < N\} \sim \frac{l(\mathcal{I})l(\mathcal{J})}{4\pi^2 V(\Lambda)} e^N,$$

where $l$ is the standard arclength measure.
Joint equidistribution of Lattices of infinite Covolume in Cartan Decomposition

Theorem (Bourgain-Kontorovich-Sarnak, Oh-Shah, Mohammadi-Oh)

Let $\Lambda$ be a lattice of $SU(1,1)$ of infinite covolume, with critical exponent $\delta$. Let $I, J$ be intervals in $[0,2\pi)$. As $N \to \infty$,

$$\#\{\gamma \in \Lambda : \phi_1(\gamma) \in I, \phi_2(\gamma) \in J, t(\gamma) < N\} \sim \frac{\nu(I)\nu(J)}{4\pi^2 V(\Lambda)} e^{\delta N},$$

where $\nu$ is the Patterson-Sullivan measure on $[0,2\pi)$. 
Motivating problems: How are the circles from an Apollonian circle packings distributed?


*There is a finite Borel measure $\mu$ on the plane, such that for any region $\mathcal{R}$ with smooth boundary, $K_{\mathcal{R}}(N)$ the number of circles in $\mathcal{R}$ with curvature bounded by $N$ has asymptotic growth*

$$K_{\mathcal{R}}(N) \sim \mu(\mathcal{R}) N^{\delta_0}$$

where $\delta_0 \approx 1.305688$ is the Hausdorff dimension of the circle packing.
Figure: A region $\mathcal{R}$ with smooth boundary
Beyond equidistribution, what else can we say? Let $X_N$ be the centers of circles from $\mathcal{P}$. We want to study the spatial statistics on $X_N$.

**Figure:** Centers
Electrostatic energy

The electrostatic energy of $X_N$ is defined to be

$$E(X_N) = \sum_{\substack{p, q \in X_N \\ p \neq q}} \frac{1}{|p - q|}$$

The energy $E$ depends on both the global distribution of points as well as a moderate penalty if two points are too close to each other. More generally, one can consider the Riesz $s$-energy:

$$E_s(X_N) = \sum_{\substack{p, q \in X_N \\ p \neq q}} \frac{1}{|p - q|^s}$$

**Question**

What’s the behavior of $E_s(X_N)$ as $N \to \infty$? Is there an asymptotic growth?
Nearest neighbor spacing statistics

Let $d_{p,N}$ denote the distance of $p$ to the remaining points of $X_N$. A typical $d_{p,N}$ should have scale $1/N$. We define the nearest spacing measure $\nu(X_N)$ on $[0, \infty)$ by

$$\nu(X_N) := \frac{1}{\#X_N} \sum_{p \in X_N} \delta_{d_{p,N}}.$$ 

where $\delta_{\xi}$ is a delta mass at $\xi \in \mathbb{R}^+$. 

Question

Is there a limiting distribution for $\nu(X_N)$ as $N \to \infty$?