Beyond Apollonian Circle Packings: Expander Graphs, Number Theory and Geometry

1 Introduction

In the last decade tremendous effort has been put in the study of the Apollonian circle packings. Given the great variety of mathematics it exhibits, this topic has attracted experts from different fields: number theory, homogeneous dynamics, expander graphs, group theory, to name a few. The principle investigator (PI) contributed to this program in his PhD studies. The scenery along the way formed the horizon of the PI at his early mathematical career. After his PhD studies, the PI has successfully applied tools and ideas from Apollonian circle packings to the studies of topics from various fields, and will continue this endeavor in his proposed research. The proposed problems are roughly divided into three categories: number theory, expander graphs, geometry. Each of which will be discussed in depth in later sections.

Since Apollonian circle packing provides main inspirations for this proposal, let’s briefly review how it comes up and what has been done. We start with four mutually circles, with one circle bounding the other three. We can repeatedly inscribe more and more circles into curvilinear triangular gaps as illustrated in Figure 1, and we call the resultant set an Apollonian circle packing, which consists of infinitely many circles.

![Figure 1: Apollonian Circle Packing](image)

The number-theoretic significance of the Apollonian circle packings comes from an early observation by Descartes, that fixing a packing $P$, if $a, b, c, d$ are curvatures from four
mutually tangent circles, then \(a, b, c, d\) satisfy the following quadratic relation

\[
Q(a, b, c, d) = 2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 = 0.
\]

From this fact one can work out a group \(\Lambda_P \subset O_Q(\mathbb{Z})\), such that if \(\vec{r}\) is the vector of curvatures from any four mutually tangent circles, then the components of the orbit of vectors \(\vec{r} \cdot \Lambda\) give all curvatures from \(P\). A consequence is that if the original four curvatures are integers, then all circles are integers (see Figure 2a for an example of an integer Apollonian circle packing), and the study of these integers boils down to the study of the group \(\Lambda_P\). A key feature for \(\Lambda_P\) is that \(\Lambda_P\) is thin, in the sense that \(\Lambda_P\) is of infinite-covolume in its Zariski closure \(O_Q(\mathbb{R})\). By comparison, we say \(\Lambda_P\) is arithmetic if \(\Lambda_P\) is of finite covolume in its Zariski closure. The signature for \(Q\) is 3-1 so \(O_Q(\mathbb{R})\) can be realized as the isometry group of \(\mathbb{H}^3\).

We assume the gcd of all curvatures from \(P\) is 1, otherwise we can rescale. Built on the works from many people, Bourgain and Kontorovich proved the following striking theorem [8] using Hardy-Littlewood circle method:

**Theorem 1.1 (Bourgain-Kontorovich)** Let \(P_{24}\) be the set of residue classes mod 24 of curvatures from \(P\), then almost every integer \(x\) in \(P_{24}\) is actually a curvature.

The magic number 24 is determined by Fuchs in her thesis [14], after a careful study of local reduction of \(\Lambda\).

\[
\begin{array}{ccc}
18 & 27 & 23 \\
35 & 47 & 63 \\
62 & 78 & 83
\end{array}
\]

\[
\begin{array}{ccc}
T & Nv & Nc \\
Ok & Nv & Ok \\
Nv & Ok & Nc
\end{array}
\]

\[
\begin{array}{ccc}
2 & 4 & 3 \\
6 & 7 & 7 \\
10 & 8 & 12
\end{array}
\]

(a) An integer Apollonian circle Packing  (b) An integer Apollonian 3-circle packing  (c) An integer circle packing from [31]

Figure 2: Integer Circle Packings of Different Conformal Types

Some of the key ingredients of the proof of Theorem 1.1 are:

1. Strong approximation property for \(\Lambda\);

2. An asymptotic growth for elements in \(\Lambda\) and all of its “congruence” subgroups \(\Lambda(q) = \{\gamma \in \Lambda : \lambda \equiv I(\text{mod } q)\}\);
3. A geometric spectral gap for \( \Lambda \), which is a uniform lower bound of the distance between the two smallest eigenvalues of the hyperbolic laplacian \( \Delta \) on \( L^2(\Lambda(q)\backslash \mathbb{H}^2) \) for \( q \in \mathbb{N} \).

Theorem 1.1 has been extended to Apollonian 3-circle packings in the PI’s thesis [34]. Apollonian 3-circle packings was discovered by Guettler and Mallows [16]. Unlike an Apollonian circle packing, to form an Apollonian 3-circle packing, in each iteration we inscribe three circles instead of one into each triangular gap (see Figure 2b for an example).

**Theorem 1.2 (Zhang)** Let \( P_8 \) be the set of residue classes mod 8 of curvatures from an Apollonian 3-circle packing, then almost every integer in \( P_8 \) is a curvature.

Recently, Stange constructed a large collection of circle configurations using the Bianchi group \( SL(2, \mathcal{O}_d) \), where \( \mathcal{O}_d \) is the ring of integers of the imaginary quadratic field \( \mathbb{Q}[\sqrt{-d}] \). She considered the orbit of \( \hat{\mathbb{R}} \) under the Möbius transform of \( SL(2, \mathcal{O}_d) \), which she called Schmidt arrangement (see Figure 3 for some examples). Then she defined \( \mathbb{Q}[\sqrt{-d}]-\text{Apollonian packings} \) to be some maximal subset of circles from \( SL(2, \mathcal{O}_d) \cdot \hat{\mathbb{R}} \), subject to the following conditions:

1. the set of circles is connected.
2. for any circle \( C \) in this set, the two disconnected components of \( \mathbb{R}^2 - C \) can not simultaneously contain circles.

In this way she discovered many new circle packing with integer curvatures (see Figure 2c for an example), and in each such packing \( P \), there exists a group \( \Lambda_P < SL(2, \mathcal{O}_d) \) which is thin and acts on the circles with finitely many orbits.

It is thus tantalizing to ask the following question:

**Question 1** Can one prove a density-one theorem for the curvatures from the \( \mathbb{Q}[\sqrt{-d}] \) circle packings?

Currently the PI and his collaborators (Elena Fuchs and Kate Stange) are making progress towards an affirmative answer to Question 1. We note that in order to apply the Bourgain-Kontorovich machinery to this setting, several issues need to be addressed. We highlight two here.

The first issue is that, in order to count lattice point growth for \( \Lambda_P \) with a power savings rate, we need \( \Lambda_P \) to be geometrically finite. So the PI proposes

**Question 2** Verify the geometric finiteness for all \( \Lambda_P \) appearing in Kate Stange’s circle packings.
We have verified the geometric finiteness for some but not all $\Lambda_P$. The PI believes that a complete solution to Question 1 will lead to some general criteria to determine the geometric finiteness of a class of Kleinian groups.

Perhaps the second issue is more interesting. In order for the circle method to work, we need to count $\Lambda_P(q)$ for all positive integers $q$, with a uniform power savings error term. This will follow from establishing a geometric spectral gap for $\Lambda_P$. It is exactly this point that draws the PI into the world of expander graphs. The PI has established spectral gap for all these $\Lambda_P$, and expect the method can be adapted to prove spectral gap property for a much larger class of groups. We will expand this in Section 2.

The Apollonian problem can be put in a much more general setting. In [5], [6] Bourgain, Gamburd and Sarnak asked the following question: let $\Lambda \subset GL(n, \mathbb{Z})$, $\vec{v} \in \mathbb{Z}^n$ and $f$ be a polynomial of $n$ variables, then what integers appear in the set $f(\vec{v} \cdot \Lambda) = \{f(\vec{v} \cdot \gamma) : \gamma \in \Lambda\}$? They coined the term affine linear sieve, which stresses the application of classical sieve methods to find primes or almost primes in the set $f(\vec{v} \cdot \Lambda)$. However, for a stronger result such as Theorem 1.1 and Theorem 1.2, affine linear sieve is insufficient and new methods must be introduced, usually exploiting some extra structure of the group. The PI has successfully proved density-one theorems in some settings, and proposes some new problems to study in this direction in Section 3.

Section 4 and Section 5 focus on the application of thin group to some distribution problems of fractal nature. This is a new and seemingly promising area that hasn’t been exploited before the PI’s work [35].
2 Expander Graphs

Expander graphs play an important role in various areas of mathematics and computer science. See [21],[18] for some wonderful surveys on this topic. In this section, we explain how expander graphs are related to our geometric spectral gap problems. Let \( \Lambda \subset GL_n(\mathbb{Z}) \) be finitely generated, with a finite symmetric generating set \( S \) (this is a standard assumption through this proposal).

We can define the Markov operator \( T_{\Lambda,S,q} \) on the space \( L^2(\Lambda/\Lambda(q)) \) by

\[
T_{\Lambda,S,q}f(g) = \frac{1}{|S|} \sum_{\gamma \in S} f(\gamma g).
\]

A combinatorial spectral gap is a uniform positive lower bound for the distance between the two maximal eigenvalues of \( T_{G,S,q} \), for all \( q \in A \), where \( A \) is a subset of \( \mathbb{N} \). Geometrically, a combinatorial spectral gap gives a rate on how fast a random walk on the Cayley graph \( Cay(\Lambda/\Lambda(q),S) \) converges to the uniform distribution. If such a gap exists, we call \( \{Cay(\Lambda/\Lambda(q),S) : q \in A\} \) a family of expander graphs. In the following, we assume \( A = \mathbb{N} \) if we don’t specify.

By the previous discussion, we hope to establish a geometric spectral gap for \( \Lambda_P \subset SL(2, \mathcal{O}_d) \) which is the symmetry group for \( \mathbb{Q}[\sqrt{-d}] \)-Apollonian circle packings. By taking a proper spin homomorphism, one can also view \( \Lambda \) as a subgroup of \( SO(3,1,\mathbb{Z}) \). Several key features of \( \Lambda_P \) are

1. The group \( \Lambda \) is finitely generated, Zariski dense, yet of infinite-covolume in \( SL(2, \mathbb{C}) \).
2. The group \( \Lambda \) contains a congruence subgroup of \( SL(2, \mathbb{Z}) \).

After verifying the critical exponent of \( \Lambda \) exceeds 1, one can use a theorem of Bourgain, Gamburd and Sarnak [6] to translate the geometric spectral gap problem to a combinatorial spectral gap problem for \( \Lambda \). The existence of one will imply the existence of the other. For succinctness sometimes we just say spectral gap, without specifying “geometric” or “combinatorial” if no confusion is caused.

The study of spectral gap over congruence families of algebraic groups has a vivid history. Selberg proved a geometric spectral gap 3/16 for \( SL(2, \mathbb{Z}) \), which also implies a combinatorial spectral gap for \( SL(2, \mathbb{Z}) \). Later, combinatorial spectral gaps are found in many arithmetic lattices (e.g. higher rank arithmetic lattices enjoying Kazhdan’s Property \( T \)). However, if \( \Lambda \) is thin, many classical methods for establishing spectral gap fail to work. To stress the issue, Lubotzky proposed the famous 1-2-3 problem: whether

\[
\left\{ Cay\left(SL_2(\mathbb{F}_p); \begin{pmatrix} 1 & \pm 3 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ \pm 3 & 1 \end{pmatrix}\right) : \ p > 3 \ prime \right\}
\]

5
is a family of expanders.

Based on Helfgott’s triple product theorem [17], together with some representation theory technique, Bourgain and Gamburd proved a combinatorial spectral gap for any \( \Lambda \subset SL(2, \mathbb{Z}) \) with Zariski closure \( Zcl(\Lambda) = SL(2, \mathbb{R}) \) and \( q \) restricted to prime numbers [4]. In particular, this gives a positive answer to Lubotzky’s 1-2-3 problem. Bourgain and Gamburd’s theorem was eventually generalized to two state-of-the-art theorems:

**Theorem 2.1 (Bourgain-Varju [9])** If \( Zcl(\Lambda) = SL_d(\mathbb{R}) \), then

\[
\{ \text{Cay}(\Lambda/\Lambda(q); S) \mid q \in \mathbb{N} \}
\]

is a family of expanders.

**Theorem 2.2 (Golsefidy-Varju [15])** For a linear group \( \Lambda \subset GL_n(\mathbb{Z}) \), there exists an integer \( B \), such that

\[
\{ \text{Cay}(\Lambda/\Lambda(q); S) \mid (q, B) = 1, q \text{ square free} \}
\]

is a family of expanders if and only if the connected component of \( Zcl(\Lambda) \) is perfect.

Theorem 2.2 already has vast applications to sieve theory. But one may expect the condition “\( q \) square free” can be removed, and indeed more and more problems require removal of this restriction. The ultimate conjecture along this line is

**Conjecture 2.3** For a linear group \( \Lambda \subset GL_n(\mathbb{Z}) \),

\[
\{ \text{Cay}(\Lambda/\Lambda(q); S) \mid q \in \mathbb{N} \}
\]

is a family of expanders if and only if the connected component of \( Zcl(\Lambda) \) is perfect.

For our purpose on the work of circle packings, neither Theorem 2.1 nor Theorem 2.2 applies, because in this setting, \( Zcl(\Lambda_P) = SO_{R}(3,1) \) and we need \( q \) to exhaust all positive integers. Conjecture 2.3 in the special setting \( Zcl(\Lambda) = SO_{R}(3,1) \) will have application to our circle packing problems.

Nevertheless, in the circle packing setting, \( \Lambda_P \) has an extra feature, that is, it has a real congruence subgroup \( \Gamma_P \subset SL(2,\mathbb{Z}) \), which is then mapped to \( SO_{Z}(2,1) \) under the spin homomorphism. From Selberg’s 3/16 theorem we know \( \Gamma_P \) has a spectral gap. Later we realized that we just need the existence of \( \Gamma_P \) with \( Zcl(\Gamma_P) = SL(2, \mathbb{R}) \), in light of Theorem 2.1 when \( d = 2 \). We are able to prove the following general theorem:

**Theorem 2.4 (Fuchs-Stange-Zhang)** Let \( \Lambda \subset SO_{Z}(3,1) \) be a Zariski dense subgroup of \( SO_{R}(3,1) \). Assume \( \Lambda \) has a subgroup \( \Gamma \) with \( Zcl(\Gamma) \cong SO_{R}(2,1) \), then

\[
\{ \text{Cay}(\Lambda/\Lambda(q); S) \mid q \in \mathbb{N} \}
\]

is a family of expanders.
The special case when $\Lambda$ is the classical Apollonian group was proved by Varju. For a while his method was considered ad hoc by experts (including Varju himself). When the PI re-investigated his method, the PI found the principle there should work universally: if $\Lambda$ has a subgroup $\Gamma$ that expands, and if reduced by $q$, $\Lambda/\Lambda(q)$ can be generated by bounded many pieces of conjugates of $\Gamma/\Gamma(q)$, then $\Lambda$ should expand too. Towards the proof of Theorem 2.4, we only use the strong approximation and expander property for $\Gamma$ and some Lie algebra considerations. (The PI is very happy to provide a preprint for the proof upon request.) This leads the PI to propose the following conjecture:

**Conjecture 2.5** Let $\Lambda \subset GL_d(\mathbb{Z})$. Suppose $\Lambda$ has a finitely generated subgroup $\Gamma$ with a finite symmetric generating set $S'$ that satisfies

1. the group $\Gamma$ satisfies the strong approximation property;
2. $\{\text{Cay}(\Gamma/\Gamma(q); S') | q \in \mathbb{N}\}$ is a family of expanders;
3. $\text{Span}(\text{Ad}(Zcl(\Lambda))\text{Lie}(Zcl(\Gamma))) = \text{Lie}(Zcl(\Lambda))$.

where $\text{Ad}$ is the standard adjoint representation. Then

$$\{\text{Cay}(\Lambda/\Lambda(q); S) | q \in \mathbb{N}\}$$

is a family of expanders.

The PI proposes and has reasonable confidence to solve

**Question 3** Prove Conjecture 2.5.

Based on the proof of Theorem 2.4, an envisioned proof of Conjecture 2.5 is just to use the strong approximation property and play with the Lie algebras of $Zcl(\Lambda)$ and $Zcl(\Gamma)$, as well as using some general structure theory of algebraic groups, which should be significantly simpler than the methods employed in Theorem 2.1 and Theorem 2.2 ($l^2$ flattening, representation theory, etc.). The point of Conjecture 2.5 is to immediately tell a group $\Lambda$ has spectral gap property, once a subgroup $\Gamma$ with spectral gap property is detected. This should be compared with a theorem of Burger and Sarnak [10], which says that under some very general setting, if $\Gamma \subset \Lambda$ and both $\Lambda, \Gamma$ are of finite covolume in their Zariski closures $Zcl(\Lambda), Zcl(\Gamma)$, then $\Gamma$ has spectral gap implies $\Lambda$ has spectral gap.

Of course, a proof of Conjecture 2.3 will render Conjecture 2.5 obsolete. But before this happens, the PI believes that a proof of Conjecture 2.5 might shed some new light for Conjecture 2.3. The PI proposes to study:

**Question 4** Can the method in the proof of Theorem 2.4 be integrated to the current standard methods, to establish even more congruence families of expanders (with $q \in \mathbb{N}$)?
3 Number Theory

The problem of determining curvatures from circle packings can be put in a more general context considered by Bourgain, Gamburd and Sarnak [6]:

Let \( \Lambda \subset GL_n(\mathbb{Z}) \), \( \vec{v} \in \mathbb{Z}^n \) and \( f \) be an integral polynomial of \( n \) variables. What integers appear in the set \( f(\vec{v} \cdot \Lambda) = \{ f(\vec{v} \cdot \gamma) : \gamma \in \Lambda \} \)?

The notion “affine linear sieve” was introduced in [6]. Indeed in great generality almost primes have been found by carrying out the classical sieve techniques [6] [30]. A key ingredient in these works is that \( \Lambda \) has a combinatorial spectral gap over \( q \) square free (see Theorem 2.2). However, so far the affine linear sieve has not produced primes. If one wants to obtain a density one theorem (e.g. Theorem 1.1, Theorem 1.2, and Question 1 that we consider) which in particular implies the existence of infinitely many primes, affine linear sieve is insufficient, and one needs to use new techniques, usually exploiting some specific structure of the group.

Here is another example which surpassed affine linear sieve [7]:

**Theorem 3.1 (Bourgain-Kontorovich)** Let \( \Lambda \subset SL(2,\mathbb{Z}) \) be finitely generated, free and containing no parabolic elements. Let \( \vec{v}, \vec{w} \) be primitive vectors in \( \mathbb{R}^2 \) (i.e. the gcd of the components is 1). Consider the set \( K = \langle \vec{v} \cdot \Lambda, \vec{w} \rangle \). If \( \delta(\Lambda) \), the critical exponent of \( \Lambda \) exceeds 0.999995, then almost every integer admissible by \( K \) is actually in \( K \), with a power savings on the growth of the exceptional set. In particular, this implies there are infinitely many primes in \( K \).

In the setting of Theorem 3.1, we say an integer \( n \) admissible by \( K \) if \( n \in K(\mod q) \) for every \( q \in \mathbb{N} \). Strong approximation property implies that \( K \) is a finite union of arithmetic progressions. Theorem 3.1 is closely related to the Zaremba’s conjecture, which has applications to Monte Carlo methods in numerical integration. The critical exponent \( \delta(\Lambda) \) is a number in \([0,1]\), which measures the growth of the lattice points from \( \Lambda \). It is known that \( \delta(\Lambda) < 1 \) if and only if \( \Lambda \) is infinite-covolume.

Recently the PI supplemented Theorem 3.1 by dealing with the case when \( \Lambda \) has parabolic elements [36]:

**Theorem 3.2 (Zhang)** Let \( \Lambda \subset SL(2,\mathbb{Z}) \) be finitely generated and containing parabolic elements, and \( \vec{v}, \vec{w} \) be primitive vectors. Let \( K = \langle \vec{v} \cdot \Lambda, \vec{w} \rangle \). If \( \delta(\Lambda) > 0.99538 \), then almost every integer admissible by \( K \) is actually in \( K \), with a power savings on the growth of the exceptional set.

It turns out that finding primes is trivial in the setting of Theorem 3.2. If \( \Lambda \) has parabolic elements, then there are infinitely many arithmetic progressions \( An + B \) in the set \( K \), thus Dirichlet’s theorem implies there are infinitely many primes as long as \( \gcd(A,B) = 1 \). However, proving density-one still remains nontrivial in this setting. Both
Theorem 3.1 and Theorem 3.2 use Hardy-Littlewood circle method. The difference between the proof of Theorem 3.1 and Theorem 3.2 is the setup of the circle method: the setup for the parabolic case (Theorem 3.2) exploits the parabolic structure, which makes the corresponding exponential sum decay very fast in the minor arcs, except for a few tiny regions where we have to require $\delta(\Lambda)$ close to 1 in order to get control over; on the other hand, existence of such parabolic elements would cause high multiplicity among the input vectors in the Bourgain-Kontorovich setup for the non-parabolic case, which would eventually explode the minor arc analysis.

In the setting of Theorem 3.2, numerical data for some examples seem to suggest that possibly the restriction on $\delta(\Lambda)$ can be removed:

**Conjecture 3.3** Let $\Lambda \subset SL(2, \mathbb{Z})$ be non-elementary, finitely generated and containing parabolic elements, and $\vec{v}, \vec{w}$ be two primitive vectors. Let $K = \langle \vec{v} \cdot \Lambda, \vec{w} \rangle$. Then all but finitely many integers admissible by $K$ are actually in $K$.

It doesn’t seem likely to the PI that the method (and minor variations) in Theorem 3.2 can give a proof for Conjecture 3.3 or even a slightly weaker density-one version. And we propose

**Question 5** Continue to investigate Conjecture 3.3.

The PI’s plan is twofolds: first the PI plans to lead a undergraduate research group to gain more numerical evidence; second the PI will try other methods to gain some intermediate results, if not the full of Conjecture 3.3.

All the density-one results exhibited in this proposal are just some sporadic examples in the very broad setting of Bourgain-Gamburd-Sarnak. It would be very interesting to find other non-trivial examples where the density-one phenomenon holds:

**Question 6** Are there other examples under the setting of Bourgain-Gamburd-Sarnak in which a density-one theorem holds?

Of particular interest to the PI is the following problem:

**Question 7** Let $\Lambda \subset SL(2, \mathbb{Z}[[i]])$ and $f$ is a polynomial on the real and imaginary parts of entries from $SL(2, \mathbb{Z}[[i]])$, defined by

$$f \left( \begin{pmatrix} a_1 + a_2 i & b_1 + b_2 i \\ c_1 + c_2 i & d_1 + d_2 i \end{pmatrix} \right) = c_1^2 + c_2^2 + d_1^2 + d_2^2.$$ 

What integers appear in the set $f(\Lambda)$? In particular, does a density-one theorem hold?
We note that Question 7 is not exactly, but can be translated to a problem under the setting Bourgain-Gamburd-Sarnak using spin homomorphism. The interest of this question comes from the fact that the Apollonian problem can be translated to a problem similar to Question 7. In the proof of Theorem 1.1, the congruence subgroup $\Gamma_P$ is heavily exploited. While we only need $\text{Zcl}(\Gamma_P) = \text{SL}(2, \mathbb{R})$ for spectral gap purpose, in the minor arc analysis we do need congruence to apply abelian harmonic analysis. As a reward, the whole analysis only requires $\delta > 1$, which is automatically given by the existence of such a congruence subgroup. But what about a general group $\Lambda \subset \text{SL}(2, \mathbb{Z}[i])$ that does not contain a congruence subgroup? The PI proposes to investigate this issue. Probably one needs to require $\delta(\Gamma)$ very close to 2, as Theorem 3.1 and Theorem 3.2 indicate.

4 Geometric Statistics

We again use Apollonian circle packings to motivate the discussion for this section. Fixing an Apollonian circle packing $P$, one may ask how the circles in $P$ are distributed. It is known from the work [32] that $\delta_0$, the critical exponent of the symmetry group acting on $P$ is exactly the Hausdorff dimension of $P$. McMullen computed $\delta_0 \approx 1.305688$ [25].

If we place the circle packing in the complex plane $\mathbb{C}$, Oh and Shah [26] showed that the circles are equidistributed according to some finite Borel measure $\mu$ on $\mathbb{C}$, in the following sense:

**Theorem 4.1 (Oh-Shah)** Let $\mathcal{R}$ be any region with smooth boundary in $\mathbb{C}$ (see Figure 4), and $N_{\mathcal{R}}(T)$ be the number of circles in $\mathcal{R}$ whose curvatures are bounded by $T$, then as $T \to \infty$, $N_{\mathcal{R}}(T) \sim \mu(\mathcal{R})T^{\delta_0}$.

![Figure 4: A region $\mathcal{R}$ with smooth boundary](image)

![Figure 5: Centers from $P$](image)

Beyond equidistribution, what else can one say about the distribution of these circles? To be concrete, let $X_N$ be the collection of centers of circles from $P$ (the blue points in Figure 5). Then if $N$ large, $X_N$ roughly gives a “skeleton” of $P$. The PI proposes to study
some spatial statistics on the point process $X_N, N \to \infty$. This will tell us some fine-scale structures of the Apollonian circle packing. We select two types of statistics to illustrate.

### 4.1 Electrostatic energy

The electrostatic energy of $X_N$ is defined to be

$$E(X_N) = \sum_{\substack{p, q \in X_N \atop p \neq q}} \frac{1}{|p - q|}$$

The energy $E$ depends on both the global distribution of points as well as a moderate penalty if two points are too close to each other. More generally, one can consider the Riesz $s$-energy:

$$E_s(X_N) = \sum_{\substack{p, q \in X_N \atop p \neq q}} \frac{1}{|p - q|^s}$$

The PI proposes

**Question 8** Investigate the asymptotic behavior of $E_s(X_N)$ as $N \to \infty$.

### 4.2 Nearest neighbor spacing statistics

For the point set $X_N$ and any point $p \in X_N$, we let $d_{p,N}$ be the distance of $p$ to the remaining points of $X_N$. Let $g_N : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function. Then for $X_N$, we can define the $g_N$-nearest neighbor spacing measure $\nu_{g_N}(X_N)$ on $[0, \infty)$ by

$$\nu_{g_N}(X_N) := \frac{1}{\#X_N} \sum_{p \in X_N} \delta_{g_N(d_{p,N})},$$

where $\delta_\xi$ is a delta mass at $\xi \in \mathbb{R}^+$. The role of $g_N$ is to normalize $d_{p,N}$ if $X_N$ were a random point process distributed according to the lebesgue measure on the the inner part of the bounding circle of $P$, then by choosing $g(x) = \frac{N}{4} x^2$, the measure $\nu_g(X_N)$ would weakly converge to the Poisson distribution $e^{-x} dx$. Clearly this is not the case for $X_N$: as $N$ large, $X_N$ tends to distribute near the circle packing $P$, which is a fractal set. We raise the following question to address this issue:

**Question 9** In the setting of $X_N$, how to choose proper functions $g_N$ to normalize the nearest neighbor spacing measure? After proper normalization, is there a limiting distribution?

To investigate Question 8 and Question 9, we plan to exploit the fact that there’s symmetry group $\Lambda_P$ acting on the circles. When identifying $\hat{C}$ as the boundary of $\mathbb{H}^3$, $\Lambda_P$ can be realized as a group of isometries of $\mathbb{H}^3$ and the packing $P$ can be viewed as a
Figure 6: An orbit of $\Lambda_P$. (see Figure 6 for a orbit of $\Lambda_P$ in $\mathbb{H}^3$.) We expect that Question 9 can be translated to a distribution problem of a $\Lambda_P$-orbit on the boundary.

If $\Lambda$ is an arithmetic lattice of $SL(2, \mathbb{C})$, then it is well known that any point orbit of $\Lambda$ will eventually distribute according to the Lebesgue measure on the boundary. There’s a rich literature on the spatial statistics of such orbits. See [19], [28] and [24], and also [13], [23], [29], [1], [2] for some interesting works that can be reduced to a $\Lambda$-orbit distribution problem.

But our issue is that, the symmetry group $\Lambda_P$ is of infinite-covolume, and our point orbit tends to converge to some fractal sets, instead of distributing evenly on the boundary. The methods in all above works can not be applied to our setting.

The PI found we may overcome this obstacle. In [35] The PI did the following prototype example (see Figure 7): On the Poincare disk $\mathbb{D}$, let $\Lambda_1$ be the Schottky group generated by three reflections with non-intersecting isometry half-circles $C_1, C_2, C_3$. It is clear that $\Lambda_1$ is of infinite-covolume, so $\delta(\Lambda_1) < 1$. We put a circle $C_I$ tangent to 1, and translate it by $\Lambda_1$. Let $Y_N$ be the collection of tangencies from all circles with curvature bounded by $N$. What does the nearest neighbor spacing statistics for $Y_N$ look like?

The nearest spacing in this setting is also called “gap distribution”. The set $Y_N$ has
roughly constant $\times N^{2\delta(\Lambda_1)}$ points. If $Y_N$ were distributed like a finite covolume lattice, then a typical gap is of the same scale as the average gap, which is $1/N^{2\delta(\Lambda_1)}$. But this is not the case for our problem. It turns out that a typical gap is of the scale $1/N^2$, which is much smaller than the average gap, this is because $Y_N$ tends to accumulate over some fractal regions. Therefore, we should normalize all gaps by dividing them by $1/N^2$. Once we have this proper normalization, we can show

**Theorem 4.2** As $N \to \infty$, the nearest neighbor spacing measure $\nu_{gN}(Y_N)$ weakly converges to some measure $h(x)dx$, where $h$ is a continuous function supported at $[c, \infty)$ for some $c > 0$ (this phenomenon is also called repulsion), and $\int_0^\infty h(x)dx = 1$. (See Figure 8 for numerical evidence.)

Figure 8: The histograms of $d\nu_{gN}(Y_N)(x)/dx$ of different $N$, for the example illustrated in Figure 7

The key to the proof of Theorem 4.2 is an infinite-covolume lattice point counting result, due to Oh and Shah [27], which is eventually reduced to the mixing of geodesic flow under the Bowen-Margulis-Sullivan measure [3].

The evidence from Theorem 4.2 makes the PI believe that Question 8 and Question 9 are tractable. In [24] Marklof and Vinogradov determined the limiting behavior of a large class of spatial statistics for $\Lambda$-orbits, where $\Lambda$ is a finite-covolume lattice in $SO_\mathbb{R}(n, 1)$.

Marklof and Vinogradov’s result eventually boils down to a dynamics fact that the geodesic flow on the unit tangent bundle $T_1(\Lambda\backslash \mathbb{H}^n)$ is mixing. Since for $\Lambda$ infinite-covolume, geodesic flow is also mixing (according to the Bowen-Margulis-Sullivan measure, instead of the Lebesgue density in the finite-covolume case), the PI proposes to study

**Question 10** Generalize the result of Marklof-Vinogradov to the thin group setting.

After this is done, Question 8 and Question 9 among other spatial statistics should be within reach. Once this project is finished, the PI believes that in principle the method could be extended to study fine structures of other fractal sets with high symmetry (e.g. there is a group action).
5 Self-intersection numbers of geodesics

We conclude this proposal with a short discussion on the intersection numbers of geodesics on surfaces with constant negative curvature.

Based on earlier works from Cohen and Lustig [12], [22], in 2011 Chas ran a computer program to study the distribution of self-intersection numbers of closed geodesics on compact hyperbolic surfaces with boundary. It is well known that the intersection numbers of a randomly chosen closed geodesics of word length \( T \) grows like \( \kappa_1 T^2 \) for some \( \kappa_1 > 0 \). Chas discovered that a typical fluctuation for the self-intersection numbers is \( \asymp T^{3/2} \) and if one normalizes properly, the eventual distribution of intersection numbers is Gaussian. Soon afterwards Chas and Lalley strictly proved this empirical result [11], and their result was extended to compact surfaces without boundary by Wroten [33].

Lalley also studied the problem with geodesics ordered by geometric length instead of word length [20]. He found that if we randomly choose a geodesic segment of length at most \( T \), or randomly choose a closed geodesic of length at most \( T \) (both according to some natural measures), then the self-intersection numbers also grow like \( \kappa_2 T^2 \) for some \( \kappa_2 > 0 \), but a typical fluctuation is of order \( T \) instead of \( T^{3/2} \). And if one normalizes properly, the limiting distribution of the self-intersection numbers is given by the Gaussian quadratic forms, so mostly likely is not Gaussian.

We ask to what extend can one generalize Chas and Lalley’s results to the setting of infinite-volume hyperbolic surfaces. The PI proposes to investigate:

**Question 11** Study the distribution of self-intersection numbers of geodesics in infinite-volume hyperbolic surfaces.

In the setting of infinite-volume surfaces (so the fundamental groups \( \Lambda \) act on the universal cover \( \mathbb{H}^2 \) with infinite-covolume), if \( \Lambda \) is finitely generated and if we order geodesics by word length, the story shouldn’t change much since Chas and Lalley’s method is combinatorial in nature, which shouldn’t differentiate weather \( \Lambda \) is finite or infinite-covolume. If we order geodesics by geometric length, and if we randomly choose a geodesic according to the measure in [20], then for probability one we will find a geodesic going to the infinity without intersecting itself. So the first question in this setting is, what are good measures to define randomness in this setting? Probably these measures are related to the Patterson-Sullivan measure. The next question is, under these measures, what is a typical growth rate for self-intersection number and fluctuation? And finally, after proper normalization, is there a limiting distribution? The PI proposes to investigate all these questions.

6 Broader Impacts

The PI plans to lead three undergraduate groups to gain some numerical evidence for some of the proposed problems. These projects are beneficial to the participants as much as to
the PI. The participants can gain research experience with results potentially published at research-level journals. Some of the tools they learn along the way are at the research frontiers, which will be useful to the motivated students who are going to pursue a graduate study of mathematics.

7 Results from Prior NSF Support

The PI has not received prior NSF support.

Acknowledgement: In writing up the proposal, the PI uses several figures from other resources. Figure 1 is from [14], Figure 2c, Figure 3 are from [31], and Figure 6 is from [8].

References Cited


