STATISTICAL REGULARITY OF APOLLONIAN GASKETS

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Abstract. Apollonian gaskets are formed by repeatedly filling the gaps between three mutually tangent circles with further tangent circles. In this paper we give explicit formulas for the limiting pair correlation and the limiting nearest neighbor spacing of centers of circles from a fixed Apollonian gasket. These are corollaries of the convergence of moments that we prove. The input from ergodic theory is an extension of Mohammadi-Oh’s Theorem on the equidistribution of expanding horospheres in infinite volume hyperbolic spaces.

1. Introduction

1.1. Introduction to the problem and statement of results. Apollonian gaskets, named after the ancient Greek mathematician Apollonius of Perga (200 BC), are fractal sets formed by starting with three mutually tangent circles and iteratively inscribing new circles into the curvilinear triangular gaps (see Figure 1).

![Figure 1. Construction of an Apollonian gasket](image)

The last 15 years have overseen tremendous progress in understanding the structure of Apollonian gaskets from different viewpoints, such as number theory and geometry [16], [15], [10], [11], [20], [25]. In the geometric direction, generalizing a result of [20], Hee Oh and Nimish Shah proved the following remarkable theorem concerning the growth of circles.

Place an Apollonian gasket $\mathcal{P}$ in the complex plane $\mathbb{C}$. Let $\mathcal{P}_t$ be the set of circles from $\mathcal{P}$ with radius greater than $e^{-t}$, and let $C_t$ be the set of centers from $\mathcal{P}_t$. Oh-Shah proved:

**Theorem 1.1** (Oh-Shah, Theorem 1.6, [25]). There exists a finite Borel measure $\nu$ supported on $\mathcal{P}$, such that for any open set $E \subset \mathbb{C}$ with boundary $\partial E$ empty or piecewise
smooth (see Figure 2), the cardinality $N(E, t)$ of the set $C_t \cap E$, satisfies
\[
\lim_{t \to \infty} \frac{N(E, t)}{e^{\delta t}} = v(E),
\]
where $\delta \approx 1.305688$ [22] is the Hausdorff dimension of any Apollonian gasket.

![Figure 2. A region $E$ with piecewise smooth boundary](image)

Theorem 1.1 gives a satisfactory explanation on how circles are distributed in an Apollonian gasket in large scale. In this paper we study some questions concerning the fine scale distribution of circles, for which Theorem 1.1 yields little information. For example, one such question is the following.

**Question 1.2.** Fix $\xi > 0$. How many circles in $P_t$ are within distance $\xi/e^t$ of a random circle in $P_t$?

Here by distance of two circles we mean the Euclidean distance of their centers. Question 1.2 is closely related to the pair correlation of circles. In this article, we study the pair correlation and the nearest neighbor spacing of circles, which concern the fine structures of Apollonian gaskets. In particular, Theorem 1.3 gives an asymptotic formula for one half of the expected number of circles in Question 1.2, as $t \to \infty$.

Let $E \subset \mathbb{C}$ be an open set with $\partial E$ empty or piecewise smooth as in Theorem 1.1, and with $E \cap P \neq \emptyset$ (or equivalently, $v(E) > 0$). This is our standard assumption for $E$ throughout this paper. The pair correlation function $P_{E,t}$ on the growing set $C_t$ is defined as
\[
P_{E,t}(\xi) := \frac{1}{2\#(C_t \cap E)} \sum_{\substack{p, q \in C_t \cap E \ q \neq p}} \mathbf{1}\{e^t|p - q| < \xi\},
\]
where $\xi \in (0, \infty)$ and $|p - q|$ is the Euclidean distance between $p$ and $q$ in $\mathbb{C}$. We have a factor $1/2$ in the definition (1.1) so that each pair of points is counted only once.
For any \( p \in C_t \), let \( d_t(p) = \min\{|q-p| : q \in C_t, q \neq p\} \). The nearest neighbor spacing function \( Q_{E,t} \) is defined as
\[
Q_{E,t}(\xi) := \frac{1}{\#\{C_t \cap E\}} \sum_{p \in C_t \cap E} 1\{e^t d_t(p) < \xi\}. \tag{1.2}
\]

For simplicity we abbreviate \( P_{E,t}, Q_{E,t} \) as \( P_t, Q_t \) if \( E = C \). It is noteworthy that in both definitions (1.1) and (1.2), we normalize distance by multiplying by \( e^t \). The reason can be seen in two ways. First, Theorem 1.1 implies that a random circle in \( C_t \) has radius \( \asymp e^{-t} \), so a random pair of nearby points (say, the centers of two tangent circles) from \( C_t \) has distance \( \asymp e^{-t} \), thus \( e^{-t} \) is the right scale to measure the distance of two nearby points in \( C_t \). The second explanation is more informal: if \( N \) points are randomly distributed in the unit interval \([0, 1]\), then a random gap is of the scale \( N^{-1} \); more generally, if \( N \) points are randomly distributed in a compact \( n \)-fold, the distance between a random pair of nearby points should be of the scale \( N^{-1/n} \). In our situation, as \( t \to \infty \), the set \( C_t \) converges to \( \overline{P} \), where \( \overline{P} \) has Hausdorff dimension \( \delta \approx 1.305688 \). From Theorem 1.1, we know that \( \#C_t \asymp e^{\delta t} \), so our scaling \( e^{-t} \) agrees with the heuristics that the distance between two random nearby points in \( C_t \) should be \( (e^{\delta t})^{-1/\delta} = e^{-t} \).

Before stating our main results, we introduce terminology. It is convenient for us to work with the upper half-space model of the hyperbolic 3-space \( \mathbb{H}^3 \):
\[
\mathbb{H}^3 = \{z + rj : z = x + yi \in \mathbb{C}, r \in \mathbb{R}\}.
\]
We identify the boundary \( \partial \mathbb{H}^3 \) of \( \mathbb{H}^3 \) with \( \mathbb{C} \cup \{\infty\} \). For \( q = x + yi + rj \in \mathbb{H}^3 \), we define \( \Re(q) = x + yi \) and \( \Im(q) = r \).

Let \( G = PSL(2, \mathbb{C}) \) be the group of orientation-preserving isometries of \( \mathbb{H}^3 \). We choose a discrete subgroup \( \Gamma < PSL(2, \mathbb{C}) \) whose limit set \( \Lambda(\Gamma) = \overline{P} \) such that \( \Gamma \) acts transitively on circles from \( \mathbb{P} \). It follows from Corollary 1.3, [6] that \( \Gamma \) is geometrically finite.

Without loss of generality, we can assume that the bounding circle of \( \mathbb{P} \) is \( C(0,1) \), where \( C(z,R) \subset \mathbb{C} \) is the circle centered at \( z \) with radius \( R \). Let \( S \subset \mathbb{H}^3 \) be the hyperbolic geodesic plane with \( \partial S = C(0,1) \), and \( H < PSL(2, \mathbb{C}) \) be the stabilizer of \( S \).

As an isometry on \( \mathbb{H}^3 \), each \( g \in PSL(2, \mathbb{C}) \) sends \( S \) to a geodesic plane, which is either a vertical plane or a hemisphere in the upper half-space model of \( \mathbb{H}^3 \). We define continuous maps \( \mathbf{q} : G \to \overline{\mathbb{H}^3}, \mathbf{q}_R : G \to \overline{\mathbb{C}} \) as follows:
\[
\mathbf{q}(g) := \begin{cases}
\text{the apex of } g(S), & \text{if } \infty \notin g(\partial S), \\
\infty, & \text{if } \infty \in g(\partial S),
\end{cases} \tag{1.3}
\]
\[
\mathbf{q}_R(g) := \begin{cases}
\Re(\mathbf{q}(g)), & \text{if } \infty \notin g(\partial S), \\
\infty, & \text{if } \infty \in g(\partial S).
\end{cases} \tag{1.4}
\]
We further define a few subsets of $\mathbb{H}^3$. For $\xi > 0$, let $B_\xi := \{ z \in \mathbb{C} : |z| < \xi \}$ and let $B_\xi^\ast \subset \mathbb{H}^3$ be the “infinite chimney” with base $B_\xi$, where for any $\Omega \subset \mathbb{C}$,
\[
\Omega^\ast := \{ z + rj : z \in \Omega, r \in (1, \infty) \}. \tag{1.5}
\]
Let $C_\xi$ be the cone in $\mathbb{H}^3$:
\[
C_\xi := \{ z + rj \in \mathbb{H}^3 : r|z| > 1, 0 < r \leq 1 \}. \tag{1.6}
\]
Now we can state our main theorems.

**Theorem 1.3** (limiting pair correlation). For any open set $E \subset \mathbb{C}$ with $E \cap P \neq \emptyset$ and $\partial E$ empty or piecewise smooth, there exists a continuously differentiable function $P$ independent of $E$, supported on $[c, \infty)$ for some $c > 0$, such that
\[
\lim_{t \to \infty} P_{E,t}(\xi) = P(\xi).
\]
The derivative $P'$ of $P$ is explicitly given by
\[
P'(\xi) = \frac{\delta}{2\mu_H^{PS}(\Gamma_H \setminus H)} \int_{h \in \Gamma_H \setminus H} \sum_{\gamma \in \gamma_H \setminus (\Gamma_H \setminus H)} \frac{|q_R(h^{-1} \gamma^{-1})|^\delta}{\xi^{\delta+1}} d\mu_H^{PS}(h).
\]
Here $\Gamma_H := \Gamma \cap H$, and $\mu_H^{PS}$ is a Patterson-Sullivan type measure on $H$. Besides $\mu_H^{PS}$, we will also encounter other conformal measures $\mu_N^{PS}, w, m^{BR}, m^{BMS}$, which are built on the Patterson-Sullivan densities. The measure $\mu_N^{PS}$ is a Patterson-Sullivan type measure on the horospherical group $N := \{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \}$, $w$ is the pullback measure of $\mu_N^{PS}$ on $\mathbb{C}$ under the identification $z \to n_z$, and $m^{BR}, m^{BMS}$ are the Burger-Roblin, Bowen-Margulis-Sullivan measures. We will have a detailed discussion of these measures in Section 4.

See Figure 3 and Figure 5 for some numerical evidence for Theorem 1.3. Let $P(\theta_1, \theta_2)$ be the unique Apollonian gasket determined by the four mutually tangent circles $C_0, C_1, C_2, C_3$, where $C_0 = C(0,1)$ is the bounding circle, and $C_1, C_2, C_3$ are tangent to $C_0$ at $1, e^{\theta_1}, e^{\theta_2}$. Figure 3, Figure 4 and Figure 5 are based on the gasket $P(\frac{1.8\pi}{3}, \frac{3.7\pi}{3})$. Figure 6 suggests that the limiting pair correlations for different Apollonian gaskets are the same. The reason is twofold. First, for a fixed gasket, the limiting pair correlation locally looks the same everywhere. Second, one can take any Apollonian gasket to any other one by a Möbius transformation, which locally looks like a dilation combined with a rotation, and it is an elementary exercise to check that the limiting pair correlation is invariant under these motions.
Figure 3. The plot for $P_t$ with various $t$'s

Figure 4. Pair correlations for the whole plane, half plane and the first quadrant
Figure 5. The empirical derivative $P'_t(\xi)$ for different $t$, with step=0.1

Figure 6. Pair correlation functions for different Apollonian gaskets
Theorem 1.4 (limiting nearest neighbor spacing). There exists a continuous function $Q$ independent of $E$, supported on $[c, \infty)$ for some $c > 0$, such that

$$\lim_{t \to \infty} Q_{E,t}(\xi) = Q(\xi).$$

(1.7)

The formula for $Q$ is explicitly given by

$$Q(\xi) = 1 - \frac{\delta}{\mu_H^P(H)} \int_{\Gamma_H \setminus H} \int_0^\infty e^{-st} \mathbf{1}\{#q(a_{-t}h^{-1}(\Gamma - \Gamma_H)) \cap B^*_\xi = 0\} dt d\mu_H^P(h).$$

(1.8)

Here $a_{-t}$ is the diagonal matrix $egin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}$, and see Figure 7 for numerical evidence.

**Figure 7.** The nearest neighbor spacing function $Q_t(\xi)$ for various $t$’s.

Remark 1. Figure 7 suggests that $Q$ should be differentiable. Unlike the limiting pair correlation, we have not been able to prove the differentiability of $Q$ based on our formula for $Q$.

Both Theorem 1.3 and Theorem 1.4 follow from the convergence of moments (Theorem 1.5), which we explain now.

Let $\Omega = \prod_{1 \leq i \leq k} \Omega_i \subset \mathbb{C}^k$, where $\Omega_i, 1 \leq i \leq k$ are bounded open subsets of $\mathcal{C}$ with piecewise smooth boundaries.

For $z \in \mathbb{C}$, let

$$B_t(\Omega_i, z) := (e^{-t}\Omega_i + z) \cap \mathcal{C}_t,$$

and

$$N_t(\Omega_i, z) := \#B_t(\Omega_i, z).$$
Let \( r = (r_1, \ldots, r_k), \beta = (\beta_1, \ldots, \beta_k) \) be multi-indices, where \( r_i \in \mathbb{Z}_{\geq 0}, \beta_i \in \mathbb{R}_{\geq 0}, 1 \leq i \leq k \), and at least one component of \( r, \beta \) is nonzero. We want to understand the behaviors of the following two integrals

\[
\int_{\mathbb{C}} \prod_{1 \leq i \leq k} N_t(\Omega_i, z)^{r_i} \chi_E(z)dz, \quad (1.9)
\]

and

\[
\int_{\mathbb{C}} \prod_{1 \leq i \leq k} N_t(\Omega_i, z)^{\beta_i} \chi_E(z)dz, \quad (1.10)
\]

as \( t \to \infty \), where \( \chi_E \) is the characteristic function for an open set \( E \subset \mathbb{C} \) with no boundary or piecewise smooth boundary. Both (1.9) and (1.10) capture information about the correlation of centers.

Define functions \( F_{\Omega, r}, F_{\Omega, \beta} \) on \( G \) by

\[
F_{\Omega, r}(g) := \prod_{1 \leq i \leq k} 1 \{ \#(q(g^{-1}\Gamma/\Gamma_H) \cap \Omega_i^*) = r_i \}, \quad (1.11)
\]

and

\[
F_{\Omega, \beta}(g) := \prod_{1 \leq i \leq k} \#(q(g^{-1}\Gamma/\Gamma_H) \cap \Omega_i^*)^{\beta_i}. \quad (1.12)
\]

We put inverse signs over \( g \) in the definitions (1.11) and (1.12) so that both \( F_{\Omega, r} \) and \( F_{\Omega, \beta} \) are left \( \Gamma \)-invariant functions and can be thought of as functions on \( \Gamma \backslash G \).

The following theorem holds:

**Theorem 1.5 (convergence of moments).** With notation as above, we have

\[
\lim_{t \to \infty} e^{(2-\delta)t} \int_{\mathbb{C}} \prod_{1 \leq i \leq k} 1 \{ N_t(\Omega_i, z) = r_i \} \chi_E(z)dz = \frac{m_{BR}(F_{\Omega, r})w(E)}{m_{BMS}(\Gamma \backslash G)},
\]

and

\[
\lim_{t \to \infty} e^{(2-\delta)t} \int_{\mathbb{C}} \prod_{1 \leq i \leq k} N_t(\Omega_i, z)^{\beta_i} \chi_E(z)dz = \frac{m_{BR}(F_{\Omega, \beta})w(E)}{m_{BMS}(\Gamma \backslash G)}. \quad (1.13)
\]

**1.2. An overview of the method.** To prove Theorem 1.5, we first turn the integrals (1.9) and (1.10) into forms that fit into Mohammadi-Oh’s theorem on the equidistribution of expanding horospheres (Theorem 1.6). Here in particular, for our convenience we use the \( HAN \) and \( NAH \) decompositions for \( G \). Here \( H, A, N \) are certain subgroups of \( G \) (see Section 2 for the definitions of \( H, A \) and \( N \)). These decompositions seem new to us and we name them the generalized Iwasawa decompositions.

**Theorem 1.6 (Mohammadi-Oh, Theorem 1.7, [23]).** Suppose \( \Gamma < G \) is geometrically finite. Suppose \( \Gamma \backslash \Gamma N \) is closed in \( \Gamma \backslash G \) and \( \mu_{PS}^{N} \) < \infty. For any \( \Psi \in C_c(\Gamma \backslash G) \) and any \( f \in C^\infty(\Gamma \backslash \Gamma N) \), we have

\[
\lim_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma \backslash \Gamma N} \Psi(na_t)f(n)d\mu_{PS}^{N}(n) = \frac{m_{BR}(\Psi)\mu_{PS}^{N}(f)}{m_{BMS}(\Gamma \backslash G)}. \quad (1.13)
\]
However, Theorem 1.6 can not be directly applied, because in the statement of Theorem 1.6, the test function $\Psi$ is assumed to be compactly supported and smooth, while in our situation, $\Psi$ is $F_{\Omega, r}$ or $F_{\Omega}^\beta$, which are neither continuous nor compactly supported. The smoothness condition for $f$ and $\Psi$ is for the purpose of obtaining a version of equidistribution with exponential convergence rate. This is not needed for our purpose, as we only pursue asymptotics. By the same method from [26], the restriction for $f$ can be relaxed to be in $L^1(\Gamma \backslash \Gamma_N)$ together with some mild regularity assumption, and $\Psi$ can be relaxed to be continuous and compactly supported; but this is still not enough for our purpose. We circumvent this technical difficulty by proving Proposition 5.2, illustrating some hierarchy structure in the space $W$ of pairs of test functions $(f, \Psi)$ where the conclusion of Theorem 1.6 holds.

Theorem 1.1 implies that certain pairs $(f_0, \Psi_0)$ related to counting circles are in the space $W$. An elementary geometric argument shows that $F_{\Omega, r}, F_{\Omega}^\beta$ are dominated by $\Psi_0$. This together with Proposition 5.2 give us the desired Theorem 1.7, which is an extension of Theorem 1.6.

**Theorem 1.7.** Let $\Gamma < PSL(2, \mathbb{C})$ be a discrete group with the limit set $\Lambda(\Gamma) = \overline{P}$ and acting transitively on the circles from $P$. Let $\Psi = F_{\Omega, r}$ or $F_{\Omega}^\beta$, where $F_{\Omega, r}$ and $F_{\Omega}^\beta$ are defined by (1.11) and (1.12). Then $m_{BR}^B(\Psi) < \infty$, and

$$\lim_{t \to \infty} e^{(2-\delta)t} \int_{\mathbb{C}} \chi_E(z)\Psi(n_z a_t)dz = \frac{m_{BR}^B(\Psi)w(E)}{m_{BMS}(\Gamma \backslash G)}. \quad (1.14)$$

Theorem 1.5 then follows from Theorem 1.7.

**Remark 2.** It is desirable to prove a version of Theorem 1.6 only assuming the integrality of $\Psi$ over the Burger-Roblin measure plus some mild restriction. While it is an exercise to relax the compactly-supported assumption to being in $L^1$ when the hyperbolic space has finite volume, such an extension seems much less obvious (at least to the author) if the space has infinite volume. We have made partial progress (say, $\Psi$ can be in the Schwartz space) but haven’t been able to achieve sufficient generality to encompass Theorem 1.7.

1.3. **A historical note.** Pair correlation as well as other spatial statistics have been widely used in various disciplines such as physics and biology. For instance, in microscopic physics, the Kirkwood-Buff Solution Theory [19] links the pair correlation function of gas molecules, which encodes the microscopic details of the distribution of these molecules to some macroscopic thermodynamical properties of the gas such as pressure and potential energy. In macroscopic physics, cosmologists use pair correlations to study the distribution of stars and galaxies.

Within mathematics, there is also a rich literature on the spatial statistics of point processes arising from various settings, such as Riemann zeta zeros [24], fractional parts of $\{\sqrt{n}, n \in \mathbb{Z}^+\}$ [14], directions of lattice points [9], [8], [18], [27], [21], [13], Farey sequences and their generalizations [17], [7], [5], [29], [4], [2], and translation surfaces [1], [3], [33]. Our list of interesting works here is far from inclusive. These statistics can contain rich information and yield surprising discoveries. For instance, Montgomery and Dyson’s famous discovery that the pair correlation of Riemann zeta zeros agrees
with that of the eigenvalues of random Hermitian matrices, bridges analytic number theory and high energy physics.

There is a major difference between all works mentioned above and our investigation of circles here. In the above works, the underlying point sequences are uniformly distributed in their “ambient” spaces. In our case, the set of centers is fractal in nature: it is not dense in any reasonable ambient space such as $B_1$, the disk centered at 0 and of radius 1. Consequently, we need different normalizations of parameters.

In some of the works above, the problems were eventually reduced to the equidistribution of expanding horospheres in finite volume hyperbolic spaces. In our case, we need an infinite volume version of this dynamical fact, which is Theorem 1.6, as well as to take care of certain emerging issues in the infinite volume situation. The main contribution of this paper, in the eyes of the author, is to introduce the recently rapidly developed theory of thin groups to study the fine scale structures of fractals, by displaying a thorough investigation of the well known Apollonian gaskets.

1.4. The structure of the paper. Section 2 gives some basic background in hyperbolic geometry. In Section 3 we set up the problem and reduce proving Theorem 1.5 to proving Theorem 1.7. In Section 4 we give a detailed discussion of some emerging conformal measures built up from the Patterson-Sullivan densities. We finish the proof of Theorem 1.7 in Section 5. Finally in Section 6 we explain how to deduce Theorem 1.3 and Theorem 1.4 from Theorem 1.7. We give complete detail for the limiting pair correlation; the limiting nearest neighbor spacing can be deduced in an analogous way and we sketch the proof.

1.5. Notation. We use the following standard notation. The expressions $f \ll g$ and $f = O(g)$ are synonymous, and $f \asymp g$ means $f \ll g$ and $g \ll f$. Unless otherwise specified, all the implied constants depend at most on the symmetry group $\Gamma$. The symbol $1\{\cdot\}$ is the indicator function of the event $\{\cdot\}$. For a finite set $S$, we denote the cardinality of $S$ by $\#S$.

1.6. Acknowledgement. Figures 3-7 were produced in a research project of Illinois Geometry Lab (IGL) [12], where Weiru Chen, Calvin Kessler and Mo Jiao were the undergraduate investigators, Amita Malik was the graduate mentor, and the author of this paper was the faculty mentor. Although we didn’t use the results from [21] directly, that paper together with the data produced from the IGL project gave us the main inspiration of this paper. The technique employed in this paper is mainly from [34], [26], [23]. Thanks are also due to Prof. Curt McMullen for his enlightening comments and corrections.

2. Hyperbolic 3-space and groups of isometries

We use the upper half-space model for the hyperbolic 3-space $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{x + yi + rj : x + yi \in \mathbb{C}, r \in \mathbb{R}\}.$$  

The boundary $\partial \mathbb{H}^3$ of $\mathbb{H}^3$ is identified with $\mathbb{C} \cup \{\infty\}$. 
The hyperbolic metric and the volume form on $\mathbb{H}^3$ are given by
\[ ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}, \]
\[ dV = \frac{dx dy dr}{r^3}. \]

Let $G = \text{PSL}(2, \mathbb{C})$ be the group of orientation-preserving isometries of $\mathbb{H}^3$, and let $e$ be the identity element of $G$. The action of $G$ on $\mathbb{H}^3$ is given explicitly as the following:
\[(a, b, c, d) (z + rj) = \frac{a\bar{c}|z|^2 + a\bar{d}z + b\bar{c}z + bd + r^2a\bar{c}}{|cz + d|^2 + r^2|c|^2} + \frac{r}{|cz + d|^2 + r^2|c|^2}J.\]

For any two points $q_1, q_2 \in \mathbb{H}^3$, the formula for their hyperbolic distance $d(q_1, q_2)$ is
\[ d(q_1, q_2) = \text{Arccosh} \left(1 + \frac{|q_1 - q_2|^2}{2\Im(q_1)\Im(q_2)}\right), \tag{2.1} \]
where $|q_1 - q_2|$ is the Euclidean distance between $q_1$ and $q_2$. Let $\pi_1, \pi_2$ be the maps from $G$ to $T^1(\mathbb{H}^3), \mathbb{H}^3$ defined by
\[
\begin{align*}
\pi_1(g) &:= g(X_1), \\
\pi_2(g) &:= g(j).
\end{align*}
\]

The following subgroups of $G$ will appear in our analysis:

(i) $A := \left\{ a_t = \begin{pmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix} : t \in \mathbb{R} \right\}$.

(ii) $K := \text{PSU}(2) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$.

(iii) $M := \left\{ m_\theta = \begin{pmatrix} e^{\frac{\theta}{2}} & 0 \\ 0 & e^{-\frac{\theta}{2}} \end{pmatrix} : \theta \in [0, 2\pi) \right\}$.

(iv) $N := \left\{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$.

(v) $H := \text{SU}(1, 1) \cup \text{SU}(1, 1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, where
\[
\text{SU}(1, 1) = \left\{ \begin{pmatrix} \xi & \eta \\ \bar{\eta} & \bar{\xi} \end{pmatrix} : \xi, \eta \in \mathbb{C}, |\xi|^2 - |\eta|^2 = 1 \right\}.
\]

(vi) $H_0 := \text{SU}(1, 1)$, the identity component of $H$.

(vii) $\tilde{A} := \left\{ \tilde{a}_t = \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} : t \in \mathbb{R} \right\}$.

We now explain the geometric meaning of the above groups. Let $\{X_1, X_2, X_3\}$ be an orthonormal frame based at $j$, where $X_1, X_2, X_3$ are unit vectors based at $j$ pointing to the negative $r$ direction, positive $y$ direction, and the positive $x$ direction, respectively. Let $S \subset \mathbb{H}^3$ be the hyperbolic geodesic plane with boundary $\partial S = C(0, 1)$, where $C(z, R) \subset \mathbb{C}$ is the circle centered at $z$ with radius $R$. The group $G$ can also be identified with the orthonormal frame bundle on $\mathbb{H}^3$. The flows $\{a_t(X_1) : t \in \mathbb{R}\}, \{\tilde{a}_t(X_3) : t \in \mathbb{R}\}$ are the geodesic flows containing $X_1, X_3$, respectively. The group $H$ is the stabilizer of
the geodesic plane $S$, $K$ is the stabilizer of $j$, and $M$ is the stabilizer of $X_1$. The orbit $N(X_1)$ is the expanding horosphere containing $X_1$.

In our analysis we adopt the following decomposition for $G$ which are particularly convenient for us:

$$G = NAH; G = HAN.$$  

We call these decompositions the generalized Iwasawa decompositions.

We further decompose the group $H$ via the Cartan decomposition:

$$H = M \left( \widetilde{A}^+ \cup \widetilde{A}^+ \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \right) \times M,$$

where

$$\widetilde{A}^+ =: \left\{ \widetilde{a}_t = \left( \begin{array}{cc} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{array} \right) : t \in (0, \infty) \right\}.$$

For every $h \in H - M \cup \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \times M$, we can write $h = m_1 a m_2$ with $m_1, m_2 \in M$ and $a \in \widetilde{A}^+ \cup \widetilde{A}^+ \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$ in a unique way.

Now we show that the generalized Iwasawa decompositions parametrize $G$ except for a codimension one subvariety. We first consider $G = NAH$. Let $V$ be the set of all horizontal vectors and vertical vectors in $T^1(\mathbb{H}^3)$, where a horizontal (vertical) vector is a vector parallel (perpendicular) to $C$ in the Euclidean sense. Let $G_{V} = \{ g \in G : g(X_1) \in V \}$. We claim the product map $\rho_1$:

$$\rho_1(n, a, m_1, \tilde{A}, m_2) := nam_1 \tilde{A} m_2$$

is a homeomorphism.

Indeed, we notice first that the map $\pi_2 \circ \rho$ on the set

$$L^1 := \{ e \} \times \{ e \} \times M \times \left( \tilde{A} \cup \tilde{A} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \right) \times \{ e \}$$

gives an identification of $L^1$ with all non-vertical vectors in the unit normal bundle $N^1(S)$. For any vector $u \in T^1(\mathbb{H}) - V$, we can find unique elements $m_1 \in M, \tilde{a} \in \tilde{A}$ such that $m_1 \tilde{a}(X_1)$ and $u$ point to the same Euclidean direction. Next we can find a unique element $a \in A$ such that $am_1 \tilde{a}(X_1)$ and $u$ are based in the same horizontal plane. After that, we can find a unique element $n \in N$ so that $nam_1 \tilde{a}(X_1)$ and $u$ are based at the same point. We observe that the actions of $N, A$ on $T^1(\mathbb{H}^3)$ preserve Euclidean directions. Thus we have $nam_1 \tilde{a}(X_1) = u$. The group $M$ preserves $X_1$, and acts transitively and faithfully on all vectors in $T^1_e(\mathbb{H}^3)$ normal to $X_1$, so $M$ can be identified with all orthonormal frames based at $j$ with the first reference vector $X_1$. As a result, choosing a unique $m_2 \in M$ for the rightmost factor $M$ on the left hand side of (2.3), we can take the frame $\{ X_1, X_2, X_3 \}$ at $e$ to any frame at $\pi(u)$ with the first reference vector $u$, by the action of $nam_1 \tilde{a} m_2$. So the claim is established. Similarly,
we have a decomposition $G = HAN$ induced from the decomposition $G = NAH$ by the inverse map of $G$. This decomposition parametrizes all elements in $G - G_{V^{-1}}$.

3. Setup of the problem

Let $P \subset \mathbb{C}$ be a bounded Apollonian gasket, and $C = C_P$ be the collection of all centers from $P$. Let $P_t$ be the set of the circles from $P$ with curvatures $< e^{-t}$ and $C_t$ be the set of centers of $P_t$.

Fix an open set $E \subset \mathbb{C}$ with $E \cap P \neq \emptyset$ and $\partial E$ empty or piecewise smooth, and fix a multi-set $\Omega = \prod_{1 \leq i \leq k} \Omega_i \subset \mathbb{C}^k$, where $\Omega_i, 1 \leq i \leq k$ are bounded open subsets of $\mathbb{C}$ with piecewise smooth boundaries.

Let

$$B_t(\Omega_i, z) := (e^{-t}\Omega_i + z) \cap C_t,$$

and

$$N_t(\Omega_i, z) := \#B_t(\Omega_i, z).$$

We want to study

$$\int_{\mathbb{C}} \prod_{1 \leq i \leq k} 1\{N_t(\Omega_i, z) = r_i\} \chi_E(z)dz. \quad (3.1)$$

and

$$\int_{\mathbb{C}} \prod_{1 \leq i \leq k} N_t(\Omega_i, z)^{\beta_i} \chi_E(z)dz. \quad (3.2)$$

as $t \to \infty$.

To proceed, first we choose a Kleinian group $\Gamma < \text{PSL}(2, \mathbb{C})$ whose limit set $\Lambda(\Gamma) = P$, such that $\Gamma$ transitively on the circles from $P$. The existence of $\Gamma$ can be seen as follows: let $\Gamma_0 = \langle \text{PSL}(2, \mathbb{Z}), \left( \begin{smallmatrix} i & 1 \\ 0 & -i \end{smallmatrix} \right) \rangle$.

One can check that the limit set of $\Gamma_0$ is the closure of the unbounded Apollonian packing $P_0$, determined by three mutually tangent circles $R, R + i, C(i/2, 1/2)$, and $\Gamma_0$ acts transitively on the circles from $P_0$. Since any Apollonian packing $P$ can be mapped to $P_0$ by a Möbius transform, the symmetry group $\Gamma$ of $P$ can then be taken as a conjugate of $\Gamma_0$.

Recall that $S$ is the geodesic plane with $\partial S = C(0, 1)$, then for any isometry $g \in G$, $g(S)$ is also a geodesic plane, so in the upper half-space model, $g(S)$ is either a hemisphere or a vertical plane.

Recall the maps $q$ from $G$ to $\mathbb{H}^3$, $q_\mathbb{R}$ from $G$ to $\hat{\mathbb{C}}$ defined at (1.3), (1.4). If $\infty \notin g(\partial S)$, there exists a unique geodesic $l_q$ which traverses $g(S)$ perpendicularly. Then $q(g)$ is the intersection of $l_q$ and $g(S)$, and $q_\mathbb{R}(g)$ is the other end point of $l(g)$ besides $\infty$, whence we can see that the definitions for $q$ and $q_\mathbb{R}$ at $g$ with $\infty \in g(\partial S)$ are continuous extensions. Therefore, both $q$ and $q_\mathbb{R}$ are continuous everywhere.

Let $r = \langle r_1, \ldots, r_k \rangle, \beta = \langle \beta_1, \ldots, \beta_k \rangle$ be multi-indices, where $r_i \in \mathbb{Z}_{\geq 0}, \beta_i \in \mathbb{R}_{\geq 0}, 1 \leq i \leq k$, and at least one component of $r, \beta$ is nonzero. Let $\Omega^*_i \subset \mathbb{H}^3$ be the “chimney”

$$\Omega^*_i := \{ z + r j : z \in \Omega_i, r > 1 \},$$
for \(1 \leq i \leq k\).

Let \(\Gamma_H = \Gamma \cap H\). Since \(\text{Stab}(C(0,1)) = H\) and \(\Gamma\) acts transitively on the circles from \(\mathcal{P}\), we have

\[
\mathcal{C} = \{\Re(q(\gamma)) : \gamma \in \Gamma/\Gamma_H\},
\]

and

\[
\mathcal{C}_t = \{\Re(q(\gamma)) : \gamma \in \Gamma/\Gamma_H, \Im(q(\gamma)) > e^{-t}\}.
\]

Therefore, we can rewrite \(\mathcal{N}_t(\Omega_i, z)\) as

\[
\mathcal{N}_t(\Omega_i, z) = \#(e^{-i}\Omega_i + z) \cap \mathcal{C}_t
\]

\[
= \#\{\gamma \in \Gamma/\Gamma_H : \Re(q(\gamma)) \in e^{-i}\Omega_i + z, \Im(q(\gamma)) > e^{-t}\}
\]

\[
= \#\{\gamma \in \Gamma/\Gamma_H : \Re(a_{-i}n_zq(\gamma)) \in \Omega_i, \Im(a_{-i}n_zq(\gamma)) > 1\}
\]

\[
= \#\{\gamma \in \Gamma/\Gamma_H : q(a_{-i}n_z\gamma) \in \Omega_i^*\}. \tag{3.3}
\]

Recall the definitions for the functions \(F_{\Omega_r}, F_{\Omega}^\beta\) on \(G\) defined by (1.11) and (1.12):

\[
F_{\Omega_r}(g) := \prod_{1 \leq i \leq k} 1 \left\{ \#(q^{-1}\Gamma/\Gamma_H) \cap \Omega_i^* = r_i \right\},
\]

\[
F_{\Omega}^\beta(g) := \prod_{1 \leq i \leq k} \#(q^{-1}\Gamma/\Gamma_H) \cap \Omega_i^*)^{\beta_i}.
\]

Collecting (3.3),(1.11),(1.12), we have

\[
\int_{\mathcal{C}} \prod_{1 \leq i \leq k} 1\{\mathcal{N}_t(\Omega_i, z) = r_i\} \chi_E(z)dz = \int_{\mathcal{C}} F_{\Omega_r}(n_z a_t) \chi_E(z)dz, \tag{3.4}
\]

\[
\int_{\mathcal{C}} \left( \prod_{1 \leq i \leq k} \mathcal{N}_t(\Omega_i, z)^{\beta_i} \right) \chi_E(z)dz = \int_{\mathcal{C}} F_{\Omega}^\beta(n_z a_t) \chi_E(z)dz. \tag{3.5}
\]

At this point, we have rephrased our problem in the setting of Theorem 1.6. We restate it here:

**Theorem 3.1** (Mohammadi-Oh, [23]). Suppose \(\Gamma < G\) is geometrically finite. Suppose \(\Gamma \backslash \Gamma N\) is closed in \(\Gamma \backslash G\) and \(|\mu_N^{PS}| < \infty\). For any \(\Psi \in C_C^\infty(\Gamma \backslash G)\) and any \(f \in C^\infty(\Gamma \backslash \Gamma N)\), we have

\[
\lim_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma \backslash \Gamma N} \Psi(na_t)f(n)d\mu_N^{Leb}(n) = \frac{m^{BR}(\Psi)\mu_N^{PS}(f)}{m^{BMS}(\Gamma \backslash G)}. \tag{3.6}
\]

Here \(m^{BR}, \mu_N^{PS}, m^{BMS}\) are certain conformal measures for which we are going into detail in the next section. In our situation, \(\Gamma\) is the symmetry group of the Apollonian gasket \(\mathcal{P}\), \(f\) is the characteristic function \(\chi_E\), and \(\Psi\) is \(F_{\Omega_r}\) or \(F_{\Omega}^\beta\). We have \(\Gamma \backslash \Gamma N = N\) as \(\Gamma \cap N = \{e\}\). Since \(\Gamma\) is geometrically finite, we have \(0 < m^{BMS}(\Gamma \backslash G) < \infty\) (Corollary 1.3, [6]). We will also see that \(\mu_N^{PS}(\chi_E) < \infty\). The issue for us to apply Theorem 1.6 is, none of the functions \(f, F_{\Omega_r}\) or \(F_{\Omega}^\beta\) is continuous. Moreover, \(F_{\Omega_r}, F_{\Omega}^\beta\) are not compactly supported, so a priori \(m^{BR}(F_{\Omega_r}), m^{BR}(F_{\Omega}^\beta)\) can be \(\infty\). The purpose
of the next two sections is to prove Theorem 1.7, which is an extended version of Theorem 1.6. Along the way we will show that \( m^{\text{BR}}(F_{\Omega_0}), m^{\text{BR}}(F_{\Omega}^3) < \infty \).

4. Conformal Measures

We keep all notation from previous sections. Let \( \Gamma < G \) be a discrete group with the limit set \( \Lambda(\Gamma) = \overline{P} \) and acting transitively on the circles from \( P \). A family of finite measures \( \{\mu_x : x \in \mathbb{H}^3\} \) on \( \partial \mathbb{H}^3 \) is called a \( \Gamma \)-invariant conformal density of dimension \( \delta_\mu > 0 \) if for any \( x, y \in \mathbb{H}^3, u \in \partial \mathbb{H}^3 \),

\[
\gamma^* \mu_x = \mu_{\gamma x}, \quad \text{and} \quad \frac{d\mu_x(u)}{d\mu_y(u)} = e^{-\beta_u(x,y)\delta_\mu},
\]

where for any Borel set \( F \subset \partial \mathbb{H}^3 \), \( \gamma^* \mu_x(F) = \mu_x(\gamma^{-1}F) \). The function \( \beta_u \) is the Busemann function defined as:

\[
\beta_u(x,y) = \lim_{t \to \infty} d(u_t,x) - d(u_t,y),
\]

where \( u_t \) is any geodesic ray tending to \( u \) as \( t \to \infty \).

Two particularly important densities are the Lebesgue density \( \{m_x : x \in \mathbb{H}^3\} \) and the Patterson-Sullivan density \( \{\nu_x : x \in \mathbb{H}^3\} \). The Lebesgue density is a \( G \)-invariant density of dimension 2, and for each \( x \), \( m_x \) is \( \text{Stab}(x) \)-invariant. The Patterson-Sullivan density \( \{\nu_x\} \) is supported on the limit set \( \overline{P} \), and of dimension \( \delta [31] \). Both densities are unique up to scaling. We normalize these densities so that \( |\nu_j| = 1 \) and \( |m_j| = \pi \).

Write \( z = x + yj \). We have an explicit formula for \( m_j \) in the \( \mathbb{C} \) coordinate:

\[
dm_j(z) = \frac{dxdy}{(1 + x^2 + y^2)^2}.
\]

Therefore, \( \dzm_j(z) \approx dx\,dy \) near 0.

The formula for \( \nu_j \) is explicitly given as the weak limit as \( s \to \delta^+ \) of the family of measures

\[
\nu_{j,s} := \frac{1}{\sum_{\gamma \in \Gamma} e^{-s d(\gamma,j)}} \sum_{\gamma \in \Gamma} e^{-s d(\gamma,j)} \delta_j,
\]

where \( \delta_j \) is the Dirac delta measure supported at the point \( \gamma j \).

We have the following estimate for \( \nu_j(B(z,r)) \), where \( B(z,r) \subset \mathbb{C} \) is the Euclidean ball centered at \( z \) with radius \( r \) (see Sec. 7 of [32]):

\[
\nu_j(B(z,r)) \ll \min\{r^\delta, 1\}.
\]

By a simple packing argument, (4.2) implies \( \nu_j(l) = 0 \) for any differentiable curve \( l \subset \mathbb{C} \). So by our assumption for \( E \), we have \( \nu_j(\partial E) = 0 \).

We also need to work with certain measures related to the conformal densities \( \{m_x : x \in \mathbb{H}^3\} \) and \( \{\nu_x : x \in \mathbb{H}^3\} \). For any \( u \in T^1(\mathbb{H}^3) \), let \( u^-, u^+ \in \mathbb{C} \) be the starting and ending points of \( u \). We can identify \( N \) with \( \partial \mathbb{H}^3 - \{0\} \) via the map \( g \to g(X_0)^+ \). Let \( H_1 = H/M \), then \( H_1 \) can be identified with \( \partial \mathbb{H}^3 - \partial S \) via the map \( g \to g(X_0)^- \). We
define measures $\mu^P_N, \mu^P_{H_1}$ as:

$$d\mu^L_{N}(n) := e^{2\beta_n(x_1^+) + (j, n(j))} dm_j(n(x_1^+)), \quad (4.3)$$

$$d\mu^P_N(n) := e^{\delta_n(x_1^+) + (j, n(j))} d\nu_j(n(x_1^+)), \quad (4.4)$$

$$d\mu^P_{H_1}(h_1) := e^{\delta_{h_1(x_1^-)}(j, n(j))} d\nu_j(h_1(x_1^-)). \quad (4.5)$$

Later on it will follow from Lemma 4.12 that $\mu^L_{N}(n_\beta) = dz$, so $\mu^L_{N}$ is in fact a Haar measure on $N$.

We can lift the measure $\mu^P_{H_1}$ to a unique right $M$-invariant measure $\mu^P_{H}$ on $H$ satisfying: for any $f \in C_c(H_1)$, define $\hat{f} \in C_c(H)$ as

$$\hat{f}(h) = f(hM).$$

Then

$$\int_H \hat{f}(h) d\mu^P_{H}(h) = \int_{H_1} f(h_1) d\mu^P_{H_1}(h_1).$$

We can view $H$ as a circle bundle over $H_1$. Under this viewpoint, from the definition of $\mu^P_{H_1}$ we have

$$\mu^P_{H} = d\mu^P_{H_1} \cdot dm^\text{Haar}_M,$$

where $m^\text{Haar}_M$ is the Haar measure of $M$ with $|m^\text{Haar}_M| = 1$.

4.1. **Finiteness of $\mu^P_{N}(N), \mu^P_{H_1}(\Gamma_H \setminus H_1)$ and $\mu^P_{H}(\Gamma_H \setminus H)$**. In this section we are going to show $0 < \mu^P_{N}(N), \mu^P_{H_1}(\Gamma_H \setminus H_1), \mu^P_{H}(\Gamma_H \setminus H) < \infty$. The $>0$ part is trivial and we focus on the $<\infty$ part. We begin with a calculation:

**Lemma 4.1.** For any $q \in \mathbb{H}^3$, we have $\beta_{\infty}(j, q) = \log \Im(q)$.

**Proof.** By the definition of the Buseman function,

$$\beta_{\infty}(j, q) = \lim_{t \to \infty} d(e^t j, j) - d(e^t j, q) = t - \lim_{t \to \infty} d(j, e^{-t} q). \quad (4.6)$$

From the hyperbolic distance formula (2.1),

$$d(j, e^{-t} q) = \text{Arccosh} \left( 1 + \frac{e^{t \cdot |j - e^{-t} q|^2}}{2 \Im(q)} \right)$$

$$= \text{Arccosh} \left( 1 + \frac{e^{t - \log \Im(q)}}{2} (1 + O_q(e^{-t})) \right)$$

$$= t - \log \Im(q) + O_q(e^{-t}). \quad (4.7)$$

Applying (4.7) to (4.6), we obtain

$$\beta_{\infty}(j, q) = \lim_{t \to \infty} t - (t - \log \Im(q) + O_q(e^{-t})) = \log \Im(q). \quad \square$$

Returning to (4.4), we have

$$e^{\delta_{n(x_1^+) + (j, n(j))}} = e^{\delta_{n(0)(j, n(j))}} = e^{-\delta_{n(0)}(j, n^{-1}(j))} = e^{-\delta_{\infty} \left( j, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} n^{-1}(j) \right)} = (|n^{-1}(0)|^2 + 1)^{\delta} \quad (4.8)$$
Since $\Lambda(\Gamma) = \overline{\mathcal{D}}$ is compact, the term $(|n^{-1}(0)|^2 + 1)^{\delta}$ is bounded on the support of $\Lambda(\Gamma) = \overline{\mathcal{D}}$ of $\mu_{PS}^{\Gamma}$. As $|v_1|$ is finite, we have $\mu_{PS}^{\Gamma}(N) < \infty$.

Now we consider $\mu_{PS}^{\Gamma}(\Gamma_H \backslash H_1)$ and $\mu_{PS}^{\Gamma}(\Gamma_H \backslash H)$. Both $\Gamma_H \backslash H$ and $\Gamma \backslash H$ have one cusp, whose ranks in both $\Gamma_H$ and $\Gamma$ are equal to 1. Therefore $\mu_{PS}^{\Gamma}(\cdot)$ is compactly supported in $\Gamma_H \backslash H_1$ from Theorem 6.3 [26]. Thus the term $e^{\delta \beta_{h_1}(x_1) - (j_{H_1}(j))}$ from (4.5) is bounded on the support of $\mu_{PS}^{\Gamma_H}(\cdot)$, so that $\mu_{PS}^{\Gamma_H}(\Gamma_H \backslash H) = \mu_{PS}^{\Gamma_H}(\Gamma_H \backslash H_1) < \infty$.

4.2. Quasi-product Conformal Measures on $T^1(\mathbb{H}^3)$ and $G$. Following Roblin [28], given two conformal measures $\{\mu_x\}, \{\mu'_x\}$, we can define a quasi-product measure $\tilde{m}^{\mu,\mu'}$ on $T^1(\mathbb{H}^3)$ by

$$d\tilde{m}^{\mu,\mu'}(u) = e^{\delta \mu u + (\delta, \pi(u))} e^{\delta \mu' u - (\delta, \pi(u))} d\mu_u(u^+) d\mu'_u(u^-) ds,$$

where $o$ is any point in $\mathbb{H}^3$, $u \in T^1(\mathbb{H}^3)$, $u^-, u^+ \in \partial\mathbb{H}^3$ are the starting and ending points of the geodesic ray containing $u$, and $s = \beta_u(o, \pi(u))$. It is an exercise to check that

i) The definition of $\tilde{m}^{\mu,\mu'}$ is independent of the chosen base point $o$.

ii) The measure $\tilde{m}^{\mu,\mu'}$ is left $\Gamma$-invariant.

We can lift the measure $m^{\mu,\mu'}$ to a unique right $M$-invariant measure on $G$ satisfying: for any $f \in C_c(T^1(\mathbb{H}^3))$, define $\tilde{f} \in C_c(G)$ as

$$\tilde{f}(g) = f(g(X_1)).$$

Then

$$\int_G \tilde{f}(g) dm^{\mu,\mu'}(g) = \int_{T^1(\mathbb{H}^3)} f(u) d\tilde{m}^{\mu,\mu'}(u).$$

We can view $G$ as a circle bundle over $T^1(\mathbb{H}^3)$, and the right action of $M$ on $G$ preserves fibers. From the right $M$-invariance of $m^{\mu,\mu'}$, we have

$$dm^{\mu,\mu'} = d\tilde{m}^{\mu,\mu'} \cdot dm_M^{\text{Haar}}.$$

By the $\Gamma$-invariance, the measures $\tilde{m}^{\mu,\mu'}$, $m^{\mu,\mu'}$ naturally descend to measures on $\Gamma \backslash T^1(\mathbb{H}^3)$, $\Gamma \backslash G$, for which we keep the same notation. For a left $\Gamma$-invariant function $F$ on $\Gamma \backslash G$, we denote the integral $\int_{\Gamma \backslash G} F(g) dm^{\mu,\mu'}(g)$ by $m^{\mu,\mu'}(F)$.

We choose the base point $o = j$. The following two quasi-product measures will appear in our analysis:

1. $\mu = m_j, \mu' = \nu_3$: we denote the measure on $T^1(\mathbb{H}^3)$ by $\tilde{m}^{\text{BR}}$, and the measure on $G$ by $m^{\text{BR}}$. These measures are called the Burger-Roblin measures.

2. $\mu = \nu_3, \mu' = \nu_3$: we denote the measure on $T^1(\mathbb{H}^3)$ by $\tilde{m}^{\text{BMS}}$, and the measure on $G$ by $m^{\text{BMS}}$. These measures are called the Bowen-Margulis-Sullivan measures.

We point out a few useful properties of these quasi-product measures.

The Burger-Roblin measures and the Bowen-Margulis-Sullivan measures are locally finite and regular Borel measures, which vanish on a countable union of submanifolds of $T^1(\mathbb{H}^3)$ or $G$ of codimension $\geq 1$ (for instance, algebraic subvarieties of $G$ of codimension $\geq 1$). This is because locally, the Burger-Roblin measures and the
Bowen-Margulis-Sullivan measures are products of measures \((m_j, \nu_j, m^\text{Harr}_j, m^\text{Harr}_M)\), each of which is locally finite, regular and vanishes on submanifolds of codimension \(\geq 1\) of its corresponding measure space.

Finally, we have \(0 < m^{\text{BMS}}(\Gamma \backslash G) < \infty\), which follows from the geometrically finiteness of \(\Gamma\) (see Page 270 of [32]).

4.3. **Computation of \(m^{\text{BR}}\) in the generalized Iwasawa Coordinates.** The purpose of this section is to compute \(m^{\text{BR}}\) in the \(\text{HAN}\) coordinates (Proposition 4.3). We further write \(H\) into its Cartan decomposition (2.2). This decomposition provides an explicit fibration of \(H\) over \(H_1\), with the first two factors \(M \times \left(\tilde{A}^+ \cup \tilde{A}^+ \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \right)\) of (2.2) parametrize \(H_1 = H/M\) except for two points \(M, \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) M\). For this reason and for simplicity we abuse notation, writing

\[
H_1 = M \times \left(\tilde{A}^+ \cup \tilde{A}^+ \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \right),
\]

ignoring the two points \(M\) and \(\left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) M\).

We first observe that the product map \(\rho_2: \)

\[
L_2 := M \times \left(\tilde{A}^+ \cup \tilde{A}^+ \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \right) \times A \times N \to T^1(\mathbb{H}^3),
\]

\[
\rho_2(m, \tilde{a}, m, n) := m\tilde{a}m(X_1)
\]

embeds \(L_2\) into an Zariski-open subset of \(T^1(\mathbb{H}^3)\), by a consideration similar to an earlier one for the \(\text{NAH}\) decomposition below (2.3). Under the \(H_1\text{AN}\) coordinates of \(T^1(\mathbb{H}^3)\), we can compute

\[
\tilde{m}^{\text{BR}}(h_1 a_{t_1} n_z(X_1)) = e^{2\beta_{h_1 a_{t_1} n_z}}(\mathbb{H}^3) e^{\delta_{h_1 a_{t_1} n_z}}(\mathbb{H}^3) dm_1(h_1 a_{t_1} n_z X_1^+) d\nu_1(h_1 a_{t_1} n_z X_1^-) dt
eq e^{\delta_{h_1 a_{t_1} n_z}}(\mathbb{H}^3) e^{\delta_{h_1 a_{t_1} n_z}}(\mathbb{H}^3) dm_1(h_1 a_{t_1} n_z 0) d\nu_1(h_1 \infty) dt,
\]

where \(t = \beta_{h_1 \infty}(j, h_1 a_{t_1} n_z j)\).

Applying Lemma 4.1 to \(t = \beta_{h_1 \infty}(j, h_1 a_{t_1} n_z j)\), we obtain

\[
t = \beta_{h_1 \infty}(j, h_1 a_{t_1} n_z j) = \beta_{h_1}(h_1^{-1} j, a_{t_1} n_z j) = \beta_{h_1}(h_1^{-1} j, j, a_{t_1} n_z j) = -l \log(h_1^{-1} j) - t_1.
\]

Combining (4.9) and (4.10), we obtain

\[
\tilde{m}^{\text{BR}}(h_1 a_{t_1} n_z(X_1)) = \frac{1}{\left(\delta(h_1^{-1} j)^\beta \right) e^{2\beta_{h_1 a_{t_1} n_z}}(\mathbb{H}^3) e^{\delta_{h_1 a_{t_1} n_z}}(\mathbb{H}^3)} e^{\delta t_1} dm_1(h_1 a_{t_1} n_z 0) d\nu_1(h_1 \infty) dt_1.
\]
Lemma 4.2. For any $g \in G$, consider the measure $\lambda_g$ on $N$ given by
\[ d\lambda_g(n_z) = e^{2\beta_{n_z}(g)n_z} dm_1(g n_z). \] (4.12)
Then $\lambda_g = \lambda_e$ and $\lambda_e$ is a Haar measure on $N$.

Proof. By the $G$-invariance of $\{m_x\}$,
\[ dm_1(g n_z) = dm_{g^{-1}}(n_z) = e^{2\beta_{n_z}(g^{-1})} dm_1(n_z), \] (4.13)
Therefore,
\[ d\lambda_g(n_z) = e^{2\beta_{n_z}(g^{-1})} \cdot e^{2\beta_{n_z}(g)} dm_1(n_z) = d\lambda_e(n_z). \] (4.14)
Combining (4.12), (4.14) and (4.1), we have
\[ d\lambda_e(n_z) = d\lambda_e(e) = dz, \] so $\lambda_e$ is a Haar measure on $N$. □

Recall the definition (4.5) for $\mu_{PS}^{H_1}$. We can use Lemma 4.1 to compute
\[ d\mu_{PS}^{H_1}(h_1) = \frac{1}{\Im(h_1^{-1})^d} d\nu(h_1 \infty) \] (4.15)
Collecting (4.11), Lemma 4.2 and (4.15), we obtain
\[ d\tilde{m}_{BR}(h_1 a_t n_z) = e^{-\delta t} d\mu_{PS}^{H_1}(h_1) dz dt. \]
Therefore, in the $H_1ANM$ decomposition for $G$, for any $h_1 \in H_1, a_t \in A, n_z \in N, m \in M$, we have
\[ dm_{BR}(h_1 a_t n_z m) = e^{-\delta t} d\mu_{H_1}^{PS}(h_1) dz dt dm_{HAAR}(m), \]
by the right $M$-invariance of $m_{BR}$.

The decompositions $H_1ANM$ and $H_1MAN$ are related as follows: If $h_1 a_t n_z m = h'_1 m'a_t' n_{z'}$, then $h'_1 = h_1, t'_1 = t_1, m' = m, z' = m^{-1} z$. Therefore, in the $H_1MAN$ decomposition, let $h_1 \in H_1, m \in M, a_t \in A, n_z \in N$, then the Burger-Roblin measure $m_{BR}$ is given by
\[ dm_{BR}(h_1 a_t n_z) = e^{-\delta t} d\mu_{H_1}^{PS}(h_1) dz dt dm. \]
Write $h = h_1 m$. Since
\[ d\mu_{H}^{PS}(h_1 m) = d\mu_{H_1}^{PS}(h_1) \cdot dm, \]
we obtain

Proposition 4.3. In the $HAN$ decomposition, let $h \in H, a_t \in A, n_z \in N$. Then the Burger-Roblin measure $m_{BR}$ is given by
\[ dm_{BR}(h a_t n_z) = e^{-\delta t} d\mu_{H}^{PS}(h) dz dt. \]
5. Equidistribution of Expanding Horospheres

The purpose of this section is to prove Theorem 1.7, which is an extension of Theorem 1.6.

Let \( \mathcal{W} \) be the set of pairs \((f, \Psi)\) satisfying:
\[
\mathcal{W} := \left\{ (f, \Psi) : f \in L^1(\Gamma \backslash \Gamma N, \mu_{N}^{PS}), \Psi \in L^1(\Gamma \backslash G, m^{BR}), \lim_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma \backslash \Gamma N} \Psi(na_t)f(n)d\mu_{N}^{Leb}(n) = \frac{m^{BR}(\Psi)\mu_{PS}^{N}(f)}{m^{BMS}(\Gamma \backslash G)} \right\}. \tag{5.1}
\]

We first observe that \( \mathcal{W} \) inherit some linear structure:
(i) If \((f, \Psi) \in \mathcal{W}\), then for any \(\alpha_1, \alpha_2 \in \mathbb{C}\), \((\alpha_1 f, \alpha_2 \Psi) \in \mathcal{W}\).
(ii) If \((f_1, \Psi), (f_2, \Psi) \in \mathcal{W}\), then \((f_1 + f_2, \Psi) \in \mathcal{W}\).
(iii) If \((f, \Psi_1), (f, \Psi_2) \in \mathcal{W}\), then \((f, \Psi_1 + \Psi_2) \in \mathcal{W}\).

The smoothness assumption in Theorem 1.6 is for obtaining an effective convergence rate. This is not needed for our purpose here.

By the same method from [25], one can extend Theorem 1.6 to \(\Psi \in C_c(\Gamma \backslash G)\) and \(f \in L^1(\Gamma \backslash \Gamma N, \mu_{N}^{PS})\) with \(\lim_{\epsilon \to 0} \mu_{N}^{PS}(f^{+}_{\epsilon} - f^{-}_{\epsilon}) = 0\), where
\[
f^{+}_{\epsilon}(n_z) := \sup_{|w-z|<\epsilon} f(n_w), \tag{5.2}\]
\[
f^{-}_{\epsilon}(n_z) := \inf_{|w-z|<\epsilon} f(n_w). \tag{5.3}\]

However, this is still not enough for our purpose. We need to extend Theorem 1.6 to cover some nonnegative functions \(f\) and \(\Psi\), with \(\Psi \in L^1(\Gamma \backslash G, m^{BR})\) and non-compactly supported. Indeed, in the lattice case, the measure \(m^{BMS}\) on \(\Gamma \backslash G\) is just the Haar measure, and Shah obtained Theorem 1.6 for any \(f \in L^1(\Gamma \backslash \Gamma N, m_{N}^{\text{Haar}})\) and \(\Psi \in L^1(\Gamma \backslash G, m_{G}^{\text{Haar}})\) [30]. However, it seems that removing the compactly supported assumption for \(\Psi\) in the infinite co-volume situation is a much more delicate issue. In fact in the works [20], [26], [23], which deal with the infinite co-volume situation, the compactly supported assumption seems crucially used in proving the equidistribution theorems of expanding horospheres. To see one subtlety here, compared to the lattice case, in the statement of Theorem 1.6, we have an extra factor \(e^{(2-\delta)t}\), which goes to infinity as \(t\) does. We haven’t been able to fully extend Theorem 1.6 to cover \(\Psi \in L^1(\Gamma \backslash G, m^{BR})\), and we circumvent this difficulty by observing some hierarchy structure in the set \(\mathcal{W}\) (Proposition 5.2), which is enough for our purpose.

In Section 5.1 we prove the membership of certain pairs in \(\mathcal{W}\) using Theorem 1.1, in Section 5.2 we prove some hierarchy structure in \(\mathcal{W}\), and in Section 5.3 we finish the proof of Theorem 1.7.

5.1. Membership of certain pairs in \(\mathcal{W}\). Let \(E \subset \mathbb{C}\) be an open set with \(\partial E\) empty or piecewise smooth, and let \(\Omega \subset \mathbb{C}\) be a bounded open set with \(\partial \Omega\) piecewise smooth. First we claim that \((f_0, \Psi_0) \in \mathcal{W}\), where
\[
f_0(n_z) := \chi_E(z), \tag{5.4}\]
\[
\Psi_0(g) := \sum_{\gamma \in \Gamma / \Gamma H} 1\{q(g^{-1}\gamma) \in \Omega^*\}, \tag{5.5}\]
recalling that $\Omega^*$ is the infinite chimney based at $\Omega$ (see (1.5)). We will see shortly that the pair $(f_0, \Psi_0)$ is related to counting circles in $E$.

We first calculate the right hand side of (3.6) with $f = f_0$ and $\Psi = \Psi_0$.

Write $g = h_1ma_tn_z$ in the $H_1MAN$ coordinate. From Proposition 4.3,

$$m^{BR}(\Psi_0) = \int_{g \in \Gamma \setminus G} \sum_{\gamma \in \Gamma \setminus \Gamma} 1\{q((\gamma g)^{-1}) \in \Omega^*\} dm^{BR}(g) \quad (5.6)$$

Next, we have $\mu^N_\Psi(f_0) = w(E)$, recalling that the measure $w$ on $\mathbb{C}$ is the pull back measure of $\mu^N_\Psi$ under the map $z \to n_z$. We also have $m^{BMS}(\Gamma \setminus G) = \bar{m}^{BMS}(\Gamma \setminus H^3)$ and $\mu^H_\Psi(\Gamma \setminus H) = \mu^H_{BMS}(\Gamma \setminus H_1)$. Therefore,

$$m^{BR}(\Psi_0)\mu^N_\Psi(f_0) = \frac{\text{Area}(\Omega)\mu^H_{BMS}(\Gamma \setminus H_1)w(E)}{\delta \cdot \bar{m}^{BMS}(\Gamma \setminus H^3)} \quad (5.7)$$

We now turn to the left hand side of (3.6). Recall that $\Gamma \setminus \Gamma N = N$ as $\Gamma \cap N = \{e\}$. We have

$$e^{(2-\delta)t} \int_N f_0(n_z) \Psi_0(n_z a_t) dz$$

$$= e^{(2-\delta)t} \int_N \chi_E(z) \sum_{\gamma \in \Gamma / \Gamma} 1\{q(a_{-t}n_{-z}\gamma) \in \Omega^*\} dz$$

$$= e^{(2-\delta)t} \int_N \chi_E(z) \sum_{\gamma \in \Gamma / \Gamma} 1\{z \gamma - z \in e^{-t}\Omega; r_\gamma > e^{-t}\} dz \quad (5.8)$$

where we wrote $q(\gamma) = z_\gamma + r_\gamma j$.

Let $N(E, t) := \#C_t \cap E$, and denote the diameter of $\Omega$ by $D(\Omega)$. For any $\epsilon > 0$, we let

$$E_{e^+:} = \{x \in C : d(x, E) < \epsilon\},$$

$$E_{e^-} := \{x \in E : d(x, \partial E) > \epsilon\}.$$

We have

$$e^{-\delta t} \text{Area}(\Omega) \cdot N(E_{e^{-tD(\Omega)+}}, t) \leq e^{-\delta t} \text{Area}(\Omega) \cdot N(E_{e^{-tD(\Omega)-}}, t) \leq e^{-\delta t} \text{Area}(\Omega) \cdot N(E_{e^{-tD(\Omega)^+}}, t) \quad (5.9)$$

The quantity $N(\ast, t)$ can be estimated via the following more detailed version of Theorem 1.1:
Theorem 5.1 (Oh-Shah, Theorem 1.6, [25]). Let $\mathcal{P}$ be a bounded Apollonian circle packing. Let $E \subseteq \mathbb{C}$ be an open set with no boundary or piecewise smooth boundary. Then

$$\lim_{t \to \infty} \frac{N(E, t)}{e^{\delta t}} = \frac{\mu_{H_1}^P(G \setminus H_1)w(E)}{\delta \cdot m_{BMS}(G \setminus \mathbb{H}^3)}.$$ 

So comparing Theorem 1.1 and Theorem 5.1, we can see $v(E)$ and $w(E)$ are off by a constant factor:

$$v(E) = \frac{\mu_{H_1}^P(G \setminus H_1)}{\delta \cdot m_{BMS}(G \setminus \mathbb{H}^3)}w(E).$$

Applying Theorem 5.1 to (5.9) with $E$ replaced by $E_{\epsilon \pm}$, we have

$$\lim_{t \to \infty} \frac{N(E_{\epsilon \pm}, t)}{e^{\delta t}} = \frac{\mu_{H_1}^P(G \setminus H_1)w(E_{\epsilon \pm})}{\delta \cdot m_{BMS}(G \setminus \mathbb{H}^3)}. \tag{5.10}$$

Noting that $\lim_{\epsilon \to 0} w(E_{\epsilon \pm}) = 0$ as $E$ is piecewise smooth, and letting $t$ goes to infinity for (5.9), we obtain

$$\lim_{t \to \infty} e^{(2-\delta)t} \int_N f(0(n_z)\Psi_0(n_z)\alpha_z)dz = \frac{\text{Area}(\Omega)}{\mu_{H_1}^P(G \setminus H_1)}w(E), \tag{5.11}$$

which agrees with (5.7).

5.2. The hierarchy structure in $\mathcal{W}$. For any $\Psi \in L^1(G \setminus E)$, let $\text{Supp}(\Psi)$ be the support of $\Psi$ and $\text{Disc}(\Psi)$ be the set of discontinuities of $\Psi$. We aim to prove the following proposition.

Proposition 5.2. Suppose $f \in L^1(G \setminus G, \mu_{H_1}^P)$, nonnegative, and $\lim_{t \to 0} \mu_{H_1}^P(f_{\epsilon \pm} - f_{\epsilon -}) = 0$. Suppose $\Psi \in L^1(G \setminus G, m_{BR})$, nonnegative, $\|\Psi\|_{L^1} < \infty$, and $m_{BR}(\text{Disc}(\Psi)) = 0$. If $(f, \Psi) \in \mathcal{W}$, then for any Borel measurable function $\tilde{\Psi}$ with $0 \leq \tilde{\Psi} \leq \Psi$ and $m_{BR}(\text{Disc}(\tilde{\Psi})) = 0$, we have $(f, \tilde{\Psi}) \in \mathcal{W}$.

Proof. First we prove the following claim.

Claim: for an $\epsilon > 0$, there exits $\Psi_{\epsilon} \in C_c(G \setminus G, m_{BR})$ such that $0 \leq \Psi_{\epsilon} \leq \Psi$, $m_{BR}(\Psi - \Psi_{\epsilon}) < \epsilon$ and $\Psi_{\epsilon}$ is supported away from the discontinuities of $\Psi$ and $\tilde{\Psi}$.

Since $G \setminus G$ is second countable and $m_{BR}$ is a regular Borel measure on $G \setminus G$, we can find a compact set $K_{\epsilon} \subseteq G \setminus G$ such that

$$\int_{G \setminus G - K_{\epsilon}} \Psi(g)dm_{BR}(g) < \epsilon/2.$$ 

We also choose a relatively compact open set $V_{\epsilon} \subseteq G \setminus G$ such that $K_{\epsilon} \subseteq V_{\epsilon}$.

Since $m_{BR}$ is a regular Borel measure on $G \setminus G$ and

$$m_{BR}(\text{Disc}(\Psi) \cup \text{Disc}(\tilde{\Psi})) = 0,$$

we can find two open sets $U_{\epsilon}, U'_{\epsilon} \subseteq G \setminus G$ such that

$$\overline{\text{Disc}(\Psi) \cup \text{Disc}(\tilde{\Psi})} \subseteq U_{\epsilon} \subset \overline{U_{\epsilon}} \subset U'_{\epsilon}.$$
and
\[ m^{BR}(U') < \frac{\epsilon}{2 \max\{1, \|\Psi\|_{L^\infty}\}}. \]

From the Tietze Extension Theorem, there exists a function \( \Phi_\epsilon \subset C(\Gamma \setminus G) \) such that
\[ 0 \leq \Phi_\epsilon \leq 1, \quad \Phi_\epsilon \equiv 1 \text{ on } K_\epsilon - U' \text{ and } \Phi_\epsilon \equiv 0 \text{ on } \overline{U_\epsilon} \cup (\Gamma \setminus G - V_\epsilon). \]

Now set \( \Psi_\epsilon = \Psi \cdot \Phi_\epsilon \), then we can see that \( \Psi_\epsilon \) is compactly supported as \( \Phi_\epsilon \) is, \( \Psi_\epsilon \) is continuous as \( \text{Supp}(\Psi_\epsilon) \cap \text{Disc}(\Psi) = \emptyset \), and \( 0 \leq \Psi_\epsilon \leq \Psi \). Therefore,
\[
\int_{\Gamma \setminus G} \Psi(g) - \Psi_\epsilon(g) dm^{BR}(g) \leq \int_{\Gamma \setminus G - K_\epsilon} \Psi(g) - \Psi_\epsilon(g) dm^{BR}(g) + \int_{U_\epsilon'} \Psi(g) - \Psi_\epsilon(g) dm^{BR}(g) \\
< \epsilon/2 + \epsilon/2 = \epsilon,
\]
finishing the proof of the claim.

Next, according to the comment around (5.2), for each \( \epsilon \), \( (f, \Psi_\epsilon) \in \mathcal{W} \). Therefore, \( (f, \Psi - \Psi_\epsilon) \in \mathcal{W} \), so that
\[
\lim_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma \setminus \Gamma N} f(n)(\Psi - \Psi_\epsilon)(n z_a t) d\mu^\text{Leb}_N(n) \leq \frac{\epsilon \cdot \mu^\text{PS}_N(f)}{m^{BMS}(\Gamma \setminus G)}. \]

Define \( \bar{\Psi}_\epsilon(g) := \min\{\Psi_\epsilon(g), \bar{\Psi}(g)\} \). We have \( \bar{\Psi}_\epsilon \in C_c(\Gamma \setminus G) \), so that \( (f, \bar{\Psi}_\epsilon) \in \mathcal{W} \), or
\[
\lim_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma \setminus \Gamma N} f(n)\bar{\Psi}_\epsilon(n z_a t) d\mu^\text{Leb}_N(n) = \frac{m^{BR}(\bar{\Psi}_\epsilon)\mu^\text{PS}_N(f)}{m^{BMS}(\Gamma \setminus G)}. \quad (5.13)
\]

We also have
\[
\int_{\Gamma \setminus G} (\bar{\Psi}(g) - \bar{\Psi}_\epsilon(g)) dm^{BR}(g) \leq \int_{\Gamma \setminus G} (\Psi(g) - \Psi_\epsilon(g)) dm^{BR}(g) < \epsilon, \quad (5.14)
\]
and
\[
\limsup_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma \setminus \Gamma N} f(n)(\bar{\Psi} - \bar{\Psi}_\epsilon)(n z_a t) d\mu^\text{Leb}_N(n) \\
\leq \lim_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma \setminus \Gamma N} f(n)(\Psi - \Psi_\epsilon)(n z_a t) d\mu^\text{Leb}_N(n) \\
\leq \frac{\epsilon \cdot \mu^\text{PS}_N(f)}{m^{BMS}(\Gamma \setminus G)}. \quad (5.15)
\]

Combining (5.14), (5.15) and (5.13), and letting \( \epsilon \to 0 \), we obtain
\[
\lim_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma \setminus \Gamma N} f(n)\bar{\Psi}(n z_a t) d\mu^\text{Leb}_N(n) = \frac{m^{BR}(\bar{\Psi})\mu^\text{PS}_N(f)}{m^{BMS}(\Gamma \setminus G)}, \quad (5.16)
\]
so that \( (f, \bar{\Psi}) \in \mathcal{W} \).
5.3. Finishing the proof of Theorem 1.7. We begin with an elementary geometric observation, which implies that any pair of points from $C_t$ can not get too close.

**Observation:** For any two non-intersecting hemispheres based on $C$, the Euclidean distance of their apices $q_1, q_2$ satisfies

$$|\Re q_1 - \Re q_2| \geq 3q_1 + 3q_2. \tag{5.17}$$

And from the hyperbolic distance formula,

$$d(q_1, q_2) = \text{Arccosh} \left(1 + \frac{|q_1 - q_2|^2}{2\Im q_1 \Im q_2}\right) \geq \text{Arccosh} \left(1 + \frac{(3q_1 + 3q_2)^2}{2\Im q_1 \Im q_2}\right) \geq \text{Arccosh}(3). \tag{5.18}$$

From the observation (5.17), if $q(g^{-1} \gamma_1), q(g^{-1} \gamma_2) \in \Omega_t^*$ for $\gamma_1 \neq \gamma_2 \in \Gamma/\Gamma_H$, then $|\Re(q_1) - \Re(q_2)| \geq 2$. For each $\gamma \in \Gamma/\Gamma_H$ with $\Im(q(g^{-1} \gamma)) > 1$, place a circle of radius 1 centered at $q(\Re(g^{-1} \gamma))$, then these circles are disjoint. By an elementary packing argument, we have

$$\# q(g^{-1} \Gamma) \cap \Omega_i^* < \frac{\pi(D(\Omega_i) + 1)^2}{\pi} = (D(\Omega_i) + 1)^2. \tag{5.19}$$

The functions we are interested in are $f = \chi_E$ and $\Psi = F_{\Omega,r}, F_{\Omega}^\beta$.

Suppose $r_j$ is a nonzero component of $r$, then we have

$$F_{\Omega,r}(g) = \prod_{1 \leq i \leq k} 1\{\# q(g^{-1} \Gamma/\Gamma_H) \cap \Omega_i^* = r_i\} \leq \#(q(g^{-1} \Gamma/\Gamma_H) \cap \Omega_i^*), \tag{5.20}$$

and

$$F_{\Omega}^\beta(g) = \prod_{1 \leq i \leq k} \#(q(g^{-1} \Gamma/\Gamma_H) \cap \Omega_i^*)^\beta_i = \sum_{r>0} r^\beta F_{\Omega,r}, \tag{5.21}$$

where $r^\beta(g) = \prod_{1 \leq i \leq k} r_i^{\beta_i}$, and $r > 0$ means all components of $r$ are nonnegative, and at least one component of $r$ is positive.

We notice that the right hand side of (5.20) is of the form $\Psi_0$ (see (5.5)), and the rightmost sum in (5.21) is a finite sum because of (5.19). So both $F_{\Omega,r}$ and $F_{\Omega}^\beta$ are dominated by (a finite linear combination of) $\Psi_0$. Therefore, we can apply Proposition 5.2 to $f = \chi_E, \Psi = F_{\Omega,r}, F_{\Omega}^\beta$, once we have verified that $m^{\text{BR}}(\overline{\text{Disc}(F_{\Omega,r})})$, $m^{\text{BR}}(\overline{\text{Disc}(F_{\Omega}^\beta)}) = 0$. It is enough to show $m^{\text{BR}}(\overline{\text{Disc}(F_{\Omega,r})}) = 0$.

Let $\mathcal{M}_{\Omega_t} := \{g \in G : q(g^{-1}) \in \partial \Omega_t^*\}$. Using the NAH decomposition, we can see that $\mathcal{M}_{\Omega_t}$ is a closed submanifold of $G$ of codimension 1, thus $m^{\text{BR}}(\mathcal{M}_{\Omega_t}) = 0$.

Next, we show that

**Lemma 5.3.** The immersion $\mathcal{M}_{\Omega_t} \to \pi_1(\mathcal{M}_{\Omega_t})$ is proper: for each $g \in \mathcal{M}_{\Omega_t}$, there does not exist infinitely many $g_j \in \Gamma_H \setminus \Gamma$, $g_j \in M$, $1 < j < \infty$, such that $\lim_{j \to \infty} g_j = g$.

**Proof.** We argue by contradiction. Suppose there exist infinitely many $g_j \in \Gamma_H \setminus \Gamma$, $g_j \in M$, $1 < j < \infty$, such that $\lim_{j \to \infty} g_j = g$. Since $q$ is continuous, we
have \( \lim_{j \to \infty} q(g_j^{-1} \gamma_j^{-1}) = q(g^{-1}) \). We note that \( q(g_j^{-1} \gamma_j^{-1}) \) are apices from disjoint hemispheres. Let \( L := \{ z + rf \in \mathbb{H}^3 : z \in C(q\mathcal{R}(g^{-1}), 1), r \in \left( \frac{3}{q(g^{-1})}, \infty \right) \} \). Then \( q(g^{-1}) \in L \). But (5.17) implies that there can be at most \( \frac{1 + 3(q(g^{-1}))^2}{3(q(g^{-1})^2)} \) many points in \( L \). Thus we have a contradiction. □

Lemma 5.3 implies that \( \pi_1(M_{\Omega}) = \pi_1(M_{\Omega}') \), so that \( m^{BR}(\pi_1(M_{\Omega}')) = 0 \). Let \( M_{\Omega} := \bigcup_{i=1}^{k} M_{\Omega_i} \). As a finite union of \( M_{\Omega_i} \), \( M_{\Omega} \) is closed in \( G \) and the immersion \( M_{\Omega} \to \Gamma \backslash \Gamma M_{\Omega} \) is proper, so that \( \Gamma \backslash \Gamma M_{\Omega} \) is closed in \( \Gamma \backslash G \) and \( m^{BR}(\Gamma \backslash \Gamma M_{\Omega}) = 0 \).

Our next lemma shows that \( F_{\Omega,r} \) is continuous outside \( \Gamma \backslash \Gamma M_{\Omega} \), and as a corollary, \( m^{BR}(Dist(F_{\Omega,r})), m^{BR}(Dist(F'_{\Omega,r})) \) \( \leq m^{BR}(\Gamma \backslash \Gamma M_{\Omega}) = 0 \), whence we can obtain Theorem 1.7 by applying Proposition 5.2 with \( f = \chi_E, \Psi = \Psi_0, \tilde{\Psi} = F_{\Omega,r}, F'_{\Omega} \).

**Lemma 5.4.** Let \( M_{\Omega} = \bigcup_{i=1}^{k} M_{\Omega_i} \), then the function \( F_{\Omega,r} \) is continuous in \( \Gamma \backslash G \to \Gamma \backslash \Gamma M_{\Omega} \).

**Proof.** Since the immersion \( M_{\Omega} \to \Gamma \backslash \Gamma M_{\Omega} \) is proper, for any \( g \in G \) \( \cap \Gamma M_{\Omega} \), there exists a simply connected open neighborhood \( O_g \subset G \) of \( g \) such that \( O_g \cap \Gamma M_{\Omega} = \emptyset \). We claim that \( F_{\Omega} \) is constant on \( \Gamma \backslash \Gamma O_g \), by showing that for each \( 1 \leq i \leq k \) and each \( \gamma \in \Gamma \backslash \Gamma \), \( 1 \{ q((\gamma g)\gamma^{-1}) \in \Omega_i \} \) is constant in \( O_g \). We argue by contradiction. Suppose \( 1 \{ q((\gamma g)\gamma^{-1}) \in \Omega_i \} \) is not constant in \( O_g \), then there exists \( g_1, g_2 \in O_g \) such that \( q((\gamma g_1)\gamma^{-1}) \in \Omega_i \) and \( q((\gamma g_2)\gamma^{-1}) \notin \Omega_i \). We observe that

\[
\Omega_i \cap G = \Omega_i = \partial \Omega_i \cup \{ \infty \}.
\]

Let \( p : [0, 1] \to O_g \) be a path with \( p(0) = g_1 \) and \( p(1) = g_2 \). Then for some \( s \in (0, 1] \), we have \( q((\gamma p(s))\gamma^{-1}) \in \partial \Omega_i \cup \{ \infty \} \). If \( q((\gamma p(s))\gamma^{-1}) \in \partial \Omega_i \), then \( p(s) \in O_g \cap \Gamma M_{\Omega_i} \), violating \( O_g \cap \Gamma M_{\Omega} = \emptyset \). Thus \( q((\gamma p(s))\gamma^{-1}) = \infty \) and we let

\[
s_0 = \inf \{ s \in (0, 1] : q((\gamma p(s))\gamma^{-1}) = \infty \}.
\]

By the continuity of \( p \) and \( q \), we have \( q((\gamma p(s_0))\gamma^{-1}) = \infty \). By the definition of \( s_0 \), for \( s < s_0 \), we have \( q((\gamma p(s))\gamma^{-1}) \in \Omega_i \). Therefore, as \( \Omega_j \) is bounded, \( q\mathcal{R}((\gamma p(s))\gamma^{-1}) \) is bounded for \( s \in (0, s_0) \). But \( q\mathcal{R}((\gamma p(s_0))\gamma^{-1}) = \infty \), and this is impossible as \( q\mathcal{R} \) is a continuous map. Thus we arrived at a contradiction.

Therefore, for any \( \gamma \), the function \( 1 \{ q((\gamma g)\gamma^{-1}) \in \Omega_i \} \) is constant in \( O_g \) as desired. This implies \( F_{\Omega,r} \) is constant in \( \Gamma \backslash \Gamma O_g \). The lemma is thus proved.

□

6. Pair correlation and nearest neighbor spacing

In this section we deduce Theorem 1.3 (limiting pair correlation) and Theorem 1.4 (limiting nearest neighbor spacing) from Theorem 1.7. We give full detail for the limiting pair correlation; the proof for the limiting nearest neighbor spacing is similar and we give a sketch.
6.1. Pair correlation. The purpose of this section is to prove Theorem 1.3. Let \( E \subset \mathbb{C} \) be an open set with \( E \cap P \neq \emptyset \) and \( \partial E \) empty or piecewisewise smooth. The pair correlation function \( P_{E,t}(\xi) \) on the set \( C_t \) is defined as

\[
P_{E,t}(\xi) = \frac{1}{2\#(C_t \cap E)} \sum_{p,q \in C_t \cap E} 1\{|p-q| < e^{-t}\xi\}. \tag{6.1}
\]

Let \( B_r \) be the disk in \( \mathbb{C} \) centered at 0 with radius \( r \). We analyze the pair correlation function \( P_{E,t} \) via the following mixed 1-moment function \( P_{E,t,\epsilon} \):

\[
P_{E,t,\epsilon}(\xi) := \frac{e^{2t}}{2\pi \epsilon^2 \cdot \#(C_t \cap E)} \int_{\mathbb{C}} \chi_E(z)N_t(B_\epsilon, z)N_t(B_\xi, z)dz - \frac{1}{2}. \tag{6.2}
\]

Here \( \epsilon \) is taken as a small enough positive number, say \( \epsilon < \min\{\frac{1}{10}, \frac{\xi}{10}\} \), and (5.17) implies that \( N_t(B_\epsilon, z) \leq 1, \forall z \in \mathbb{C} \).

The function \( P_{E,t,\epsilon} \) is an approximate to \( P_{E,t} \). Indeed,

\[
\int_{\mathbb{C}} \chi_E(z)N_t(B_\epsilon, z)N_t(B_\xi, z)dz
\]

\[
= \sum_{p \in C_t} \int_{\mathbb{C}} 1\{z \in e^{-t}B_\epsilon + p\}N_t(B_\xi, z)\chi_E(z)dz
\]

\[
\leq \sum_{p \in C_t} e^{-2t} \pi \epsilon^2 N_t(B_{\xi+\epsilon}, p)\chi_{E+}(p)
\]

\[
eq e^{-2t} \pi \epsilon^2 \sum_{p \in C_t} \chi_{E+}(p) + e^{-2t} \pi \epsilon^2 \sum_{p \in C_t} \chi_{E+}(p) \sum_{q \in C_t} 1\{|q-p| < e^{-t}(\xi + \epsilon)\}
\]

\[
\leq e^{-2t} \pi \epsilon^2 \#(C_t \cap E) + e^{-2t} \pi \epsilon^2 \sum_{p \in C_t \cap E} \sum_{q \in C_t} 1\{|q-p| < e^{-t}(\xi + \epsilon)\}. \tag{6.3}
\]

Putting (6.3) back to (6.2), we have

\[
P_{E,t,\epsilon}(\xi) \leq \frac{\#(C_t \cap E)}{2\#(C_t \cap E)} \frac{1}{2} + \frac{\#(C_t \cap E)}{2\#(C_t \cap E)} P_{E,t,\epsilon}(\xi + \epsilon). \tag{6.4}
\]

Similarly, we have

\[
P_{E,t,\epsilon}(\xi) \geq \frac{\#(C_t \cap E)}{2\#(C_t \cap E)} \frac{1}{2} + \frac{\#(C_t \cap E)}{2\#(C_t \cap E)} P_{E,t,\epsilon}(\xi - \epsilon). \tag{6.5}
\]

We can work out from (6.4) and (6.5) that

\[
P_{E,t}(\xi) \leq \frac{\#(C_t \cap E)}{\#(C_t \cap E)} P_{E,t,\epsilon}(\xi) + \frac{\#(C_t \cap E)}{2\#(C_t \cap E)} - \frac{1}{2} \tag{6.6}
\]

and

\[
P_{E,t}(\xi) \geq \frac{\#(C_t \cap E)}{\#(C_t \cap E)} P_{E,t,\epsilon}(\xi) + \frac{\#(C_t \cap E)}{2\#(C_t \cap E)} - \frac{1}{2}. \tag{6.7}
\]
Letting $t \to \infty$ and then $\epsilon \to 0^+$ in (6.6) and (6.7), Theorem 1.3 is proved once we have shown
\[
\lim_{\epsilon \to 0^+} \lim_{t \to \infty} P_{E_{t}, t, \epsilon}(\xi + \epsilon) = \lim_{\epsilon \to 0^+} \lim_{t \to \infty} P_{E_{t}, t, \epsilon}(\xi - \epsilon) = P(\xi) \tag{6.8}
\]
for some continuously differentiable function $P(\xi)$.

Now we analyze the limit of $P_{E_{t}, t, \epsilon}(\xi)$, as $t \to \infty$. From Theorem 5.1, we have
\[
\lim_{t \to \infty} \frac{\#C_t \cap E}{e^{\delta t}} = \frac{\mu_{PS}^{H}(\Gamma_{H} \setminus H)w(E)}{\delta \cdot m_{BMS}(\Gamma \setminus G)}. \tag{6.9}
\]
From Theorem 1.7, we have
\[
\lim_{t \to \infty} e^{(2-\delta)t} \int_{C} \chi_{E}(z)N_{t}(B, x)N_{t}(B_{1}, x)dx = \frac{w(E)}{m_{BMS}(\Gamma \setminus G)} \cdot \int_{\Gamma_{H} \setminus H} \left( \sum_{\gamma = G} 1\{q((\gamma g)^{-1}) \in B_{e}\} \right) \cdot \# \{ \gamma \in \Gamma_{H} \setminus H : q((\gamma g)^{-1}) \in B_{e}^{*}\} dm_{BR}(g)
\]
\[
= \frac{w(E)}{m_{BMS}(\Gamma \setminus G)} \cdot \int_{\Gamma_{H} \setminus H} 1\{q(g^{-1}) \in B_{e}^{*}\} \cdot \# \{ \gamma \in \Gamma_{H} \setminus H : q((\gamma g)^{-1}) \in B_{e}^{*}\} dm_{BR}(g). \tag{6.10}
\]
Writing $g = h_{a}u_{z}$ in the HAN decomposition, from Proposition 4.3, we have
\[
(6.10) = \frac{w(E)}{m_{BMS}(\Gamma \setminus G)} \cdot \int_{\Gamma_{H} \setminus H} \int_{z \in B_{e}} \int_{0}^{\infty} e^{-\delta t} \# \{ \gamma \in \Gamma_{H} \setminus H : q(n_{-}a_{t}h^{-1}g^{-1}) \in B_{e}^{*}\} dt dz dm_{PS}^{H}(h). \tag{6.11}
\]
The conditions $z \in B_{e}$ and $q(n_{-}a_{t}h^{-1}g^{-1}) \in B_{e}^{*}$ imply that $q(a^{-1}h^{-1}g^{-1}) \in B_{e}^{*}$. Therefore, we have
\[
(6.10) \leq \frac{\pi e^{2}w(E)}{m_{BMS}(\Gamma \setminus G)} \cdot \int_{\Gamma_{H} \setminus H} \int_{0}^{\infty} e^{-\delta t} \# \{ \gamma \in \Gamma_{H} \setminus H : q(a^{-1}h^{-1}g^{-1}) \in B_{e}^{*}\} dt dm_{PS}^{H}(h) \tag{6.12}
\]
and similarly,
\[
(6.10) \geq \frac{\pi e^{2}w(E)}{m_{BMS}(\Gamma \setminus G)} \cdot \int_{\Gamma_{H} \setminus H} \int_{0}^{\infty} e^{-\delta t} \# \{ \gamma \in \Gamma_{H} \setminus H : q(a^{-1}h^{-1}g^{-1}) \in B_{e}^{*}\} dt dm_{PS}^{H}(h). \tag{6.13}
\]
Define
\[
P(\xi) := \frac{\delta}{2\mu_{PS}^{H}(\Gamma_{H} \setminus H)} \int_{\Gamma_{H} \setminus H} \int_{0}^{\infty} e^{-\delta t} \# \{ \gamma \in \Gamma_{H} \setminus H : q(a^{-1}h^{-1}g^{-1}) \in B_{e}^{*}\} dt dm_{PS}^{H}(h) - \frac{1}{2}
\]
\[
= \frac{\delta}{2\mu_{PS}^{H}(\Gamma_{H} \setminus H)} \int_{\Gamma_{H} \setminus H} \int_{0}^{\infty} e^{-\delta t} \# \{ \gamma \in \Gamma_{H} \setminus (\Gamma - H) : q(a^{-1}h^{-1}g^{-1}) \in B_{e}^{*}\} dt dm_{PS}^{H}(h). \tag{6.14}
\]
Combining (6.9), (6.10), (6.12), (6.13), we obtain
\[
P(\xi - \epsilon) \leq \liminf_{t \to \infty} P_{E,t,\epsilon}(\xi) \leq \limsup_{t \to \infty} P_{E,t,\epsilon}(\xi) \leq P(\xi + \epsilon). \tag{6.15}
\]
The definition of \(P\) is independent of the set \(E \subset \mathbb{C}\), so (6.15) also holds with \(E\) replaced by \(E_{c\pm}\). Thus the relation (6.8) is established once we have shown \(P\) is continuously differentiable.

First, we observe that \(P(\xi)\) is indeed finite, as \(\mu^\PS_H(\Gamma_H \setminus H)\) is finite and the integrand of (6.14) is bounded: for each fixed \(h\) and \(t\), from (5.19) we have
\[
\# \{ \gamma \in \Gamma_H \setminus \Gamma : q(\gamma^{-1}) \in B^*_h \} \leq (2\xi + 1)^2.
\]
Next, we show that the pair correlation function \(P\) is continuously differentiable.

We observe that if there exists \(t > 0\) such that \(q(\gamma^{-1}) \in B^*_h\), then \(q(\gamma^{-1}) \in C_\xi \cup B^*_h\), where \(C_\xi\) is the cone defined at (1.6).

We thus write \(P(\xi)\) into two parts:
\[
P(\xi) = \frac{\delta}{2\mu^\PS_H(\Gamma_H \setminus H)} \int_{\Gamma_H \setminus H} \sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} \int_0^\infty e^{-\delta t} \mathbf{1}_{\{q(\gamma^{-1}) \in B^*_h\}} dt d\mu^\PS_H(h)
\]
\[
= \frac{1}{2\mu^\PS_H(\Gamma_H \setminus H)} \int_{\Gamma_H \setminus H} \sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} \mathbf{1}_{\{q(\gamma^{-1}) \in B^*_h\}} \left( 1 - \left(\frac{|q(t^\gamma \gamma^{-1})|}{\xi}\right)^\delta \right) + \mathbf{1}_{\{q(\gamma^{-1}) \in C_\xi\}} \left( \Re(q(\gamma^{-1}))^\delta - \left(\frac{|q(\gamma^{-1})|}{\xi}\right)^\delta \right) d\mu^\PS_H(h). \tag{6.16}
\]

To proceed, we need the following lemma:

**Lemma 6.1.** Define
\[
p(h, \xi) = \sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} \mathbf{1}_{\{q(\gamma^{-1}) \in B^*_h\}} + \mathbf{1}_{\{q(\gamma^{-1}) \in C_\xi\}} \Re(q(\gamma^{-1}))^\delta.
\]
Fixing \(\xi\), then \(p(h, \xi)\) is bounded for \(h \in \Gamma_H \setminus H\).

**Proof.** First, from (5.19), we have
\[
\sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} \mathbf{1}_{\{q(\gamma^{-1}) \in B^*_h\}} < (2\xi + 1)^2. \tag{6.17}
\]
Next, let \(C^t_{\xi, t_1, t_2}\) be the truncated cone
\[
C^t_{\xi, t_1, t_2} := \{ z + rj \in \mathbb{H}^3 : \frac{r}{|z|} > \frac{1}{\xi}, t_1 < r \leq t_2 \}.
\]
Recall the definition of \(C_\xi\) at (1.6). An elementary exercise in hyperbolic geometry shows that the 2-neighborhood of \(C_\xi\) (the set of all points in \(\mathbb{H}^3\) having hyperbolic distance < 2 to \(C_\xi\)) is contained in the cone
\[
\tilde{C}_\xi := \left\{ z + rj \in \mathbb{H}^3 : \frac{r}{|z|} > \frac{1}{e^2\xi} : 0 < r \leq e^2 \right\}, \tag{6.18}
\]
and the 2-neighborhood of $C_{\xi}^{t_1,t_2}$ is contained in the truncated cone

$$\tilde{C}_{\xi}^{t_1,t_2} := \left\{ z + rj \in \mathbb{H}^3 : \frac{r}{|z|} > \frac{1}{e^{2\xi}}, \frac{t_1}{e^2} < r \leq t_2 e^2 \right\}.$$  

Therefore, for each $0 < t < 1$,

$$\sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} 1\{q(h^{-1}\gamma^{-1}) \in C_{\xi} \} \Im(q(h^{-1}\gamma^{-1}))^\delta$$

$$= \sum_{n=0}^{\infty} \sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} 1\{q(h^{-1}\gamma^{-1}) \in C_{\xi}^{\frac{1}{2n+1},\frac{1}{2n}} \} \Im(q(h^{-1}\gamma^{-1}))^\delta$$

$$\leq \sum_{n=0}^{\infty} \frac{\text{Vol}(\tilde{C}_{\xi}^{\frac{1}{2n+1},\frac{1}{2n}})}{4\pi} \frac{1}{2^{n\delta}}$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{2^{n\delta}} < \infty,$$

(6.19)

where in (6.19) we used a packing (by hyperbolic balls) argument combined with (5.18), and here $\text{Vol}(\tilde{C}_{\xi}^{\frac{1}{2n+1},\frac{1}{2n}})$ is the hyperbolic volume of $\tilde{C}_{\xi}^{\frac{1}{2n+1},\frac{1}{2n}}$. \(\square\)

Now we show that $P(\xi)$ is differentiable. For small $\epsilon > 0$,

$$\frac{P(\xi + \epsilon) - P(\xi)}{\epsilon} =$$

$$\frac{1}{2\mu_H^{\text{PS}}(\Gamma_H \setminus H)} \int_{h \in \Gamma_H \setminus H} \sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} 1\{q(h^{-1}\gamma^{-1}) \in B_{\xi}^* \cup C_{\xi} \} \cdot \frac{|q_R(h^{-1}\gamma^{-1})|^\delta \left( \frac{1}{\xi^\epsilon} - \frac{1}{(\xi + \epsilon)^\delta} \right)}{\epsilon} d\mu_H^{\text{PS}}(h)$$

$$+ \frac{1}{2\mu_H^{\text{PS}}(\Gamma_H \setminus H)} \int_{h \in \Gamma_H \setminus H} \sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} 1\{q(h^{-1}\gamma^{-1}) \in (B_{\xi+\epsilon}^* - B_{\xi}^*) \} \cdot 1 - \left( \frac{|q_R(h^{-1}\gamma^{-1})|}{\xi+\epsilon} \right)^\delta$$

$$+ 1\{q(h^{-1}\gamma^{-1}) \in (C_{\xi+\epsilon} - C_{\xi}) \} \cdot \frac{\Im(q(h^{-1}\gamma^{-1}))^\delta - \left( \frac{|q_R(h^{-1}\gamma^{-1})|}{\xi+\epsilon} \right)^\delta}{\epsilon} d\mu_H^{\text{PS}}(h)$$

$$= \frac{\delta}{2\mu_H^{\text{PS}}(\Gamma_H \setminus H)} \int_{h \in \Gamma_H \setminus H} \sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} 1\{q(h^{-1}\gamma^{-1}) \in B_{\xi}^* \cup C_{\xi} \} \cdot \frac{|q_R(h^{-1}\gamma^{-1})|^\delta}{\xi^{\delta+1}} (1 + O_\xi(\epsilon)) d\mu_H^{\text{PS}}(h)$$

$$+ O_\xi \left( \int_{h \in \Gamma_H \setminus H} p(h, \xi + \epsilon) - p(h, \xi) d\mu_H^{\text{PS}}(h) \right).$$  

(6.21)

Noting that

$$\sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} 1\{q(h^{-1}\gamma^{-1}) \in B_{\xi}^* \cup C_{\xi} \} \cdot \frac{|q_R(h^{-1}\gamma^{-1})|^\delta}{\xi^{\delta+1}} \ll_\xi p(h, \xi),$$
and letting $\epsilon \to 0^+$, we have

$$
\lim_{\epsilon \to 0^+} \frac{P(\xi + \epsilon) - P(\xi)}{\epsilon} = \frac{\delta}{2\mu^\text{PS}(\Gamma_H \setminus H)} \int_{h \in \Gamma_H \setminus H} \sum_{\gamma \in \Gamma - \Gamma_H \cap \gamma H} \frac{|q_R(h^{-1}\gamma^{-1})|\delta}{\xi^{\delta+1}} d\mu^\text{PS}_H(h),
$$

(6.22)

once we have shown that the term $O(\cdot)$ from (6.21) goes to 0 as $\epsilon \to 0^+$. Indeed, since $p(h, \xi)$ is bounded with respect to $h$ and monotone with respect to $\xi$, by Lebesgue’s Dominated Convergence Theorem,

$$
\lim_{\epsilon \to 0^+} \int_{h \in \Gamma_H \setminus H} p(h, \xi + \epsilon) - p(h, \xi) d\mu^\text{PS}_H(h) = \sum_{\gamma \in \Gamma_H \setminus \Gamma} \int_{h \in \Gamma_H \setminus H} 1\{q(h^{-1}\gamma^{-1}) \in \partial(B^*_\xi \cup C_\xi)\} \cdot \max\{\Im(q(h^{-1}\gamma^{-1}))^\delta, 1\} d\mu^\text{PS}_H(h).
$$

(6.23)

We can check that for each $\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)$, the set $H_{\gamma, \xi} := \{h \in H : q(h^{-1}\gamma^{-1}) \in \partial(B^*_\xi \cup C_\xi)\}$ is contained in an algebraic subvariety of $H$ of codimension 1. Therefore, $\mu^\text{PS}_H(H_{\gamma, \xi}) = 0$, so that (6.23)=0, and (6.22) is established.

By a similar consideration, we can also show

$$
\lim_{\epsilon \to 0^+} \frac{P(\xi) - P(\xi - \epsilon)}{\epsilon} = \frac{\delta}{2\mu^\text{PS}(\Gamma_H \setminus H)} \int_{h \in \Gamma_H \setminus H} \sum_{\gamma \in \Gamma_H \setminus \Gamma \cap \gamma H} \frac{|q_R(h^{-1}\gamma^{-1})|\delta}{\xi^{\delta+1}} d\mu^\text{PS}_H(h).
$$

Therefore, $P$ is differentiable. The continuity of $P'$ follows from that, by the Dominated convergence theorem,

$$
\limsup_{\epsilon \to 0^\pm} |P'(\xi + \epsilon) - P'(\xi)| \leq \sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} \int_{\Gamma_H \setminus H} 1\{q(h^{-1}\gamma^{-1}) \in \partial(B^*_\xi \cup C_\xi)\} d\mu^\text{PS}_H(h) = 0.
$$

Finally, the reason that $P$ is supported away from 0 is due to the elementary observation (5.17). Theorem 1.3 is thus completely proved.

6.2. Nearest Neighbor Spacing. As usual we let $E \subset \mathbb{C}$ be an open set with no boundary or piecewise smooth boundary, and with $E \cap P \neq \emptyset$. For any $p \in C_t$, let $d_t(p) = \min\{|p - q| : q \in C_t, q \neq p\}$. The nearest neighbor spacing function $Q_{E,t}$ is defined by

$$
Q_{E,t}(\xi) = \frac{1}{\#(C_t \cap E)} \sum_{p \in C_t \cap E} 1\{d_t(p) < e^{-t}\xi\}.
$$

(6.24)

We sketch our analysis for $Q_{E,t}$, which is in a very similar fashion as we did for the pair correlation function. The function $Q_{E,t}(\xi)$ can be approximated by the following
function

\[ Q_{E,t,\epsilon}(\xi) := 1 - \frac{e^{2t}}{\pi \epsilon^2 \cdot \# (C_t \cap E)} \int_{\mathbb{C}} \chi_E(z) \mathbf{1} \left\{ (a_{-t} n_z q(\Gamma) \cap B_\epsilon^*) = 1 \right\} \mathbf{1} \left\{ (a_{-t} n_z q(\Gamma) \cap B_\epsilon^*) = 1 \right\} dz. \] (6.25)

Indeed, one can check that

\[ Q_{E,t}(\xi - \epsilon) \leq Q_{E,t,\epsilon}(\xi) \leq Q_{E,t}(\xi + \epsilon). \] (6.26)

Applying Theorem 1.7 to \( Q_{E,t,\epsilon} \) and letting \( \epsilon \to 0^+ \), we obtain Theorem 1.4. The continuity of \( Q \) follows from that, by the Dominated Convergence Theorem,

\[ \limsup_{\epsilon \to 0^+} |Q(\xi + \epsilon) - Q(\xi)| \ll \xi \sum_{\gamma \in \Gamma_H \setminus (\Gamma - \Gamma_H)} \int_{\Gamma_H \setminus H} \int_0^\infty e^{-\delta t} \mathbf{1} \left\{ q(a_{-t} h^{-1} \gamma^{-1}) \in \partial B_\epsilon^* \right\} dt d\mu_H^{PS}(h) = 0. \]

\section*{References}


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