1 Introduction

We consider the problem of decomposing a compactly supported morse distribution \( f : \mathbb{R}^n \rightarrow [0, \infty) \) into the sum of a minimal number of unimodal distributions. We call the minimal number of unimodal summands the unimodal category, denoted by \( \text{ucat}(f) \). We will outline a strategy using a basic topological and constructive approach to determine the unimodal category of such distributions in the case of \( n=2 \). The results described in this report were obtained by close collaboration of the authors.

2 Terminology

Definition 1. A Compactely Supported Morse Distribution (Hereafter referred to as a "Distribution") is a function \( f : \mathbb{R}^n \rightarrow [0, \infty) \) which is Morse and for which there exists a compact set \( K \) containing the support of \( f \).

Definition 2. A Unimodal Distribution (which we will refer to as a "unimodal function") \( f \) is a distribution for which the sets \( \{ x \in \mathbb{R}^n : f(x) \geq c \} \) are contractible for each real number \( c \).

Note that this means Unimodal functions of \( \mathbb{R}^2 \) cannot have disconnected level sets, they can have only a single maximal region and no minima.

Definition 3. A Unimodal Decomposition of a distribution, \( f \), is a set of functions \( f_i \) such that \( \sum_{i=1}^{n} f_i = f \)

Definition 4. The Unimodal Category of a distribution, \( f \), is the minimal number of functions needed to create a unimodal decomposition.
3 Previous Work

The case when \( n = 1 \) has been studied and solved in [1]. In this case the algorithm is quite simple. Choose the left most peak of the graph of the distribution, \( f \). To the left of the peak let the unimodal function take the same values as the initial distribution. To the right of the peak, let the unimodal function have the same slope as the distribution when the derivative is negative and have zero slope otherwise. This unimodal function gives you the first summand \( f_1 \). You then repeat this process on \( f - f_1 \) to determine \( f_2 \), and so on. More detail and a proof of the process can be found in [1]. An illustration of an example can be found in figure 1.

\[ f = f_1 + f_2 \] and \( \text{ucat}(f) = 2 \)

Figure 1

4 Main Results

4.1 Some Definitions and Observations

Definition 5. A Morse – Smale Graph of a distribution \( f \) is a weighted graph embedded in \( \mathbb{R}^2 \) which has vertices at the local maxima of \( f \) and such that a vertex \( M \) has weight \( f(M) = m \). Edges are formed by drawing a path through each saddle point \( S \) from one maxima to another such that the path from each maxima to \( S \) is decreasing. Crossing edges is not allowed. An edge associated with a saddle point \( S \) is similarly given a weight \( f(S) \).

Note that the level sets containing all of the maxima and saddles can be reconstructed from a Morse-Smale Graph. Also note that the Morse-Smale
Graph of a function $f$ is not unique (figure 2).

By Convention we will use $M_i \ (i = 1, \ldots, |V(G)|)$ to denote the vertices of a Morse-Smale Graph (and consequently also the Maxima of $f$ since the vertices of the graph are identified with corresponding maxima in $\mathbb{R}^2$).

**Definition 6.** *The Path Value from $M_i$ to $M_j$, denoted $PV(M_i, M_j)$ is the value of the least weighted edge on a path from $M_i$ to $M_j$. If there are multiple paths, select the one with the greatest such value.*

**Lemma 4.1.** Suppose $M_1, M_2, M_3$ are vertices in the Morse-Smale graph of a function $f$ and the edge associated with the saddle $S$ connects $M_1$ and $M_2$. Then there exists another Morse-Smale Graph of $f$ in which the vertex associated with $S$ instead connects $M_1$ and $M_3$ where the graphs are otherwise identical, if and only if the weight of $S$ is less than or equal to $PV(M_2, M_3)$.

**Proof.** \( \Rightarrow \) If $PV(M_2, M_3) < s$ then any path from $S$ to $M_3$ would have to pass through a level curve less than the weight of $S$ and therefore the path would not be increasing.

\( \Leftarrow \) If $PV(M_2, M_3) \geq s$ then there is no such level curve and an increasing path can be found.

Using this lemma we can easily see that it is possible to turn any graph without cycles into a single linear path. As seen in figure 3.
Corollary 4.2. For any pair of local maxima $M_1$ and $M_2$ of $f$, $PV(M_1, M_2)$ is the same in every Morse-Smale Graph of $f$.

Proof. Clearly, any two Morse-Smale Graphs of $f$ can be transformed into each other via this lemma. Finally, it is easy to check that this process of switching the vertices attached to each edge does not alter path values. \qed

4.2 The Case of No Cycles

We now turn our attention determining the unimodal category of functions with Morse-Smale graphs which are trees. The following proposition is our primary result.

Proposition 4.3. For a Morse distribution $f$, with a Morse-Smale graph, $G$, then if $G$ has no cycles:

1. $\text{ucat}(f) \leq n \Rightarrow \exists P_1, ..., P_n \in V(G)$ not necessarily distinct, s.t. $\forall M_j \in V(G)$, where $M_j \neq P_1, ..., P_n$, $\sum_{i=1}^{n} PV(P_i, M_j) \geq m_j$,

2. $\exists P_1, ..., P_n \in V(G)$ not necessarily distinct, s.t. $\forall M_j \in V(G)$, where $M_j \neq P_1, ..., P_n$, $\sum_{i=1}^{n} PV(P_i, M_j) > m_j \Rightarrow \text{ucat}(f) \leq n$
Proof. For (1), suppose it were not true, then there would exist some function $f$ with a Morse-Smale Graph $G$ such that for any choice of $P_1, ..., P_n \in G$ 
\[\sum_{i=1}^{n} PV(P_i, M_j) < m_j\] but such that $f$ has unimodal category $\leq n$. Now let $\sum_{i=1}^{n} f_i = f$ such that each $f_i$ is unimodal.
Let $R(M_i)$ be the set containing $M_i$ inside of the following boundary. Take any Morse-Smale Graph of $G$ which is a path (We know this exists via our previous corollary). At each saddle point draw decreasing path in each direction until the path intersects $\mathbb{R} \setminus \text{Supp}(f)$. Connect these via curves outside of the support of $f$ (See figure 4).

![Figure 4](image-url)

**Figure 4:** We can see here that a unimodal function with maxima outside of $M_4$ and $M_5$ should be bounded inside the red box by $s_3$

Let $\text{Max}(f_i)$ to be the set of points where $f_i$ is maximal.

Associate with each $f_i$ a vertex $Q_i$, where $R(Q_i)$ intersects $\text{Max}(f_i)$. Choose $M$ not equal to any $Q_i$. We know that we can make such a choice, otherwise we could take $P_i = Q_i$ and note that then these $P_i$ would encompass every vertex and contradict our assumption on path values.

Now we claim that, on $R(M)$, $f_i$ is bounded by $PV(Q_i, M)$. This can be shown by choosing a vertex, on the path from $Q_i$ to $M$ with the value $PV(Q_i, M)$. Now look at the domain defined by the union of all $R(M_i)$ on the same side of the vertex as $M$. Notice that on the boundary of this domain, $f$ is bounded by $PV(Q_i, M)$ and therefore $f_i$ is bounded by $PV(Q_i, M)$.
Then we can conclude that by our assumption:
\[ f(M) = \sum_{i=1}^{n} f_i(M) \leq \sum_{i=1}^{n} PV(Q_i, M) \leq m = f(M), \]

which is a contradiction.

For (2), we will prove via construction. Because it makes no difference to the Morse-Smale Graph of \( f \), we assume that \( f \) is such that \( M_i = (i, 1) \) and \( S_i = (i + 1/2, 1) \) in the x,y-plane, up to re-labeling of the critical values of \( f \). Furthermore, we can assume \( f \) is a piecewise linear function defined by linear transitions between critical values on the plane \( y=1 \) (figure 5). We also assume that our function decays linearly to zero at \( y=0 \) and \( y=2 \) as in the following figure 6.

![Cross section at y = 1](image1)

**Figure 5**

![Cross section at x = i](image2)

**Figure 6**

Now that we understand how our general distribution looks, we can now
define a unimodal function $f_1$ in the following way. Suppose $P_1 = M_k$. Let $t$ be such that $M_k - t = Max(s_k, s_{k-1})$. Then we will define $f_1$ by the cross section in figure 7 at $x = k$.

Now we will use cross-sections to define $f_1$ around each critical point. For our saddles, if $i \geq k$ then we will take $t = PV(M_k, M_{i+1}) - \epsilon$, where $\epsilon$ is small enough that $\sum_{i=1}^{n} PV(P_i, M_j) - n\epsilon > m_j$, and use the cross-section in figure 10 (We use epsilon because after subtracting $f_1$ we wish to preserve the height of the saddles but without sacrificing so much we cannot sum to more than our maxima). If $i < k$ then we instead use $t = PV(M_k, M_i) - \epsilon$. 

Figure 7

Figure 8

7
If \( m_i - PV(M_k, M_i) \leq Max(s_i, s_{i-1}) \) then we use the cross-section in figure 9 for \( M_i \). Where \( h \) is a number such that \( Max(s_i, s_{i-1}) = M_i - h \), taking \( t = PV(M_k, M_i) - \epsilon \).

![Cross section for \( M_i \)](image)

**Figure 9**

If \( m_i - PV(M_k, M_i) > Max(s_i, s_{i-1}) \) then we instead use the following figure 10, using the same \( t \) as before:

![Cross section for \( M_i \)](image)

**Figure 10**

At all of the intermediary points, not accounted for in these cross sections, we define \( f_i \) to transition linearly in the obvious fashion (as we did with \( f \)). This creates a unimodal function with a peak in \( R(M_k) \) and ridges running under each saddle. From each peak, the unimodal function either collapses the peak into a simple decreasing path from the higher saddle to the lower saddle, or subtracts \( PV(M_k, M_i) - \epsilon \) from it. This function is clearly
unimodal since as the cross sections get further from $M_k$ the value of $t$ which is the height of the ridge is decreasing.

We now apply this same method to $f - f_1$ to obtain $f_2$, and so on with $f_3$ until our function, $f$, has been transformed into a unimodal function itself. This will occur since at each iteration peaks are collapsed until there is only one peak remaining. And because of $\sum_{i=1}^{n} PV(P_i, M_j) > m_j$ it should take at most $n - 1$ steps to accomplish this. Therefore, the unimodal category of $f$ is less than or equal to $n$.

We can use this proposition to determine the unimodal category of almost all unimodal functions whose Morse-smale graphs are trees. The sole exceptions follow from observing that (2.1) only gives an upper bound on unimodal category when the path values sum to strictly more than the height of the peaks. It is easy to see, however that when $\sum_{i=1}^{n} PV(P_i, M_j) = m_j$ then $\text{ucat}(f) \leq n + 1$.

4.3 Reeb Graphs

**Definition 7.** The Reeb Graph of a distribution $f$ is a measured graph formed by identifying points lying in the same connected component of a level set. In other words, it is the quotient space defined by this equivalence relation.

We can see that in the case of reeb graphs the presence of cycles in the Morse-Smale graph of a distribution corresponds with the presence of multiple branches of the Reeb graph of the distribution pointing downward (there is always at least one). Also it is simple to check that $PV(M_i, M_j)$ is the same as the lowest point one must travel on a Reeb graph to get from $M_i$ to $M_j$. This gives us a method of determining the unimodal category of a distribution from its Reeb graph, similar to how we do with Morse-Smale graphs as was predicted in [1] (Proposition 12 pg. 4).

4.4 The Trouble With Cycles

Unfortunately, our lemma does not work when the graph of our function contains cycles. For example, consider the Morse-Smale Graph of 12 shown in 11.
This certainly satisfies our proposition’s hypothesis regarding path values for unimodal category 1. However, this is clearly not the case as 12 is certainly not a unimodal function. Other examples of this phenomenon can be found in the below figures as well.
One potentially useful result regarding Morse-Smale graphs with cycles has been made, however.

**Proposition 4.4.** Every distribution $f$ has a Morse-Smale graph which is a single path with loops attached at vertices.

*Proof.* This result is quickly obtained through the use of the Lemma above, which applies to all Morse-Smale graphs. We simply reassign the smallest edge $S$ on each cycle to connect to connect to an adjacent peak in both directions. This can be done because being the smallest edge means that there is a path around this cycle connecting this peak never going below the value of $S$. Repeating this iteravely gives a Morse-Smale graph which is a
tree. We can then linearize this tree in the same manner as we did in our corollary.

This new proposition allows us to reduce the number of graphs we must consider in developing an algorithm for determining the unimodal category of a distribution. However, we were not able to find a general algorithm, even with this simplification.

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References