Math 213 1-Hour Exam, Fall 2012

Nov 1st, 7:00pm – 8:20pm, 101 Armory, UIUC

Name (PRINTED): ________________

UIN: ________________

Rules and Instructions:

• Please put your I-card face-up on your desk.

• Write down your name and UIN on this page and your ID on the top of each page. Remember that do NOT detach any paper from the test book! Do NOT write your answer on any other papers!

• This exam is a Closed Book Exam. Any Books, Notes, or Scratch paper (other than that provided), are NOT permitted. Students may not begin to write anything, before a closed book exam begins.

• Electronic devices (such as calculators, laptops, cell phones) are NOT permitted during the exam.

• You can use the back of the test paper as scratch paper. If you need more, ask the instructor. Do Not use your own paper!

• Write clearly and in order, if you want to get full points. Box your final answer.

• Please be quiet before you hand in the paper and leave the room. Discussion is NOT permitted during the exam and after it before you leave the room. If you have any questions, ask the instructor.

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Score: ________________
1. (4 points) How many different permutations are there of the letters ”Mississippi”?

Solution: There are 11 letters in total in ”Mississippi”, 4 repeated ”s”, 4 repeated ”i”, and 2 ”p”. Thus, the total is

\[ \frac{11!}{4!4!2!} = 34650. \]

(It is not necessary to calculate the exact number in the exam itself.)

2. Answer the following questions.

(a) (3 points) How many bit strings of length \( n \) are there? (Your answer should be a simple function of \( n \).)

Solution: \( 2^n \).

(b) (4 points) How many bit strings of length \( n \) are there which contain exactly \( k \) 0’s? (Your answer should be a simple function of \( k \) and \( n \).)

Solution: \( \binom{n}{k} \).

(c) (5 points) Prove the following identity:

\[ 2^n = \sum_{k=0}^{n} \binom{n}{k}. \]

Solution: The total number of bit strings of length \( n \) can be counted by adding up all the bit strings of length \( n \) which contain exactly \( k \) 0’s, over all possible values of \( k \) (so \( 0 \leq k \leq n \)). Thus by the previous two parts,

\[ 2^n = \sum_{k=0}^{n} \binom{n}{k}. \]

3. (10 points) We know that the word ”enhance” appears in 200 out of 1000 spam messages, and in 30 out of 1000 messages that are not spam. Assuming that a randomly chosen message is just as likely to be spam as not to be spam, what is the probability that a message is spam given that it contains the word ”enhance”?

Solution: Let \( E \) be the event that the word ”enhance” appears in an email message, and \( F \) be the event that an email message is spam. Then by Bayes’ Theorem, the desired
probability is
\[ P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|\bar{F})P(\bar{F})} \]
\[ = \frac{\frac{200}{1000} \frac{1}{2}}{\frac{200}{1000} \frac{1}{2} + \frac{30}{1000} \frac{1}{2}} \]
\[ = \frac{\frac{1}{5} + \frac{3}{100}}{\frac{1}{5} + \frac{3}{100}} \]
\[ = \frac{20}{23}. \]

4. Suppose box A contains 10 apples and 40 oranges and box B contains 18 apples and 12 oranges. We first randomly pick a box and then randomly pick a fruit from that box.

(a) (4 points) What is the probability of the event that we pick an apple?

Solution: \( \frac{10}{20} + \frac{18}{30} = \frac{2}{5}. \)

(b) (5 points) Suppose the fruit we pick is an orange. What is the probability of the event that box A was chosen?

Solution: Let \( F \) be the event that box A is chosen, and \( E \) be the event that the fruit picked is an orange. Then by Bayes' Theorem, the desired probability is

\[ P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|\bar{F})P(\bar{F})} \]
\[ = \frac{\frac{4}{5} \frac{1}{2}}{\frac{4}{5} \frac{1}{2} + \frac{12}{30} \frac{1}{2}} \]
\[ = \frac{\frac{4}{5} + \frac{2}{5}}{\frac{4}{5} + \frac{2}{5}} \]
\[ = \frac{2}{3}. \]

5. Let \( a_n = 5a_{n-1} - 6a_{n-2} + 2^n \) \( \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdOTS
(c) (5 points) Suppose the particular solution has the form $a_n^{(p)} = qn2^n$ where $q$ is a constant. Find $q$.

*Solution:* Substituting the given form into the recurrence relation, we have

$$qn2^n = 5q(n - 1)2^{n-1} - 6q(n - 2)2^{n-2} + 2^n.$$  

Dividing both sides by $2^{n-2}$, we have

$$4qn = 10q(n - 1) - 6q(n - 2) + 4.$$  

Simplifying, we have $0 = 2q + 4$ so $q = -2$.

(d) (2 points) Write down the general solution of the nonhomogeneous recurrence relation $(\star)$.

*Solution:* We add the homogeneous solution to the particular solution to get

$$a_n = a_{(h)}n + \alpha_12^n + \alpha_23^n - 2n2^n.$$  

6. Let $a_n = \sum_{k=1}^{n} (k + 1)$.

(a) (4 points) Find a recurrence relation for $a_n$.

*Solution:* $a_n = a_{n-1} + n + 1$ for $n \geq 2$.

(b) (6 points) Solve the recurrence relation in part a.

*Solution:* Note we have $a_1 = 2$. The homogenous part is $a_n = a_{n-1}$ which has characteristic equation $r = 1$, and so $a_n^{(h)} = \alpha$ for some constant $\alpha$.

The nonhomogeneous part is $(n + 1) \times 1^n$ and since $1$ is a root of the characteristic equation of multiplicity one, the form of the particular solution is $a_n^{(p)} = n(p_1n + p_0)$. Substituting this into the recurrence gives

$$p_1n^2 + p_0n = p_1(n - 1)^2 + p_0(n - 1) + n + 1$$  

$$= p_1n^2 - 2p_1n + p_1 + p_0n - p_0 + n + 1.$$  

This gives

$$0 = -2p_1n + p_1 + n - p_0 + 1 = (1 - 2p_1)n + p_1 - p_0 + 1.$$  

Thus $1 - 2p_1 = 0$ which gives $p_1 = 1/2$, and $p_1 - p_0 + 1 = 0$, which gives $p_0 = 3/2$.

The general solution is

$$a_n = \alpha + n(1/2n + 3/2).$$  

Since $a_1 = 2$, we get $\alpha = 0$, so we have

$$a_n = \frac{n(n + 3)}{2}.$$
7. (10 points) How many solutions does the equation \( x_1 + x_2 + x_3 = 5 \) have, where \( x_1, x_2, x_3 \) are nonnegative integers and \( x_1 \leq 3, x_2 \leq 2, x_3 \leq 1 \)?

\textit{Solution 1:} If \( x_1 = 3 \), then there are two solutions to \( x_2 + x_3 = 2 \) with the above constraints, namely \( x_2 = 1 = x_3 \) or \( x_2 = 2 \) and \( x_3 = 0 \). It is impossible to have \( x_2 = 0 \) since \( x_3 \leq 1 < 2 \).

On the other hand, if \( x_1 = 2 \), then there is only one solution to \( x_2 + x_3 = 3 \), namely \( x_2 = 2 \) and \( x_3 = 1 \). Here, no smaller values of \( x_2 \) or \( x_3 \) can be a solution.

Note that if \( x_1 < 2 \), then \( x_1 + x_2 + x_3 < 2 + 2 + 1 = 5 \) is never equal to 5. Thus the total number of solutions is 3.

\textit{Solution 2:} Let \( P_1, P_2, \) and \( P_3 \) be the properties that \( x_1 \geq 4, x_2 \geq 3 \) and \( x_3 \geq 2 \) respectively. Then by the alternate form of Inclusion-Exclusion, the quantity we want is

\[
N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)
\]

\[
= \binom{7}{2} - \binom{3}{2} - \binom{4}{2} - \binom{5}{2} + 0 + 0 + \binom{2}{2} - 0
\]

\[
= \binom{7}{2} - \binom{3}{2} - \binom{4}{2} - \binom{5}{2} + \binom{2}{2}
\]

\[
= 3.
\]

8. (a) (4 points) How many derangements of 12345 are there?

\textit{Solution:}

\[
D_5 = 5!(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}) = 60 - 20 + 5 - 1 = 44.
\]

(b) (4 points) How many permutations of 12345 are there which start with 1 and end with 5, but the positions of 2, 3 and 4 are changed?

\textit{Solution:}

\[
D_3 = 3!(1 - 1 + 1/2 - 1/6) = 3 - 1 = 2.
\]

(c) (5 points) How many permutations of 12345 are there which changes the positions of 3, 4, and 5, with no condition on the positions of 1 and 2? (For instance, 13452 is such a permutation.)

\textit{Solution:} If the positions of 1 and 2 remain unchanged, we get \( D_3 = 2 \) permutations. If the position of 1 is unchanged but the position of 2 changes, we have \( D_4 = \)}
4!(1 − 1 + 1/2 − 1/6 + 1/24) = 12 − 4 + 1 = 9 permutations, and similarly if the position of 1 changes but the position of 2 remains unchanged. If all the positions change, then we have $D_5 = 44$ permutation. In total, we have

$$D_5 + 2D_4 + D_3 = 44 + 18 + 2 = 64$$

permutations.

9. (10 points) **Bonus:** There are two boxes filled with red and blue marbles. Alice picks a marble by first picking a box at random and then picking a marble from that box at random. When Alice told Bob that she had picked a red marble, Bob deduced that she picked the second box with probability $2/3$. Prove that at most half of the marbles in the first box are red.

**Solution:** Let $p$ and $q$ be the proportion of red marbles in the first and second boxes respectively. Let $E$ and $F$ be the events that Alice picks a red marble and Alice picks the second box respectively. Then by Bayes Theorem

$$2/3 = P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|\bar{F})P(\bar{F})} = \frac{q}{p + q}.$$ 

Thus, $p = \frac{q}{2} \leq 1/2$, as desired.