Math 415 Exam I

Calculators, books and notes are not allowed!

Name: ________________________________

Student ID: __________________________
(20pts) 1. Let $A$ be a square matrix satisfying $A^2 = 2A$. Find the determinant of $A$.

*Sol.* From $A^2 = 2A$, we get

$$(\det A)(\det A) = \det(2A) = 2^n\det A,$$

where $n$ is the number of columns in $A$. Thus $\det A = 0$ or $\det A = 2^n$. 
(20 pts) 2. Let $\mathcal{M}_{3 \times 3}$ be the vector space of all real matrices of size $3 \times 3$, equipped with the matrix addition and scalar multiplication. Let $W$ be the set of all $3 \times 3$ upper triangular real matrices. Prove that $W$ is a subspace of $\mathcal{M}_{3 \times 3}$.

Proof. We only need to prove that $W$ is closed under the matrix addition and scalar multiplication.

First, let
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}$$
be any two matrices in $W$. From the definition of the matrix addition,
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ 0 & a_{22} + b_{22} & a_{23} + b_{23} \\ 0 & 0 & a_{33} + b_{33} \end{pmatrix},$$
which is an upper triangular matrix in $W$.

Second, let $c$ be any real number (a scalar). From the definition of the scalar multiplication,
$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ 0 & ca_{22} & ca_{23} \\ 0 & 0 & ca_{33} \end{pmatrix},$$
which is an upper triangular matrix in $W$.

Therefore $W$ is closed under the matrix addition and scalar multiplication and we prove that $W$ is a subspace of $\mathcal{M}_{3 \times 3}$. \qed
(20pts) 3. Find the LU factorization of \( A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ 0 & 4 & 4 \end{pmatrix} \).

**Sol.** Perform the following elementary row operations for \( A \):
1) Add \((-2)\) row 1 to row 2,
2) Add \((-1)\) row 2 to row 3.

By operations 1) and 2), \( A \) can be reduced to \( U = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} \). Let \( L_1 \) be the elementary matrix associated to operation 1). And Let \( L_2 \) be the elementary matrix associated to operation 2). Then

\[
L_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.
\]

Thus

\[
A = L_1^{-1}L_2^{-1}U = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix},
\]

which yields the \( LU \) factorization of \( A \).
(25pts) 4. Let $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 2 & 3 & -1 \\ 0 & 4 & 3 & -3 \end{pmatrix}$.

a) Find a basis for $\text{Range} A$.

b) Find a basis for $\text{Ker} A$.

c) Find the general solution to the linear system $A\vec{x} = \begin{pmatrix} 0 \\ b \\ b \end{pmatrix}$, where $b$ is a real number.

**Sol.** a) Perform the following elementary row operations for $A$:
1) Add $(-2)\text{row} 1$ to $\text{row} 2$,
2) Add $(-1)\text{row} 2$ to $\text{row} 3$.

By operations 1) and 2), $A$ can be reduced to $U = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since the pivots 1 and 4 are in the first column and the second column respectively, the first column and the second column in $A$ are linearly independent. Therefore a basis of $\text{Range} A$ is $\{(1, 2, 0)^T, (-1, 2, 4)^T\}$.

b) To obtain $\text{Ker} A$, we need to solve the homogeneous linear system $U\vec{x} = \vec{0}$. This homogeneous linear system can be written as

\[
\begin{align*}
0 &= x_1 - x_2 + x_4 \\
4x_2 + 3x_3 - 3x_4 &= 0
\end{align*}
\]

Then we get $x_1 = \frac{-3}{4}x_3 - \frac{1}{4}x_4$ and $x_2 = \frac{-3}{4}x_3 + \frac{3}{4}x_4$ for fixed $x_3, x_4$. Thus

\[
\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{-3}{4}x_3 - \frac{1}{4}x_4 \\ \frac{-3}{4}x_3 + \frac{3}{4}x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3/4 \\ -3/4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1/4 \\ 3/4 \\ 0 \\ 1 \end{pmatrix}
\]

The fundamental theorem of Linear Algebra yields $\dim(\text{Ker} A) = 4 - \dim(\text{Range} A) = 4 - 2 = 2$. Thus a basis of $\text{Ker} A$ is $\{(−3/4, −3/4, 1, 0)^T, (1/4, 3/4, 0, 1)^T\}$.

c) Since we know $\text{Ker} A$ is span$\{(−3/4, −3/4, 1, 0)^T, (1/4, 3/4, 0, 1)^T\}$ from part b), we only need to find a particular solution to $A\vec{x} = (0, b, b)^T$ in order to obtain the general solution. By performing the operations 1) and 2) in part a), we reduce the augmented matrix $\begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 2 & 2 & 3 & -1 & b \\ 0 & 4 & 3 & -3 & b \end{pmatrix}$ to $\begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 4 & 3 & -3 & b \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The corresponding linear system is

\[
\begin{align*}
x_1 - x_2 + x_4 &= 0 \\
4x_2 + 3x_3 - 3x_4 &= b
\end{align*}
\]

Clearly $\vec{x} = (b/4, b/4, 0, 0)^T$ is a particular solution to this linear system. Thus the general solution is

\[
\vec{x} = (x_1, x_2, x_3, x_4)^T = (b/4, b/4, 0, 0)^T + c_1(-3/4, -3/4, 1, 0)^T + c_2(1/4, 3/4, 0, 1)^T.
\]
(15pts) 5. Let $A$ be an invertible matrix of $n \times n$. And let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ be (column) vectors in $\mathbb{R}^n$.

a) Prove that if $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ are linearly independent, then $A\vec{v}_1, A\vec{v}_2, \cdots, A\vec{v}_k$ are linearly independent.

b) Let $M_1 = (\vec{v}_1 \cdots \vec{v}_k)$ (the matrix generated by $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$), and $M_2 = (A\vec{v}_1 \cdots A\vec{v}_k)$ (the matrix generated by $A\vec{v}_1, A\vec{v}_2, \cdots, A\vec{v}_k$). Prove that $\text{rank} M_1 = \text{rank} M_2$.

**Proof.** Proof of Part a).

Let $c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + \cdots + c_k A\vec{v}_k = \vec{0}$, where $c_1, c_2, \cdots, c_k$ are scalars. To prove the linear independence of $A\vec{v}_1, A\vec{v}_2, \cdots, A\vec{v}_k$, we only need to prove $c_1 = c_2 = \cdots = c_k = 0$. Notice that

$$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + \cdots + c_k A\vec{v}_k = A(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k).$$

Thus we obtain

$$A(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k) = \vec{0}. \tag{1}$$

Since $A$ is invertible, $A^{-1}$ exists. Multiply both sides of (1) by $A^{-1}$ to get

$$A^{-1}A(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k) = A^{-1}\vec{0},$$

which yields

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = \vec{0}.$$

Since in part a) $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ are linearly independent, we get that $c_1 = c_2 = \cdots = c_k = 0$ from the definition of linear independence. Thus the proof of part a) is completed.

Proof of Part b).

**Method 1.** From Part a), one can conclude that $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ are linearly independent if and only if $A\vec{v}_1, A\vec{v}_2, \cdots, A\vec{v}_k$ are linearly independent. This is because $\vec{v}_j = A^{-1}(A\vec{v}_j)$ for all $j$. Thus we can conclude $\text{rank} M_1 = \text{rank} M_2$ from the definition of the rank.

**Method 2.** Set $r_1 = \text{rank} M_1$ and $r_2 = \text{rank} M_2$. First, we prove $r_1 \leq r_2$. From the definition of the rank, we know there are $r_1$ linearly independent vectors among $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$. Let us name these $r_1$ linearly independent vectors by $\vec{u}_1, \cdots, \vec{u}_{r_1}$ respectively. From Part a), $A\vec{u}_1, \cdots, A\vec{u}_{r_1}$ are linearly independent. Thus we get $r_1$ linearly independent vectors among $A\vec{v}_1, A\vec{v}_2, \cdots, A\vec{v}_k$. Thus the maximal number of linearly independent vectors among $A\vec{v}_1, A\vec{v}_2, \cdots, A\vec{v}_k$ must be at least $r_1$, which gives $r_1 \leq r_2$ by the definition of the rank.

Now we prove $r_2 \leq r_1$. This can be done similarly. In fact, let $\vec{w}_1 = A\vec{v}_1$, $\vec{w}_2 = A\vec{v}_2, \cdots, \vec{w}_k = A\vec{v}_k$. Since $A$ is invertible, we have $v_1 = A^{-1}w_1$, $v_2 = A^{-1}w_2$,
\[ v_k = A^{-1} w_k. \] Repeat the same argument as in the previous paragraph to conclude
\[ \text{rank}(\vec{w}_1 \cdots \vec{w}_k) \leq \text{rank}(A^{-1} \vec{w}_1 \cdots A^{-1} \vec{w}_k), \]
which gives \( r_2 \leq r_1 \) since \( (\vec{w}_1 \cdots \vec{w}_k) = M_2 \) and \( (A^{-1} \vec{w}_1 \cdots A^{-1} \vec{w}_k) = M_1. \)

Finally we conclude \( r_1 = r_2 \) since \( r_1 \leq r_2 \) and \( r_2 \leq r_1. \) And this completes the proof.