ON A LIPSCHITZ VARIANT OF THE KAKEYA MAXIMAL FUNCTION

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1. The Lipschitz Kakeya Maximal Functions

This paper is part of a series of papers of the authors [10,11] concerning certain degenerate Radon transforms. In this paper, we are concerned with a maximal function estimate over rectangles in the plane, which have a prescribed maximal length, but arbitrary orientation and width. That is, we are concerned with a certain variant of the Kakeya maximal function. The rectangles used in the maximal function will be specified in a particular way by a Lipschitz choice of directions in the plane.

Our results are stated first; afterwords we place it in the context of the literature. We set notations and conventions. A *rectangle* is determined as follows. Fix a choice of unit vectors in the plane $(\mathrm{e}, \mathrm{e}^{\perp})$, with $\mathrm{e}^{\perp}$ being the vector e rotated by $\pi/2$. Using these vectors as coordinate axes, a rectangle is a product of two intervals $R = I \times J$. We will insist that $|I| \ge |J|$, and use the notations

$$\mathsf{L}(R) = |I|, \qquad \mathsf{W}(R) = |J| \tag{1.1}$$

for the length and width respectively of R.

The *interval of uncertainty of* R is the subarc $\mathsf{EX}(R)$ of the unit circle in the plane, centered at e, and of length $\mathsf{W}(R)/\,\mathsf{L}(R)$. See Figure 1.

We now fix a Lipschitz map v of the plane into the unit circle. We only consider rectangles R with

$$\mathsf{L}(R) \le (100\|v\|_{\mathrm{Lip}})^{-1}\,. \tag{1.2}$$

For such a rectangle R, set $\mathsf{V}(R) = R \cap v^{-1}(\mathsf{EX}(R))$.

For $0 < \delta < 1$, we consider the maximal functions

$$\mathrm{M}_{v,\delta}\, f(x) \stackrel{\mathrm{def}}{=} \sup_{|\mathsf{V}(R)| \ge \delta |R|} \frac{\mathbf{1}_R(x)}{|R|} \int_R |f(y)|\, dy. \tag{1.3}$$

The authors are supported in part by NSF grants. M.L. is a Guggenheim Fellow.

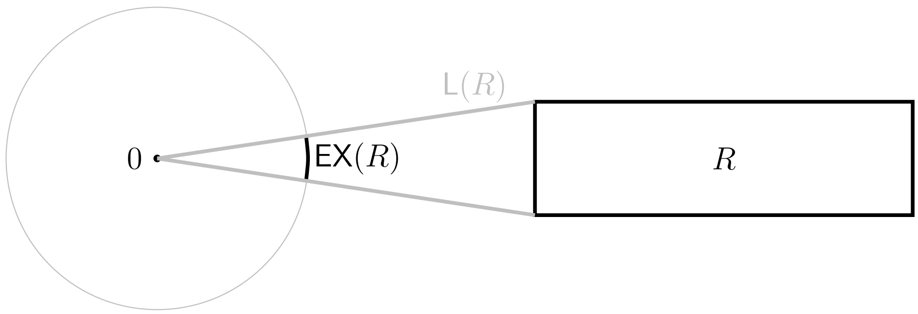

FIGURE 1. An example eccentricity interval $\mathsf{EX}(R)$. The circle on the left has radius one.

That is we only form the supremum over rectangles for which the vector field lies in the interval of uncertainty for a fixed positive proportion δ of the rectangle.

Theorem 1.4. *The maximal function $\mathrm{M}_{\delta,v}$ is bounded from $L^2(\mathbb{R}^2)$ to $L^{2,\infty}(\mathbb{R}^2)$ with norm at most $\lesssim \delta^{-1/2}$. That is, for any $\lambda > 0$, and $f \in L^2(\mathbb{R}^2)$, this inequality holds:*

$$\lambda^{-2}|\{x \in \mathbb{R}^2 : \mathrm{M}_{\delta,v} f(x) > \lambda\}| \lesssim \delta^{-1}\lambda^{-2}\|f\|_2^2 . \tag{1.5}$$

The norm estimate in particular is independent of the Lipschitz vector field v.

There is an alternate maximal function of interest, in which we form a supremum over rectangles of a fixed width. For $0 < \delta < 1$ and $0 < \mathsf{w} < (200\|v\|_{\mathrm{Lip}})^{-1}$ we consider

$$\mathrm{M}_{v,\delta,\mathsf{w}} f(x) \stackrel{\text{def}}{=} \sup_{\substack{|\mathsf{V}(R)|\ge\delta|R| \\ \mathsf{w}\le\mathsf{W}(R)\le 2\mathsf{w}}} \frac{\mathbf{1}_R(x)}{|R|}\int_R |f(y)|\,dy. \tag{1.6}$$

We find that in this instance, the maximal function is bounded on all L^p, for $1 < p < 2$.

Theorem 1.7. *The maximal function $\mathrm{M}_{v,\delta,\mathsf{w}}$ maps L^p into itself for all $1 < p < \infty$, with norm at most $\lesssim \delta^{-1/p}$. That is we have the inequalities*

$$\|\mathrm{M}_{v,\delta,\mathsf{w}} f\|_p \lesssim \delta^{-1/p}\|f\|_p , \qquad 1 < p < \infty . \tag{1.8}$$

The norm estimate is independent of the Lipschitz vector field v, and the choice of widths w.

The significance of both Theorems is that the norm bounds are independent of the eccentricity of the rectangles. The Lipschitz assumption is sharp; it cannot be weakened to Hölder continuity of any index less than one.

The Kakeya maximal function is typically defined as

$$\mathrm{M}_{\mathrm{K},\epsilon} f(x) = \sup_{|\mathsf{EX}(R)|\geq\epsilon} \frac{\mathbf{1}_R(x)}{|R|} \int_R |f(y)|\, dy\,, \qquad \epsilon > 0.$$

One is forced to take $\epsilon > 0$ due to the existence of the Besicovitch set. It is a critical fact that the norm of this operator admits norm bound on L^2 that is logarithmic in ϵ^{-1}. See Córdoba and Fefferman [7], and Strömberg [14, 15]. Subsequently, there have been several refinements of this observation, we cite only Nets H. Katz [8], Alfonseca, Soria and Vargas [1], and Alfonseca [2]. These papers contain additional references.

Our variant is inspired by questions of Zygmund and E.M. Stein concerning certain degenerate Radon transforms. For Lipschitz vector fields v, consider the maximal function and Hilbert transforms along line segments determined by v. For $\nu = (100\|v\|_{\mathrm{Lip}})^{-1}$, set

$$\mathrm{M}_v f(x) \stackrel{\mathrm{def}}{=} \sup_{0<t<\nu} (2t)^{-1} \int_{-t}^{t} |f(x - yv(x))|\, dy\,,$$

$$\mathrm{H}_v f(x) \stackrel{\mathrm{def}}{=} \mathrm{p.v.} \int_{-\nu}^{\nu} f(x - yv(x)) \frac{dy}{y}\,.$$

The question of the boundedness of the M_v on e.g. $L^2(\mathbb{R}^2)$ is attributed to Zygmund, motivated by constructions of the Besicovitch set which would show that the Lipschitz condition is sharp. E.M. Stein raised a similar question about the Hilbert transform analog [13].

Both operators are examples of Radon transforms. And that theory generally applies under suitable additional geometric conditions placed on the vector field v. Thus, there are positive results in the case that the vector field is analytic, due to Nagel, Stein and Wainger [12]. Bourgain [3] provided an extension to the case of real analytic vector fields, which exhibit a range of degeneracies that analytic vector fields can't possess. The theory of Radon transforms has a beautiful exposition in the article of Christ, Nagel, Stein and Wainger [6]. Another paper relevant to the Radon transforms of this paper is one by Carbery, Seeger and Wainger, Wright [4]. Both papers contain a wide set of references to the literature on Radon transforms.

Nets Katz [9] provided a partial result in the direction of Zygmund's conjecture. Lacey and Li [11] showed that the Hilbert transform H_v is bounded on L^2, assuming slightly more on the vector field, that it is $C^{1+\epsilon}$. Strikingly, this result contains Carleson's Theorem [5] on the convergence of Fourier series. (See the introduction of *op. cit.*) These are the only results known in the absence of geometric conditions on the vector field.

The modified Kakeya maximal functions of this paper, in particular (1.6), are a critical technical tool in [11]. It is likely that a better understanding of these maximal functions will prove to be a key ingredient of extensions of the results of the authors on these Radon transforms.

We speculate that the norm bounds obtained in the Theorems of this paper are optimal. Namely, that the operator $\mathrm{M}_{v,\delta}$ will in general have a weak type bound on L^2 that is at least as big as $c\delta^{1/2}$, and is unbounded on L^p for $1 < p < 2$. Likewise, we think the norm bound obtained in Theorem 1.7 is optimal in nature.

The notation $A \lesssim B$ means that $A \le KB$ for an absolute, but unspecified, constant K. $A \simeq B$ means that $A \lesssim B \lesssim A$. $\mathbf{1}_A$ means the indicator function of the set A.

2. Proof of Theorem 1.4

The Covering Lemma Conditions. We adopt the covering lemma approach of Córdoba and R. Fefferman [7]. To this end, we regard the choice of vector field v and $0 < \delta < 1$ as fixed. Let $\mathcal{R}$ be any finite collection of rectangles obeying the conditions (1.2) and $|\mathsf{V}(R)| \ge \delta |R|$. We show that $\mathcal{R}$ has a decomposition into disjoint collections $\mathcal{R}'$ and $\mathcal{R}''$ for which these estimates hold.

$$\Big\| \sum_{R \in \mathcal{R}'} \mathbf{1}_R \Big\|_2^2 \lesssim \delta^{-1} \Big\| \sum_{R \in \mathcal{R}'} \mathbf{1}_R \Big\|_1 , \tag{2.1}$$

$$\Big| \bigcup_{R \in \mathcal{R}''} R \Big| \lesssim \Big\| \sum_{R \in \mathcal{R}'} \mathbf{1}_R \Big\|_1 \tag{2.2}$$

The first of these conditions is the stronger one, as it bounds the L^2 norm squared by the L^1 norm; the verification of it will occupy most of the proof.

Let us see how to deduce Theorem 1.4. Take $\lambda > 0$ and $f \in L^2$ which is non negative and of norm one. Set $\mathcal{R}$ to be all the rectangles R of prescribed maximum length as given in (1.2), density with respect to the vector field, namely $|\mathsf{V}(R)| \ge \delta |R|$, and

$$\int_R f(y)\, dy \ge \lambda |R| .$$

We should verify the weak type inequality

$$\lambda \Big| \bigcup_{R\in\mathcal{R}} R \Big|^{1/2} \lesssim \delta^{-1/2} \,. \tag{2.3}$$

Apply the decomposition to $\mathcal{R}$. Observe that

$$\begin{aligned} \lambda \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_1 &\le \Big\langle f, \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\rangle \\ &\le \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_2 \\ &\lesssim \delta^{-1/2} \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_1^{1/2} . \end{aligned}$$

Here of course we have used (2.1). This implies that

$$\lambda \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_1^{1/2} \lesssim \delta^{-1/2}.$$

Therefore clearly (2.3) holds for the collection $\mathcal{R}'$.

Concerning the collection $\mathcal{R}''$, apply (2.2) to see that

$$\lambda \Big| \bigcup_{R\in\mathcal{R}''} R \Big|^{1/2} \lesssim \lambda \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_1^{1/2} \lesssim \delta^{-1/2} \,.$$

This completes our proof of (2.3).

The remainder of the proof is devoted to the proof of (2.1) and (2.2).

The Covering Lemma Estimates.

Construction of $\mathcal{R}'$ and $\mathcal{R}''$. In the course of the proof, we will need several recursive procedures. The first of these occurs in the selection of $\mathcal{R}'$ and $\mathcal{R}''$.

We will have need of one large constant κ, of the order of say 100, but whose exact value does not concern us. Using this notation hides distracting terms.

Let M_κ be a maximal function given as

$$\mathrm{M}_\kappa f(x) = \sup_{s>0} \max \Big\{ s^{-2} \int_{x+sQ} |f(y)| \, dy \,, \; \sup_{\omega\in\Omega} s^{-1} \int_{-s}^{s} |f(x+\sigma\omega)| \, d\sigma \Big\} \,.$$

Here, Q is the unit cube in plane, and Ω is a set of uniformly distributed points on the unit circle of cardinality equal to κ. It follows from the usual weak type bounds that this operator maps $L^1(\mathbb{R}^2)$ into weak $L^1(\mathbb{R}^2)$.

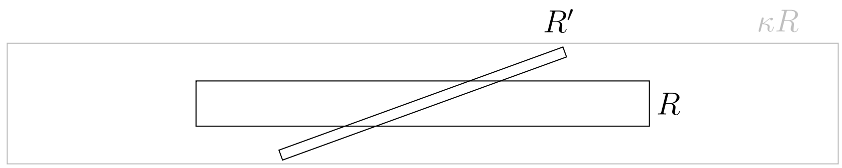

FIGURE 2.

To initialize the recursive procedure, set

$$\mathcal{R}' \leftarrow \emptyset\,,$$
$$\mathsf{STOCK} \leftarrow \mathcal{R}\,.$$

The main step is this while loop. While STOCK is not empty, select $R \in \mathsf{STOCK}$ subject to the criteria that first it have a maximal length $\mathsf{L}(R)$, and second that it have minimal value of $|\mathsf{EX}(R)|$. Update

$$\mathcal{R}' \leftarrow \mathcal{R}' \cup \{R'\}.$$

Remove R from STOCK. As well, remove any rectangle $R' \in \mathsf{STOCK}$ which is also contained in

$$\Big\{ \mathrm{M}_\kappa \sum_{R\in\mathcal{R}'} \mathbf{1}_{\kappa R} \geq \kappa^{-1} \Big\}.$$

As the collection $\mathcal{R}$ is finite, the while loop will terminate, and at this point we set $\mathcal{R}'' \stackrel{\mathrm{def}}{=} \mathcal{R} - \mathcal{R}'$. In the course of the argument below, we will refer the order in which rectangles where added to $\mathcal{R}'$.

With this construction, it is obvious that (2.3) holds, with a bound that is a function of κ. Yet, κ is an absolute constant, so this dependence does not concern us. And so the rest of the proof is devoted to the verification of (2.1).

An important aspect of the qualitative nature of the interval of eccentricity is encoded into this algorithm. We will choose κ so large that this is true: Consider two rectangles R and R' with $R \cap R' \neq \emptyset$, $\mathsf{L}(R) \geq \mathsf{L}(R')$, $\mathsf{W}(R) \geq \mathsf{W}(R')$, $|\mathsf{EX}(R)| \leq |\mathsf{EX}(R')|$ and $\mathsf{EX}(R) \subset 10\,\mathsf{EX}(R')$ then we have

$$R' \subset \kappa R\,. \tag{2.4}$$

See Figure 2.

Uniform Estimates. We estimate the left hand side of (2.1). In so doing we expand the square, and seek certain uniform estimates. Expanding the square on the left hand side of (2.1), we can estimate

$$\text{l.h.s. of (2.1)} \le \sum_{R\in\mathcal{R}'} |R| + 2 \sum_{(\rho,R)\in\mathcal{P}} |\rho \cap R|$$

where $\mathcal{P}$ consists of all pairs $(\rho, R) \in \mathcal{R}' \times \mathcal{R}'$ such that $\rho \cap R \neq \emptyset$, and ρ was selected to be a member of $\mathcal{R}'$ before R was. It is then automatic that $\mathsf{L}(R) \le \mathcal{L}(\rho)$. And since the density of all tiles is positive, it follows that $\operatorname{dist}(\mathsf{EX}(\rho), \mathsf{EX}(R)) \le 2\|v\|_{\text{Lip}} \mathsf{L}(\rho) < \frac{1}{50}$.

We will split up the collection $\mathcal{P}$ into subcollections $\{\mathcal{S}_R \mid R \in \mathcal{R}'\}$ and $\{\mathcal{T}_\rho \mid \rho \in \mathcal{R}'\}$.

For a rectangle $R \in \mathcal{R}'$, we take $\mathcal{S}_R$ to consist of all rectangles ρ such that (a) $(\rho, R) \in \mathcal{P}$; and (b) $\mathsf{EX}(\rho) \subset 10\,\mathsf{EX}(R)$. We assert that

$$\sum_{\rho\in\mathcal{S}_R} |R \cap \rho| \le |R|, \qquad R \in \mathcal{R}. \tag{2.5}$$

This estimate is in fact easily available to us. Since the rectangles $\rho \in \mathcal{S}_R$ were selected to be in $\mathcal{R}'$ before R was, we cannot have the inclusion

$$R \subset \Big\{ \mathrm{M}_\kappa \sum_{\rho\in\mathcal{S}_R} \mathbf{1}_{\kappa\rho} > \kappa^{-1} \Big\}. \tag{2.6}$$

Now the rectangle ρ are also longer. Thus, if (2.5) does not hold, we would compute the maximal function of

$$\sum_{\rho\in\mathcal{S}_R} \mathbf{1}_{\kappa\rho}$$

in a direction which is close, within an error of $2\pi/\kappa$, of being orthogonal to the long direction of R. In this way, we will contradict (2.6).

The second uniform estimate that we need is as follows. For fixed ρ, set $\mathcal{T}_\rho$ to be the set of all rectangles R such that (a) $(\rho, R) \in \mathcal{P}$ and (b) $\mathsf{EX}(\rho) \not\subset 10\,\mathsf{EX}(R)$. We assert that

$$\sum_{R\in\mathcal{T}_\rho} |R \cap \rho| \lesssim \delta^{-1} |\rho|, \qquad \rho \in \mathcal{R}'. \tag{2.7}$$

This proof of this inequality is more involved, and taken up in the next subsection.

Remark 2.8. In the proof of (2.7), it is not necessary that $\rho \in \mathcal{R}'$. Writing $\rho = I_\rho \times J_\rho$, in the coordinate basis e and $\mathrm{e}_\perp$, we could take any rectangle of the form $I \times J_\rho$.

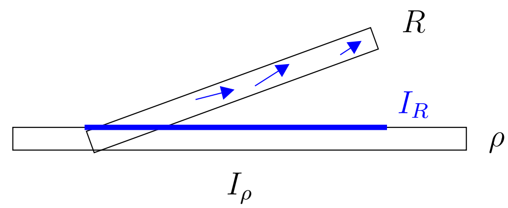

FIGURE 3. Notation for the proof of (2.7).

These two estimates conclude the proof of (2.1). For any two distinct rectangles $\rho, R \in \mathcal{P}$, we will have either $\rho \in \mathcal{S}_R$ or $R \in \mathcal{T}_\rho$. Thus (2.1) follows by summing (2.5) on R and (2.7) on ρ.

The Proof of (2.7). We fix ρ, and begin by making a decomposition of the collection $\mathcal{T}_\rho$. Suppose that the coordinate axes for ρ are given by e_ρ, associated with the long side of R, and $\mathrm{e}_\rho^\perp$, with the short side. Write the rectangle as a product of intervals $I_\rho \times J$, where $|I_\rho| = \mathsf{L}(\rho)$. Denote one of the endpoints of J as α. See Figure 3.

For rectangles $R \in \mathcal{T}_\rho$, let I_R denote the orthogonal projection R onto the line segment $2I_\rho \times \{\alpha\}$. Subsequently, we will consider different subsets of this line segment. The first of these is as follows. For $R \in \mathcal{T}_\rho$, let V_R be the projection of the set $\mathsf{V}(R)$ onto $2I \times \{\alpha\}$. We have

$$\mathsf{L}(R) \le |I_R| \le 2\,\mathsf{L}(R), \quad \text{and} \quad \delta\,\mathsf{L}(R) \lesssim |\mathsf{V}_R|. \tag{2.9}$$

A recursive mechanism is used to decompose $\mathcal{T}_\rho$. Initialize

$$\mathsf{STOCK} \leftarrow \mathcal{T}_\rho\,,$$
$$\mathcal{U} \leftarrow \emptyset\,.$$

While $\mathsf{STOCK} \neq \emptyset$ select $R \in \mathsf{STOCK}$ of maximal length. Update

$$\begin{aligned} \mathcal{U} &\leftarrow \mathcal{U} \cup \{R\}, \\ \mathcal{U}(R) &\leftarrow \{R' \in \mathsf{STOCK} \mid \mathsf{V}_R \cap \mathsf{V}_{R'} \neq \emptyset\}. \\ \mathsf{STOCK} &\leftarrow \mathsf{STOCK} - \mathcal{U}(R). \end{aligned} \tag{2.10}$$

When this while loop stops, it is the case that $\mathcal{T}_\rho = \bigcup_{R \in \mathcal{U}} \mathcal{U}(R)$.

With this construction, the sets $\{\mathsf{V}_R \mid R \in \mathcal{U}\}$ are disjoint. By (2.9), we have

$$\sum_{R \in \mathcal{U}} \mathsf{L}(R) \lesssim \delta^{-1}\,\mathsf{L}(\rho)\,. \tag{2.11}$$

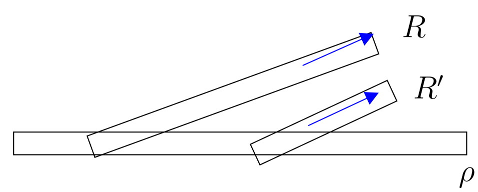

FIGURE 4. Proof of Lemma 2.13: The rectangles $R, R' \in \mathcal{U}(\rho)$, and so the angles R and R' form with ρ are nearly the same.

The main point, is then to verify the uniform estimate

$$\sum_{R' \in \mathcal{U}(R)} |R' \cap \rho| \lesssim \mathsf{L}(R) \cdot \mathsf{W}(\rho)\,, \qquad R \in \mathcal{U}. \tag{2.12}$$

Note that both estimates immediately imply (2.7).

Proof of (2.12). There are three important, and more technical, facts to observe about the collections $\mathcal{U}(R)$. We state these as Lemmas; the first two are taken from Lacey and Li [11], while the third is an instance of a Lemma of Stromberg [14, 15].

For any rectangle $R' \in \mathcal{U}(R)$, denote it's coordinate axes as $\mathrm{e}_{R'}$ and $\mathrm{e}_{R'}^{\perp}$, associated to the long and short sides of R' respectively.

Lemma 2.13. *For any rectangle $R' \in \mathcal{U}(R)$ we have*

$$|\mathrm{e}_{R'} - \mathrm{e}_R| \le \tfrac{1}{2} |\mathrm{e}_\rho - \mathrm{e}_R|$$

Proof. There are by construction, points $x \in \mathsf{V}(R)$ and $x' \in \mathsf{V}(R')$ which get projected to the same point on the line segment $I_\rho \times \{\alpha\}$. See Figure 4. Observe that

$$\begin{aligned} |\mathrm{e}_{R'} - \mathrm{e}_R| &\le |\mathsf{EX}(R')| + |\mathsf{EX}(R)| + |v(x') - v(x)| \\ &\le |\mathsf{EX}(R')| + |\mathsf{EX}(R)| + \|v\|_{\mathrm{Lip}} \cdot \mathsf{L}(R) \cdot |\mathrm{e}_\rho - \mathrm{e}_R| \\ &\le |\mathsf{EX}(R')| + |\mathsf{EX}(R)| + \tfrac{1}{100} |\mathrm{e}_\rho - \mathrm{e}_R| \end{aligned}$$

Now, $|\mathsf{EX}(R)| \le \frac{1}{5}|\mathrm{e}_\rho - \mathrm{e}_R|$, else we would have $\rho \in \mathcal{S}_R$. Likewise, $|\mathsf{EX}(R')| \le \frac{1}{5}|\mathrm{e}_{R'} - \mathrm{e}_R|$. And this proves the desired inequality.

□

Lemma 2.14. *Suppose that there is an interval $I' \subset I_\rho$ such that*

$$\sum_{\substack{R' \in \mathcal{U}(R) \\ \mathsf{L}(R') \geq 8|I|}} |R' \cap I \times J| \geq |I \times J| \,. \tag{2.15}$$

Then there is no $R'' \in \mathcal{U}(R)$ such that $\mathsf{L}(R'') < |I|$ and $R'' \cap 4I \times J \neq \emptyset$.

Proof. There is a natural angle θ between the rectangles ρ and R, which we can assume is positive, and is given by $|\mathrm{e}_\rho - \mathrm{e}_R|$. Notice that we have $\theta \geq 10|\mathsf{EX}(R)|$, else we would have $\rho \in \mathcal{S}_R$, which contradicts our construction.

Moreover, there is an important consequence of the previous Lemma 2.13: For any $R' \in \mathcal{U}(R)$, there is a natural angle θ' between R' and ρ. These two angles are close. For our purposes below, these two angles can be regarded as the same.

For any $R' \in \mathcal{U}(R)$, we will have

$$\begin{aligned} \frac{|\kappa R' \cap \rho|}{|I_\rho \times J|} &\simeq \kappa \frac{\mathsf{W}(R') \cdot \mathsf{W}(\rho)}{\theta |I_\rho| \, \mathsf{W}(\rho)} \\ &= \kappa \frac{\mathsf{W}(R')}{\theta \cdot |I_\rho|} . \end{aligned}$$

Recall M_κ is larger than the maximal function over κ uniformly distributed directions. Choose a direction e' from this set of κ directions that is closest to $\mathrm{e}_\rho^\perp$. Take a line segment Λ in direction e' of length $\kappa\theta|I|$, and the center of Λ is in $4I \times J$. See Figure 5. Then we have

$$\frac{|\kappa R' \cap \Lambda|}{|\Lambda|} \geq \frac{\mathsf{W}(R')}{\theta \cdot |I|}$$

Thus by our assumption (2.15),

$$\frac{1}{|\Lambda|} \sum_{R' \in \mathcal{U}(R)} |R' \cap \Lambda| \geq 1 \,.$$

That is, any of the lines Λ are contained in the set

$$\Big\{ \mathrm{M}_\kappa \sum_{R \in \mathcal{R}'} \mathbf{1}_{R'} > \kappa^{-1} \Big\}.$$

Clearly our construction does not permit any rectangle $R'' \in \mathcal{U}(R)$ contained in this set. To conclude the proof of our Lemma, we seek a contradiction. Suppose that there is an $R'' \in \mathcal{U}(R)$ with $\mathsf{L}(R'') < |I|$ and R'' intersects $2I \times J$. The range of line segments Λ we can permit

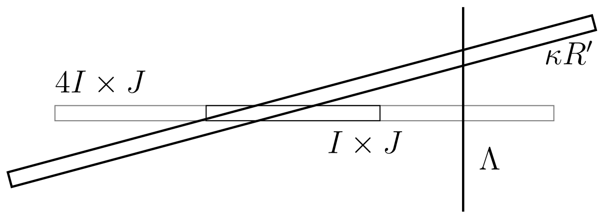

FIGURE 5.

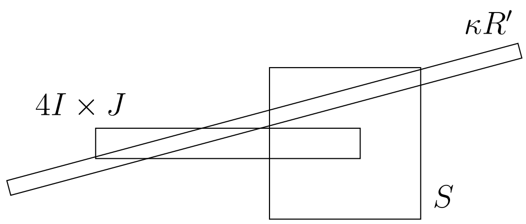

FIGURE 6. The proof of Lemma 2.16

is however quite broad. The only possibility permitted to us is that the rectangle R'' is quite wide. We must have

$$\mathsf{W}(R'') \geq \tfrac{1}{4}|\Lambda| = \tfrac{\kappa}{4} \cdot \theta \cdot |I|.$$

This however forces us to have $|\mathsf{EX}(R'')| \geq \frac{\kappa}{4}\theta$. And this implies that $\rho \in \mathcal{S}_{R''}$, as in (2.5). This is the desired contradiction.

□

Our third and final fact about the collection $\mathcal{U}(R)$ is a consequence of Lemma 2.13 and a geometric observation of J.-O. Stromberg [14, Lemma 2, p. 400].

Lemma 2.16. *For any interval $I \subset I_R$ we have the inequality*

$$\sum_{\substack{R' \in \mathcal{U}(R) \\ \mathsf{L}(R') \leq |I| \leq \sqrt{\kappa}\,\mathsf{L}(R')}} |R' \cap I \times J| \leq 5|I| \cdot \mathsf{W}(\rho)\,. \tag{2.17}$$

Proof. For each point $x \in 4I \times J$, consider the square S centered at x of side length equal to $\sqrt{\kappa} \cdot |I| \cdot |\mathrm{e}_R - \mathrm{e}_\rho|$. See Figure 6. It is Stromberg's observation that for $R' \in \mathcal{U}(R)$ we have

$$\frac{|\kappa R' \cap I \times J|}{|I \times J|} \simeq \frac{|S \cap \kappa R'|}{|S|}$$

with the implied constant being independent of κ. Indeed, by Lemma 2.13, we have that

$$\begin{aligned}\frac{|\kappa R' \cap I \times J|}{|I \times J|} &\simeq \frac{\kappa \,\mathsf{W}(R')}{|\mathrm{e}_R - \mathrm{e}_\rho| \cdot |I|} \\ &\simeq \frac{\kappa \,\mathsf{W}(R') \cdot |I| \cdot |\mathrm{e}_R - \mathrm{e}_\rho|}{(|\mathrm{e}_R - \mathrm{e}_\rho| \cdot |I|)^2} \\ &\simeq \frac{|S \cap \kappa R'|}{|S|},\end{aligned}$$

as claimed.

Now, assume that (2.17) does not hold and seek a contradiction. Let $\mathcal{U}' \subset \mathcal{U}(R)$ denote the collection of rectangles R' over which the sum is made in (2.17). The rectangles in $\mathcal{U}'$ were added in some order to the collection $\mathcal{R}'$, and in particular there is a rectangle $R_0 \in \mathcal{U}'$ that was the last to be added to $\mathcal{U}'$. Let $\mathcal{U}''$ be the collection $\mathcal{U}' - \{R_0\}$. By construction, $\mathcal{U}'$ must consist of at least three rectangles, so that $\mathcal{U}''$ is not empty. Moreover, we certainly have

$$\sum_{R' \in \mathcal{U}''} |R'' \cap I \times J| \geq 4|I \times J|.$$

Since we cannot have $\rho \in \mathcal{S}_{R_0}$, Stromberg's observation implies that

$$R_0 \subset \Big\{ \mathrm{M}_\kappa \sum_{R' \in \mathcal{U}''} \mathbf{1}_{R'} > \kappa^{-1} \Big\}.$$

Here, we rely upon the fact that the maximal function M_κ is larger than the usual maximal function over squares. But this is a contradiction to our construction, and so the proof is complete. □

The principal line of reasoning to prove (2.12) can now begin. We make a recursive decomposition of the collection $\mathcal{U}(R)$, which is indexed by a collection of intervals $\mathcal{I}$ that we now define. Initialize

$$\begin{aligned} \mathsf{STOCK} &\leftarrow \mathcal{U}(R) \\ \mathcal{I} &\leftarrow \emptyset \end{aligned} \tag{2.18}$$

While there is an interval $I \subset I_R$ satisfying

$$\sum_{\substack{R' \in \mathsf{STOCK} \\ \mathsf{L}(R) \geq 8|I|}} |R' \cap I \times J_R| \geq 10|I| \cdot \mathsf{W}(\rho),$$

we take I to be a maximal interval with this property, and update

$$\begin{gathered} \mathcal{I} \leftarrow \mathcal{I} \cup \{I\}\,; \\ \mathcal{V}(I) \leftarrow \{R' \in \mathsf{STOCK} \mid \mathsf{L}(R') \geq 8|I|\,, R' \cap I \times J_R \neq \emptyset\}\,; \\ \mathsf{STOCK} \leftarrow \mathsf{STOCK} - \mathcal{V}(I)\,; \end{gathered}$$

When the while loop terminates, we set $\mathcal{V} \leftarrow \mathsf{STOCK}$.

It is then clear that we must have

$$\sum_{R' \in \mathcal{V}} |R' \cap \rho| \lesssim |I_R| \cdot \mathsf{W}(\rho) \leq \mathsf{L}(R) \cdot \mathsf{W}(\rho)\,.$$

Lemma 2.16 implies that each $I \in \mathcal{I}$ must have length $|I| \leq \kappa^{-1/2}|I_\rho|$. But we choose intervals in I to be of maximal length, so we have

$$\sum_{R' \in \mathcal{V}(I)} |R' \cap \rho| \leq 20 \cdot |I| \cdot \mathsf{W}(\rho)\,. \tag{2.19}$$

Lemmas 2.14 and 2.16 place significant restrictions on the collection of intervals $\mathcal{I}$. If we have $I \neq I' \in \mathcal{I}$ with $2I \cap 2I' \neq \emptyset$, then we must have e.g. $\sqrt{\kappa}|I'| < |I|$. Moreover, we cannot have three distinct intervals $I, I', I'' \in \mathcal{I}$ with non empty intersection. Therefore, we must have

$$\sum_{I \in \mathcal{I}} |I| \lesssim |I_R| \lesssim \mathsf{L}(R).$$

With (2.19), this completes the proof of (2.12).

3. The Proof of Theorem 1.7

The maximal function is clearly bounded on L^∞ with norm one. By interpolation, it suffices for us to prove the weak type inequality, namely prove the estimate

$$\lambda |\{\mathrm{M}_{v,\delta,\mathsf{w}}\, f > \lambda\}|^{1/p} \lesssim \delta^{-1/p} \|f\|_p\,, \qquad 0 < \mathsf{w} < \frac{\|v\|_{\mathrm{Lip}}}{100}\,. \tag{3.1}$$

for values of $1 < p < 2$ that have a limiting value of 1.

This we do for choices of $1 < p < 2$ for which the conjugate index $q = p/(p-1)$ is an integer. This again we do by way of a covering lemma.

Covering Lemma Estimates. Let $\mathcal{R}$ be any finite collection of rectangles obeying the conditions (1.2); $|\mathsf{V}(R)| \ge \delta|R|$; and $\mathsf{w} \le \mathsf{W}(R) \le 2\mathsf{w}$. We show that $\mathcal{R}$ has a decomposition into disjoint collections $\mathcal{R}'$ and $\mathcal{R}''$ for which these estimates hold.

$$\Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_q^q \lesssim \delta^{-q+1} \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_1 , \tag{3.2}$$

$$\Big| \bigcup_{R\in\mathcal{R}''} R \Big| \lesssim \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_1 . \tag{3.3}$$

Let us deduce (3.1). Take $\lambda > 0$ and $f \in L^p$ which is non negative and of norm one. Set $\mathcal{R}$ to be all the rectangles R of prescribed maximum length as given in (1.2); of prescribed width; density with respect to the vector field, namely $|\mathsf{V}(R)| \ge \delta|R|$; and

$$\int_R f(y)\, dy \ge \lambda |R| \, .$$

We should verify the weak type inequality

$$\lambda \Big| \bigcup_{R\in\mathcal{R}} R \Big|^{1/p} \lesssim \delta^{-1/p} . \tag{3.4}$$

Apply the decomposition to $\mathcal{R}$. Observe that

$$\begin{aligned} \lambda \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_1 &\le \Big\langle f, \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\rangle \\ &\le \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_q \\ &\lesssim \delta^{-1/p} \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_1^{1/q} . \end{aligned}$$

Here of course we have used (3.2). This implies that

$$\lambda \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_1^{1/p} \lesssim \delta^{-1/p}.$$

Therefore clearly (3.4) holds for the collection $\mathcal{R}'$.

The collection $\mathcal{R}''$ is controlled by application of (3.3), to see that

$$\lambda \Big| \bigcup_{R\in\mathcal{R}''} R \Big|^{1/p} \lesssim \lambda \Big\| \sum_{R\in\mathcal{R}'} \mathbf{1}_R \Big\|_1^{1/p} \lesssim \delta^{-1/p}$$

This completes our proof of (3.4).

The proof of (3.2) and (3.3) follows the lines of argument of the previous Theorem; the main point is to observe that a variant of a BMO estimate applies in the case of the rectangles having fixed widths.

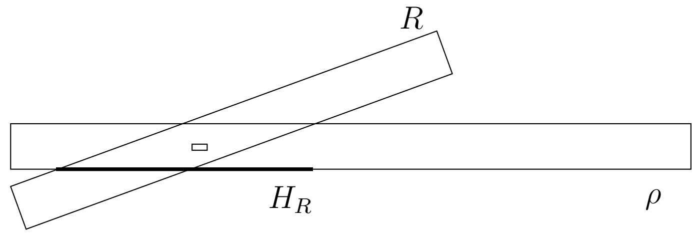

FIGURE 7. This illustrates (3.6). The smallest rectangle is $\frac{1}{8}H_R \times (\frac{1}{8}J)$.

The construction of the collections $\mathcal{R}'$ and $\mathcal{R}''$ proceeds in precisely the same manner as the previous proof.[1] As before, the condition (3.3) follows easily from the construction of $\mathcal{R}''$, and the bulk of the proof is occupied with establishing (3.2).

Uniform Estimates. Different simplifications of the previous proof accrue due to the assumption of fixed width. For instance, the collections $\mathcal{S}_\rho$ are empty, due to the condition (2.4), and that all rectangles have approximately the same width. Nevertheless, we continue to use the same notation as in the previous proof.

In particular, we prove the inequality

$$\Big\| \sum_{R \in \mathcal{T}_\rho} \mathbf{1}_{R\cap\rho} \Big\|_q^q \lesssim \delta^{-q} |\rho| \,, \qquad \rho \in \mathcal{R}' \,. \tag{3.5}$$

This proves the inequality (3.2) (with q replaced by $q+1$) upon summation over ρ.

Write ρ as a product of intervals $I_\rho \times J$, where $|I_\rho| = \mathsf{L}(\rho)$. Denote one of the endpoints of J as α. Below, when we form products of intervals, it is understood to be in the coordinate system of ρ.

For rectangles $R \in \mathcal{T}_\rho$, let I_R denote (as before) the orthogonal projection R onto the line segment $2I_\rho \times \{\alpha\}$, and let H_R the the orthogonal projection of $R \cap \rho$ onto the line segment $2I_\rho \times \{\alpha\}$. Observe that

$$\tfrac{1}{8}H_R \times (\tfrac{1}{8}J) \subset R \cap \rho \subset H_R \times J \tag{3.6}$$

with the product being in the coordinate system associated with ρ. This is a critical property of rectangles having fixed widths. See Figure 7.

[1] As all rectangles have fixed widths, the recursive procedure permits some simplifications, but not enough to justify a complete reaccounting of the procedure.

Using Remark 2.8, we see that for any subinterval $I \subset I_\rho$ we have the inequality

$$\sum_{\substack{R \in \mathcal{T}_\rho \\ I_R \subset I}} |I_R \times J| \lesssim \delta^{-1} |I| \cdot |J| \,.$$

A standard variant of the John Nirenberg inequality would then show that

$$\Big\| \sum_{R \in \mathcal{T}_\rho} \mathbf{1}_{I_R \times J} \Big\|_q \lesssim \delta^{-1} |\rho|^{1/q} \,.$$

(An argument to this effect is given in section 5 of [11].) This completes the proof.

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