M1: Euler Circuits, Eulerization

Objectives: SWBAT

- Identify the vertices and edges in a graph
- Identify if a given graph is connected
- Determine the valence of each vertex of a graph
- Determine whether or not a graph contains an Euler circuit
- Eulerize a graph which does not contain an Euler circuit

Individual Activity/Group Work: Worksheet M1.1

These pictures are examples of graphs, a finite set of dots and connecting lines. We call the dots vertices, each one is a vertex, and the links are called edges.

A path is a connected sequence of edges starting and ending at a vertex. A path that starts and ends at the same vertex is called a circuit.

Example:

How many vertices does this graph have? Edges? The vertices represent cities and the edges represent nonstop airline routes between them. Is there an edge connecting New York to Berlin? How could we travel with this airline from New York to Berlin? We could take the path NLB, can we see another path? Can anyone give me an example of a circuit? A path that covers each edge of a graph once, but not more than once, is called an Euler path, if it also starts and ends at the same vertex it is called an Euler circuit. Is there an Euler circuit in our example graph? Yes - BLNMRMB

Goal: Now that we know what Euler circuits are, is there a way to tell if a graph has an Euler circuit, and if so, can we find it easily?

The valence of a vertex in a graph is the number of edges meeting at the vertex.

A graph is said to be connected if, for every pair of its vertices, there is at least one path connecting the two vertices.
Example:

Each component of this graph is connected, but the graph itself is not. What is the valence of each vertex in this graph?

Let’s look back at the graphs from the group activity and fill in this table:

<table>
<thead>
<tr>
<th>Graph</th>
<th>Euler path?</th>
<th>Euler circuit?</th>
<th># of vertices</th>
<th># with even valence</th>
<th># with odd valence</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Graph 1]</td>
<td>No</td>
<td>No</td>
<td>5</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>![Graph 2]</td>
<td>Yes</td>
<td>No</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>![Graph 3]</td>
<td>Yes</td>
<td>No</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>![Graph 4]</td>
<td>Yes</td>
<td>Yes</td>
<td>5</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>![Graph 5]</td>
<td>Yes</td>
<td>Yes</td>
<td>7</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>![Graph 6]</td>
<td>No</td>
<td>No</td>
<td>7</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>![Graph 7]</td>
<td>Yes</td>
<td>No</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>![Graph 8]</td>
<td>Yes</td>
<td>No</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

What do we notice about the graphs where Euler paths exists? How about Euler circuits? Neither?

**Thm. Euler Circuit Theorem**

1. If $G$ is connected and has all valences even, then $G$ has an Euler circuit.
2. Conversely, if $G$ has an Euler circuit, then $G$ must be connected and all its valences must be even.

Even though a graph may not have an Euler circuit, it is possible to eulerize it so that it does.
Can any one see a better eulerization, i.e. one that adds fewer edges?

**Individual Work/Group Work**: Worksheet M1.2

So how do we find a best eulerization? A systematic procedure does exist, but it is complicated. However, in the case of *rectangular* networks, the technique is especially easy:

**Example:**

Start in one corner and walk counterclockwise, whenever you arrive at an odd-valent vertex, link it to the next with an added edge. If the next vertex is now even, skip it and continue around looking for odd vertices, if the next vertex is now odd, link it to the next.

If the graph is not rectangular, we look for all the vertices that have odd valence and pair them up based on the shortest path between them.

**Group work**: Worksheet 1.3

**Eulerization Algorithm**

1. Find all vertices with odd valence.
2. Pair them up with their nearest neighbor.
3. Find the shortest path between each pair.
4. Duplicate these edges.

**Homework**:

Ch 1: 2, 18, 44, 46