(1) Find the horizontal asymptotes of \( \frac{5x^3 + x^2 - 3}{-2x^3 + x} \)

First, we do some algebra; we divide the numerator and denominator by the highest power of \( x \):

\[
\frac{5x^3 + x^2 - 3}{-2x^3 + x} = \frac{5 + \frac{1}{x} - \frac{3}{x^3}}{-2 + \frac{1}{x^2}} \quad \text{when } x \neq 0
\]

As \( x \) goes to \( \infty \), the numerator goes to \( 5 + 0 + 0 = 5 \) while the denominator goes to \( -2 + 0 = 0 \). So the horizontal asymptote as \( x \) goes to \( \infty \) of \( f(x) \) is \( y = -\frac{5}{2} \). As \( x \) goes to \( -\infty \), we may also conclude that \( f(x) \) goes to \( -\frac{5}{2} \) (as \( \frac{1}{x^3} \) goes to zero for \( x \) going to either \( \infty \) or \( -\infty \)).

(2) Find the horizontal asymptotes of \( \frac{\sqrt{x^2 - 2}}{x + 5} \)

This one is a little trickier. We need to think about how to pull the \( x^2 \) out of the square root; this is done first by factoring an \( x^2 \) out of \( x^2 - 2 \) and then pulling the \( x^2 \) out of the square root itself:

\[
\sqrt{x^2 - 2} = \sqrt{(x^2 - 2) \left(1 - \frac{2}{x^2}\right)} = |x| \sqrt{1 - \frac{2}{x^2}}
\]

So, for the full quotient, we have

\[
\frac{\sqrt{x^2 - 2}}{x + 5} = \frac{|x| \sqrt{1 - \frac{2}{x^2}}}{x(1 + \frac{5}{x})}
\]

For \( x > 0 \), \( \frac{|x|}{x} = 1 \), so as \( x \) goes to \( \infty \), this function goes to 1. However, for \( x < 0 \), \( \frac{|x|}{x} = -1 \), so as \( x \) goes to \( -\infty \), the fraction goes to \( -1 \). So this function has horizontal asymptotes \( x = 1 \) and \( x = -1 \).

(3) Find the horizontal asymptotes of \( \frac{e^{8x} + e^{3x} - 5}{4e^{8x} - 6} \)

By multiplying both the numerator and denominator by \( e^{-8x} \), we have:

\[
\frac{e^{8x} + e^{3x} - 5}{4e^{8x} - 6} = \frac{1 + e^{-5x} - 5e^{-8x}}{4 - 6e^{-8x}} \quad \text{when } x \neq 0
\]
As $x$ goes to $\infty$, remember that $e^{-x}$ goes to 0, so this fraction goes to $1/4$. As $x$ goes to $-\infty$, the numerator of the original fraction goes to $-5$ while the denominator goes to $-6$, so the limit as $x$ goes to $-\infty$ of the quotient is $5/6$. So this function has horizontal asymptotes $1/4$ and $5/6$.

(4) Find $\lim_{x \to -\infty} (x^7 + 5x^4)$.

We factor $(x^7 + 5x^4) = x^4(x^3 + x)$. As $x$ goes to $-\infty$, $x^4$ goes to $\infty$ while $x^3 + x$ goes to $-\infty$ (since both $x^3$ and $x$ go to $-\infty$ while $x$ goes to $-\infty$). So we can use the product rule for infinite limits to conclude that the limit of $(x^7 + 5x^4)$ as $x$ goes to $-\infty$ is $-\infty$.

(5) Find all real numbers $a$, $b$, and $c$ so that:

$$f(x) = \frac{ax^2 + cx}{x^2 + bx + a}$$

has vertical asymptotes $x = 1$ and $x = 3$, horizontal asymptote $y = 3$, and only one zero.

First, we can check that $f(x)$ has a horizontal asymptote at $y = a$. So, we need $a = 3$. Next, note that the numerator factors as $3x^2 + cx = x(3x + c)$. So $f(x)$ has zeros at $x = 0$ and $x = -c/3$. So to enforce that $f$ has only one zero, we should take $c = 0$. Finally, we have a denominator $x^2 + bx + 3$. We want vertical asymptotes of $x = 1$ and $x = 3$, meaning we want the denominator to go to zero at 1 and 3. This implies we want the denominator to be $(x-1)(x-3) = x^2 - 4x + 3$. So we need $b = -4$.

The rest of the problems will be included in a different worksheet later this week.