(1) Find all possible combinations of real numbers \(a, b,\) and \(c\) so that the function:

\[
 f(x) = \begin{cases} 
 b & x < -1 \\
 ax^2 + bx + c & -1 \leq x \leq 1 \\
 cx^3 + a & 1 < x < 2 \\
 7 & 2 \leq x 
\end{cases}
\]

is continuous.

Away from the possible jump discontinuities at \(-1\) and \(1,\) \(f(x)\) is a polynomial and therefore is continuous. To be sure \(f(x)\) is continuous everywhere, we need \(\lim_{x \to -1^+} f(x) = f(-1),\) \(\lim_{x \to -1^-} f(x) = \lim_{x \to 1^-} f(x) = f(1),\) and \(\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2).\)

Let’s calculate each:

\[
\begin{align*}
\lim_{x \to -1^+} f(x) &= \lim_{x \to -1^-} b = b \\
\lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} (ax^2 + bx + c) = a - b + c \\
f(-1) &= b \\
\lim_{x \to 1^+} f(x) &= \lim_{x \to 1^-} (ax^2 + bx + c) = a + b + c \\
\lim_{x \to 1^-} f(x) &= \lim_{x \to 1^+} (cx^3 + a) = c + a \\
f(1) &= a + b + c \\
\lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (cx^3 + a) = 8c + a \\
\lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} 7 = 7 \\
f(2) &= 7
\end{align*}
\]

For the first three lines to match, we need \(a - b + c = b.\) For the second three lines to match, we need \(a + b + c = c + a.\) It necessarily follows that \(b = 0,\) so \(a - b + c = b\) implies \(a + c = 0.\) Finally, for the last three lines to match, we need \(8c + a = 7.\) Since \(a + c = 0,\) \(8c + a = 7\) means \(7c = 7,\) or \(c = 1.\) Finally, we have \(a = -1.\)

To recap, \(f\) is continuous only when \(a = -1,\) \(b = 0,\) and \(c = 1.\)

(2) Find

\[
\lim_{x \to 0} \arcsin \left( \frac{\sqrt{x^2 + 1} - 1}{x^2} \right)
\]
We can use the fact that arcsin\(x\) is continuous on its domain to conclude that, if the limit of the inside exists and is in the domain of arcsin\(x\), then
\[
\lim_{x \to 0} \arcsin \left( \frac{\sqrt{x^2 + 1} - 1}{x^2} \right) = \arcsin \left( \lim_{x \to 0} \frac{\sqrt{x^2 + 1} - 1}{x^2} \right)
\]
So, to find \(\lim_{x \to 0} \frac{\sqrt{x^2 + 1} - 1}{x^2}\), we do some algebra:
\[
\frac{\sqrt{x^2 + 1} - 1}{x^2} = \frac{\sqrt{x^2 + 1} - 1}{x^2} \cdot \frac{\sqrt{x^2 + 1} + 1}{\sqrt{x^2 + 1} + 1} = \frac{x^2 + 1 - 1}{x^2(\sqrt{x^2 + 1} + 1)} = \frac{1}{\sqrt{x^2 + 1} + 1} \quad \text{(when } x \neq 0)\
\]
So since
\[
\frac{\sqrt{x^2 + 1} - 1}{x^2} = \frac{1}{\sqrt{x^2 + 1} + 1} \quad \text{(when } x \neq 0)\
\]
, we have that
\[
\lim_{x \to 0} \frac{\sqrt{x^2 + 1} - 1}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 1} + 1} = \frac{1}{2}
\]
So since \(1/2\) is in the domain of arcsin,
\[
\lim_{x \to 0} \arcsin \left( \frac{\sqrt{x^2 + 1} - 1}{x^2} \right) = \arcsin \left( \frac{1}{2} \right) = \frac{\pi}{6}
\]
(3) For \(f(x) = x^2 \sin \left( \frac{1}{x} \right)\), find
\[
\lim_{x \to 0} e^{f(x)}
\]
We use the same logic as in the previous problem. Since \(\lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) = 0\) (we’ve shown this before using the Squeeze Theorem), we can conclude that, since \(e^x\) is continuous at every real number,
\[
\lim_{x \to 0} e^{f(x)} = e^{\lim_{x \to 0} f(x)} = e^0 = 1
\]
(4) Find
\[
\lim_{x \to 2} \sqrt{\sin \left( \frac{\pi}{2} \cdot \frac{x^2 - 3x + 2}{x - 2} \right)}
\]
We use the same logic as in the previous two problems. First, note that:
\[
\frac{x^2 - 3x + 2}{x - 2} = \frac{(x - 1)(x - 2)}{x - 2} = x - 1 \quad \text{when } x \neq 2
\]
So,
\[
\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x \to 2} (x - 1) = 1
\]
Since \(\sin(x)\) is continuous at every real number, we have that
\[
\lim_{x \to 2} \sin \left( \left( \frac{\pi}{2} \right) \frac{x^2 - 3x + 2}{x - 2} \right) = \sin \left( \lim_{x \to 2} \left( \left( \frac{\pi}{2} \right) \frac{x^2 - 3x + 2}{x - 2} \right) \right) = \sin \left( \frac{\pi}{2} \right) = 1
\]
Finally, since \(\sqrt{x}\) is continuous at \(x = 1\), we have that:
\[
\lim_{x \to 2} \sqrt{\sin \left( \left( \frac{\pi}{2} \right) \frac{x^2 - 3x + 2}{x - 2} \right)} = \sqrt{\lim_{x \to 2} \sin \left( \left( \frac{\pi}{2} \right) \frac{x^2 - 3x + 2}{x - 2} \right)} = \sqrt{1} = 1
\]

(5) Show that \(\sin^2(x) - e/3\) has a root with \(\pi < c < 3\pi/2\) (hint: remember \(e \approx 2.7\)).

Let \(f(x) = \sin^2(x) - e/3\). Since \(e \approx 2.7\), we have that \(e < 3\), so \(e/3 < 1\). Then since \(\sin^2(\pi) = 0\) and \(\sin^2(3\pi/2) = (-1)^2 = 1\), we have that \(f(\pi) = -e/3\) while \(f(3\pi/2) = 1 - e/3 > 0\). So, since \(\sin^2(x)\) is continuous, \(f(x)\) is continuous as well, and, using the Intermediate Value Theorem, we have that, since \(f(\pi) < 0 < f(3\pi/2)\), there is a value \(c\) with \(\pi < c < 3\pi/2\) for which \(f(c) = 0\).

(6) Show that \(\arccos(x) - \sin(x)\) has a root (hint: \(-\pi/2 < -1 < 0\) and \(0 < 1 < \pi/2\)).

Let \(f(x) = \arccos(x) - \sin(x)\). Then \(f\) has domain \([-1, 1]\). By the hint, we have that the angle \(-1\) is in the fourth quadrant while \(1\) is in the first. This means that \(\sin(-1) < 0\) while \(\sin(1) > 0\). So \(f(-1) = \arccos(-1) - \sin(-1) = \pi - \sin(-1) > 0\) while \(f(1) = \arccos(1) - \sin(1) = 0 - \sin(1) < 0\).

So, since \(f(1) < 0 < f(-1)\) and since \(f(x)\) is continuous on \([-1, 1]\) (since \(\arccos(x)\) and \(\sin(x)\) are both continuous on \([-1, 1]\)), we may use the Intermediate Value Theorem to conclude that \(f\) as a root \(c\) with \(-1 < c < 1\).

(7) Suppose a triangle \(\Delta\) in the \(xy\)-plane is built from the points \((-1, 0), (1, 0)\), and a point \(P\) somewhere on the circle \(x^2 + y^2 = 1\). Show that there is a point \(P\) in the first quadrant so that \(\Delta\) has area \(\pi/4\).

Recall that the area of a triangle is given by \(A = \frac{1}{2}b \cdot h\), where \(b\) is the length of the base while \(h\) is the length of the height. Let’s say the base of \(\delta\) runs along the \(x\)-axis; then this has length 2. Since the third point \(P\) is on the unit circle \(x^2 + y^2 = 1\), the height of the triangle, given by the \(y\)-coordinate for \(P\), is just \(\sin(\theta)\), where \(\theta\) is the angle of the point \(P\) on the unit circle. So we can write the area of \(\delta\) as a function of \(\theta\): \(A(\theta) = \frac{1}{2} \cdot 2 \cdot \sin(\theta) = \sin(\theta)\).

As we are looking for \(P\) in the first quadrant, we must have that \(0 \leq \theta \leq \pi/2\). So since \(\sin(0) = 0\) and \(\sin(\pi/2) = 1\) and since \(0 < \pi/4 < 1\), the Intermediate Value Theorem tells us that there is an angle \(\alpha\) satisfying \(0 < \alpha < \pi/2\) so that the area of \(\delta\) is \(\pi/4\).

(8) Show that, if the limit
\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]
exists and is finite, then $f$ is continuous at $a$.

Note that $\lim_{x \to a} (x - a) = 0$. So, as explained on a previous homework, since $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is finite, we must have that $\lim_{x \to a} (f(x) - f(a)) = 0$. Adding the constant function $f(a)$ to $(f(x) - f(a))$, we have that $\lim_{x \to a} f(x) = \lim_{x \to a} ((f(x) - f(a)) + f(a)) = \lim_{x \to a} (f(x) - f(a)) + \lim_{x \to a} f(a) = 0 + f(a) = f(a)$. 