(1) Show that, if two functions \( f(x) \) and \( g(x) \) are both one-to-one, then \((f \circ g)(x)\) is one-to-one as well.

If \( f(g(x_1)) = f(g(x_2)) \), then \( g(x_1) = g(x_2) \) as \( f \) is one-to-one. We repeat, noting that, if \( g(x_1) = g(x_2) \), then \( x_1 = x_2 \) as \( g \) is one-to-one. The above two sentences combine to tell us that, if \( f(g(x_1)) = f(g(x_2)) \), then \( x_1 = x_2 \), meaning \( f \circ g \) is one-to-one.

Here’s another, wordier way to see why this is true. Each element of the range of \( f \) corresponds to exactly one element of the domain of \( f \). In turn, each element in the domain of \( f \) that is also in the range of \( g \) corresponds to exactly one element of the domain of \( g \). Since the range of \( f \circ g \) is in the range of \( f \) and the domain of \( f \circ g \) are those things in the domain of \( g \) that get sent to things in the domain of \( f \) (this was a Worksheet 2 problem), we can conclude from the first two sentences of this paragraph that everything in the range of \( f \circ g \) corresponds to exactly one thing in the domain of \( f \circ g \).

(2) Find the inverses of the following functions (be sure to “trim” the domains, if necessary!):

(a) \( f(x) = 3x + 4 \)

We did this in class for any line with non-zero slope.

(b) \( f(x) = \sqrt{\log_2(x)} \)

First, let’s be careful and find the domain of \( f(x) \). Note that the range of \( \log_2(x) \) is \((-\infty, \infty)\), so we must cut the domain of \( \log_2(x) \) accordingly. As we show below, \( \log_2(1) = 0 \), so we can conclude that \( \log_2(x) \geq 0 \) for \( x \geq 1 \). So, \( \sqrt{\log_2(x)} \) has domain \([1, \infty)\).

Since \( \log_2(x) \) for \( x \geq 1 \) has range \([0, \infty)\), the range of \( f(x) \) must be \([0, \infty)\). Since \( \sqrt{x} \) and \( \log_2(x) \) are both one-to-one, \( \sqrt{\log_2(x)} \) is also one-to-one. We solve:

\[
\begin{align*}
    y &= \sqrt{\log_2(x)} \\
    y^2 &= \log_2(x) \\
    2y^2 &= 2^{\log_2(x)} = x
\end{align*}
\]

So \( f^{-1}(x) = 2x^2 \) with domain \([0, \infty)\) (as this is the range of \( f(x) \)).

(c) \( f(x) = \cos(x^2) \)

Here, the domain of \( f \) is all real numbers but we know that \( \cos(x) \) fails the horizontal line test without trimming its domain. Let’s use the standard trimming of the domain for \( \cos(x) \), considering only \( 0 \leq x \leq \pi \). Then, as we are first applying \( x^2 \), we should take \( 0 \leq x \leq \sqrt{\pi} \).
Now, we can solve:

\[ y = \cos(x^2) \]
\[ \arccos(y) = x^2 \]
\[ \sqrt{\arccos(y)} = x \]

As the range of \( \cos(x^2) \) is still \([-1, 1]\), we may conclude that \( f^{-1}(x) = \sqrt{\arccos(x)} \) with domain \([-1, 1]\); note that this makes sense, as our standard definition for \( \arccos(x) \) has range \([0, \pi]\).

(3) Recall the tangent function is defined as \( \tan(x) = \frac{\sin(x)}{\cos(x)} \).

(a) What’s the domain of \( \tan \)?

This one is a little tricky; essentially, it is the set of all real numbers where \( \cos(x) \neq 0 \). \( \cos(x) = 0 \) exactly when \( x = \frac{n\pi}{2} \) for \( n \) an odd integer (i.e., an odd whole number like 1, 3, 5, etc. or the negative of an odd whole number, like \(-1, -3, -5, \) etc.). This is a bit tricky to denote symbolically, but we can write this as:

\[ \ldots \cup (-3\pi/2, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \ldots \]

(b) Sketch the graph of \( \tan \).

You can plug this into your favorite graphing program.

(c) Find the inverse of \( \tan \) (called \( \arctan(x) \)), trimming the domain of \( \tan(x) \) as necessary. NOTE: beware that the book and WebAssign use \( \tan^{-1}(x) \) instead of \( \arctan(x) \).

As I mentioned in class, “find” is a bit of a misnomer; the key really is how we should trim the domain of \( \tan(x) \) to be able to get an inverse function. Well, as one can see from the graph, \( \tan(x) \) on the restricted domain \(-\pi/2 < x < \pi/2\) satisfies the horizontal line test. So, we can define an inverse function, called \( \arctan(x) \), for this restriction.

(d) What are the domain and range of \( \arctan \)?

The domain and range of \( \arctan \) are the range and domain (respectively) of \( \tan(x) \) restricted to \(-\pi/2 < x < \pi/2\). The range of \( \tan(x) \) is \((-\infty, \infty)\) and the domain is given as \(\pi/2 < x < \pi/2\).

(4) For a positive constant \( a > 0 \), we can show that \( \log_a(a) = 1 \) as follows: first, for \( f(x) = a^x \), \( \log_a(x) = f^{-1}(x) \), so \( f(f^{-1}(x)) = x \). In other words:

\[ a^{\log_a(x)} = x \]

So:

\[ a^{\log_a(a)} = a \]
and, since $a^1 = a$ and $a^x$ is one-to-one, it must be the case that $\log_a (a) = 1$. Use similar arguments to show the following rules of logarithms:

(a) $\log_a (1) = 0$

As above,

$$a^{\log_a (x)} = x$$

So, since $a^0 = 1$, we conclude that $\log_a (1) = 0$.

(b) $\log_a (xy) = \log_a (x) + \log_a (y)$

We compare:

$$a^{\log_a (xy)} = xy$$

and

$$a^{\log_a (x) + \log_a (y)} = a^{\log_a (x)}a^{\log_a (y)} = xy$$

(c) $\log_a \left( \frac{x}{y} \right) = \log_a (x) - \log_a (y)$

$$a^{\log_a \left( \frac{x}{y} \right)} = \frac{x}{y}$$

and

$$a^{\log_a (x) - \log_a (y)} = a^{\log_a (x)}a^{-\log_a (y)} = \frac{a^{\log_a (x)}}{a^{\log_a (y)}} = \frac{x}{y}$$

(d) $\log_a (x^r) = r \log_a (x)$

$$a^{\log_a (x^r)} = x^r$$

and

$$a^{r \log_a (x)} = \left(a^{\log_a (x)}\right)^r = x^r$$

(5) Recall that, for the constant $e \approx 2.71828 \ldots$, the function $f(x) = e^x$ is called the natural exponential function. Similarly, the logarithm $\log_e (x)$ is given a special name: the natural log. It’s also given the special symbol: $\log_e (x) = \ln(x)$. Use the rules of logarithms above and the equation:

$$x = b^{\log_b (x)}$$

to show the change in base equation:

$$\log_b (x) = \frac{\ln(x)}{\ln(b)}$$

(Hint: apply ln to both sides of the first equation)

Applying ln to both sides of

$$x = b^{\log_b (x)}$$
we have:

\[
\ln(x) = \ln(b^{\log_b(x)}) \\
= \log_b(x) \ln(b)
\]

Dividing both sides by \(\ln(b)\), we get what we want.

(6) Use a right triangle with sides 1, \(x\), and \(\sqrt{x^2 + 1}\) to prove that

\[
\cos(\arctan(x)) = \frac{1}{\sqrt{1 + x^2}}
\]

(hint: remember what \(\tan\) and \(\cos\) correspond to in a right triangle).

We build a right triangle with sides 1, \(x\), and hypotenuse \(\sqrt{1 + x^2}\). Suppose that \(\theta\) is the angle across from the side of length \(x\). Then \(\tan(\theta) = \frac{x}{1}\), so \(\arctan(x) = \theta\). Since \(\cos(\theta) = \frac{1}{\sqrt{x^2 + 1}}\), we can conclude the result we wanted.

(7) The population of rabbits on a remote island triples every year. If, at year zero, there are only 10 rabbits, find an exponential equation that gives the number of rabbits \(N\) after \(t\) years.

If the population is tripling every year, then we should use a function of the form \(f(t) = C3^t\) for \(t\) measured in years. Then, for \(t = 0\), we solve \(f(0) = C\) to see that \(C = 10\).

Alternatively, we can use the fact that we want our exponential function \(f(t) = Ca^t\) to go through the points \((0, 10)\) (i.e., there are 10 rabbits at time \(t = 0\)) and \((1, 30)\) (the rabbit numbers have tripled to 30 in one year). Then we conclude that \(f(0) = Ca^0 = C = 10\) and \(f(1) = 10a^1 = 10a = 30\), so \(a = 3\).

(8) Given the scenario above, find a function that gives the year \(t\) there are \(N\) rabbits on the island.

In other words, we want an inverse to \(f(t)\). This is done by solving:

\[
N = 10 \cdot 3^t \\
\frac{N}{10} = 3^t \\
\log_3 \left(\frac{N}{10}\right) = t
\]