(1) Warm-up: with your group members, verbally define the terms: function, domain, range, vertical line test, even/odd function.

These are all in Section 1.1 of the book.

(2) Sketch:
   (a) a curve that is not the graph of a function.
   (b) Sketch the graph of a function with domain \([0, 1]\) and range \([1, 2]\).
   (c) Sketch the graph of a function with domain \([1, 2)\) and range \((-\infty, 0]\).
   (d) Sketch the graph of a function with domain \((0, \infty)\) and range \((0, 1]\).

I provided a couple of solutions for these in class; see me if you are having trouble with any of these.

(3) Critique the following argument:
Since \(x^2 - 2x + 1 = (x - 1)^2\), the functions
\[
f(x) = \frac{x^2 - 2x + 1}{x - 1}
\]
and
\[
g(x) = x - 1
\]
are the same function.

The issue here is that \(g\) has domain all real numbers while \(f\) has domain all real numbers excluding \(x = 1\). So, since \(g(1) = 0\) while \(f(1)\) is not defined, we may conclude they are not the same function.

(4) State the domains of the following functions:
   (a) \(\sqrt[3]{x + 2}\)

Any number (even negative numbers!) has a cube root; for instance, \(\sqrt[3]{-1} = -1\). So this function has domain \((−\infty, \infty)\) (all real numbers.)

   (b) \(\frac{x}{|x|}\)

The only issue is when \(x = 0\), since this is the only case when \(|x| = 0\). So the domain is \((−\infty, 0) \cup (0, \infty)\).

   (c) \(\sqrt{x + 1} + \sqrt{16 - x^2}\)

The domain of this function will be the set of all real numbers that are in the domain of the first summand and in the domain of the second. The first piece has domain \([-1, \infty)\) while the second has domain \([-4, 4]\). So the full domain is \([-1, 4]\).
(d) \(\frac{\sqrt{x^2 - 2}}{x + 8}\)

The numerator has domain \((-\infty, \sqrt{2}] \cup [\sqrt{2}, \infty)\). The denominator has domain all real numbers, though, as usual, we must avoid dividing by zero. So the full function has the domain of the numerator excluding \(x = -8\). This is \((-\infty, -8) \cup (-8, -\sqrt{2}] \cup [\sqrt{2}, \infty)\).

(5) Consider the curve \(y^2 = x - 1\).

(a) Show algebraically that there is no function \(f(x)\) so that the graph \(y = f(x)\) is the curve.

Solving \(y^2 = x - 1\) for \(y\), we get:

\[y = \pm \sqrt{x - 1}\]

Since each value of \(x\) gets associated to two values of \(y\), we can see that this does not correspond to a function.

(b) Sketch a graph of this curve and use the vertical line test to confirm that no such function \(f\) (as mentioned in part a) exists.

I explained how to sketch this in class; see me if you’re having trouble (you should be able to do it without a calculator!)

(c) Note that the curve comes in two pieces. Are these two pieces the graphs of functions? If so, describe these functions.

Yes; the two pieces are: \(f(x) = \sqrt{x - 1}\) and \(g(x) = -\sqrt{x - 1}\). They both have domain \([1, \infty)\).

(6) Explain why an odd function \(f(x)\) with zero in its domain must satisfy \(f(0) = 0\).

If 0 is in the domain of \(f\), then, since 0 = \(-0\), \(f(0) = f(-0)\). Since \(f\) is odd, \(f(-0) = -f(0)\). Therefore, \(f(0) = -f(0)\). The only real number \(c\) satisfying \(-c = c\) is 0.

(7) What special properties must the domains of even or odd functions satisfy? (Hint: if, for instance, 1 is in the domain of \(f\), what can you conclude about \(-1\)?)

If a real number \(c\) is in the domain of an even function \(f\), then \(f(c)\) is defined. But this tells us that \(f(-c) = f(c)\) is defined as well. Therefore, the domain \(D\) of \(f\) must be symmetric across 0 in the real line. The same basic style of argument allows us to conclude the same for an odd function.

(8) For what values of \(m\) and \(b\) is the line \(f(x) = mx + b\) an even function? When is it an odd function? Explain your answers.
If $f(x) = mx + b$ is even, then $f(x) = f(-x)$, so:

$$mx + b = m(-x) + b$$

$$mx = -mx$$

$$m = -m$$

where in the second step we subtract $b$ from both sides and in the third step, since $f$ has domain all real numbers, we may sub in $x = 1$. Again, since $m = -m$ if and only if $m = 0$, we can conclude the only lines that are even functions are the constant functions $f(x) = b$. This makes sense geometrically as lines with non-zero slope must cross the $y$-axis at a tilt and are therefore not going to be symmetric across the $y$-axis.

If $f(x) = mx + b$ is odd, then $f(-x) = -f(x)$, so:

$$m(-x) + b = -(mx + b)$$

$$-mx + b = -mx - b$$

$$b = -b$$

So, we must conclude that $b = 0$. Geometrically, this makes sense as we’ve already shown that odd functions must pass through the origin when their domain includes 0 and the constant $b$ controls where the line $f(x)$ passes through the $y$-axis.

(9) Is there a function that is both even and odd? If so, what is it? Is it the only such function? If there isn’t such a function, can you explain why not?

Well, suppose $f(x)$ is even and odd. Then $f(-x) = f(x)$ and $f(-x) = -f(x)$ for every $x$ in the domain of $f$. Putting these together, we have that $f(x) = -f(x)$ for every $x$ in the domain of $f$. This implies $f(x) = 0$ for every $x$ in the domain of $f$. Since we concluded this for any $f$ that is both even and odd, up to difference in domain (see Question 3!), $f(x) = 0$ is the only function that is both even and odd.