Multicolor on-line degree Ramsey numbers of trees

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Abstract

In the on-line Ramsey game on a family $\mathcal{H}$ of graphs, “Builder” presents edges of a graph one-by-one, and “Painter” colors each edge as it is presented; we require that Builder keep the presented graph in $\mathcal{H}$. Builder wins the game $(G, \mathcal{H})$ if he can ensure that a monochromatic $G$ arises. The $s$-color on-line degree Ramsey number of $G$, denoted $\bar{R}_\Delta(G; s)$, is the least $k$ such that Builder wins $(G, \mathcal{H})$ when $\mathcal{H}$ is the family of graphs having maximum degree at most $k$ and Painter has $s$ colors available. More generally, $R_\Delta(G_1, \ldots, G_s)$ is the minimum $k$ such that Builder can force a copy of some $G_i$ in color $i$ while restricted to those graphs having maximum degree at most $k$.

In this paper, we prove that $\bar{R}_\Delta(T; s) \leq s(\Delta(T) - 1) + 1$ for every tree $T$; this is sharp, with equality whenever $T$ has adjacent vertices of maximum degree. We also give lower and upper bounds on $R_\Delta(G_1, \ldots, G_s)$ when each $G_i$ is a double-star. When each $G_i$ is a star, we determine $R_\Delta(G_1, \ldots, G_s)$ exactly.

1 Introduction

When every 2-edge-coloring of a host graph $H$ contains a monochromatic copy of a target graph $G$, we write $H \rightarrow G$. More generally, when every $s$-edge-coloring of $H$ contains a monochromatic $G$, we write $H \rightarrow^s G$. The central problem of graph Ramsey theory is to find the least $n$ such that $K_n \rightarrow G$. In other words, we seek $\min \{|V(H)| : H \rightarrow G\}$; this is the Ramsey number of $G$ and is denoted $R(G)$ (or $R(G; s)$ in the $s$-color setting). This interpretation of the Ramsey number suggests a natural generalization: given a graph parameter $\rho$, determine $\min \{\rho(H) : H \rightarrow G\}$. When $\rho$ is the maximum degree, this yields the degree Ramsey number, $R_\Delta(G)$. This parameter was introduced by Burr, Erdős, and Lovász [2], who determined $R_\Delta(K_{1,k})$ and $R_\Delta(K_n)$; for further results on degree Ramsey numbers, see [7] and [10].

In this paper, we consider the on-line variant of the degree Ramsey number. Consider a game played by two players, “Builder” and “Painter”, on an infinite set of vertices. In

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each round, Builder introduces an edge and Painter colors it. (The set of colors available to Painter is finite and fixed in advance.) Builder aims to create a monochromatic copy of some fixed graph $G$. When Builder is unrestricted, Ramsey’s Theorem implies that Builder can always win by presenting a large complete graph. Thus we restrict Builder by requiring that the presented graph always belongs to some family $H$; we then say that the game is played on $H$. If Builder can still force a monochromatic $G$, then we say that Builder “wins $(G; H)$”. This model was introduced by Beck [1] and studied by Grytczuk, Hałuszczak, and Kierstead [5] for several natural choices of $H$; additional results appear in [4], [6], [8], and [9].

We focus on the case where $H$ is $S_k$, the set of all graphs having maximum degree at most $k$. We define the [diagonal, $s$-color] on-line degree Ramsey number of $G$, denoted $\hat{R}_\Delta(G; s)$, to be the minimum $k$ such that Builder wins $(G, S_k)$ when Painter has $s$ colors available. When $s$ is 2, we sometimes write simply $\hat{R}_\Delta(G)$. More generally, we say that Builder “wins $(G_1, \ldots, G_s; H)$” when Builder, restricted to playing on $H$, can force a copy of some $G_i$ in some color $i$. We define $\hat{R}_\Delta(G_1, \ldots, G_s)$ to be the least $k$ such that Builder wins $(G_1, \ldots, G_s; S_k)$; this is sometimes called the “non-diagonal” case.

Butterfield et al. [3] proved several results for the diagonal case when $s = 2$; in particular, they determined the exact on-line degree Ramsey numbers for paths, stars, and double-stars (trees with diameter 3), and they proved that $\hat{R}_\Delta(T) \leq 2\Delta(T) - 1$ for every tree $T$. In this paper, we extend several of those results to the $s$-color, non-diagonal setting. Proposition 2.3 states that $\hat{R}_\Delta(P_{n_1}, \ldots, P_{n_s}) = s + 1$ when each $n_i$ is at least 4; this uses a recursive lower bound for $\hat{R}_\Delta(G_1, \ldots, G_s)$ in terms of $\hat{R}_\Delta(G_1, \ldots, G_{s-1})$. Theorem 2.5 gives somewhat technical lower and upper bounds for $\hat{R}_\Delta(G_1, \ldots, G_s)$ when each $G_s$ is a double-star; these bounds coincide when the central vertices of each $G_i$ have identical degrees (Corollary 2.7), and a refined argument shows that the upper bound holds with equality when each $G_i$ is a star (Theorem 2.8). Finally, Theorem 2.9 states that $\hat{R}_\Delta(T_1, \ldots, T_s) \leq \sum_{i=1}^s (\Delta(T_i) - 1) + 1$ whenever each $T_i$ is a tree; this bound holds with equality whenever each $T_i$ has adjacent vertices of maximum degree.

## 2 Results

We begin by introducing some conventions that greatly simplify our arguments.

In the course of a particular game, we often focus our attention on special subgraphs of the presented graph. (Usually such subgraphs are monochromatic.) In such situations, we need to distinguish between the degree of vertex within a subgraph and its degree within the full presented graph. Generally we use “degree” to mean “degree within the particular subgraph under consideration” and “global degree” to mean “degree within the full presented graph”.

Throughout the paper, we may assume that Painter behaves “consistently”. A consistent Painter chooses a color for edge $uv$ based solely on the edge-colored components containing $u$ and $v$. It was shown in [3] that for any graph $G$ and any monotone additive graph family $H$, Builder wins $(G; H)$ if and only if he wins against any consistent Painter. Thus
consistent Painters are no weaker than general Painters, so for the sake of convenience we assume a consistent Painter. By consistency, if Builder repeats the same sequence of moves several times on disjoint sets of vertices, then a consistent Painter produces the same coloring every time. This observation yields the lemma below, which we apply throughout the paper without explicit citation.

**Lemma 2.1.** If Builder can force an edge-colored graph $G$ against a consistent Painter, then he can force arbitrarily many copies of $G$. \(\square\)

Our first result is a general lower bound on $\hat{R}_\Delta(G; s)$. It uses a Painter strategy that generalizes the “greedy $S_k$-Painter” from [3], who colors an edge red when the resulting red subgraph would belong to $S_k$.

**Proposition 2.2.** For any graphs $G_1, \ldots, G_s$, we have

$$\hat{R}_\Delta(G_1, \ldots, G_s) \geq (\hat{R}_\Delta(G_1, \ldots, G_{s-1}) - 1) + \max_{uv \in E(G_s)} \min\{d_{G_s}(u), d_{G_s}(v)\}.$$

**Proof.** Let $d = (\hat{R}_\Delta(G_1, \ldots, G_{s-1}) - 1) + \max_{uv \in E(G_s)} \min\{d(u), d(v)\}$; we provide a strategy for Painter to win on $S_{d-1}$. Painter colors edges using blue and $s - 1$ shades of red. Painter behaves similarly to a greedy $S_{\hat{R}_\Delta(G_1, \ldots, G_{s-1})-1}$-Painter. However, whenever Painter colors an edge red, he chooses the particular shade of red according to some winning strategy for $(G_1, \ldots, G_{s-1}; S_{\hat{R}_\Delta(G_1, \ldots, G_{s-1})-1})$. In this way Painter avoids producing a copy of any $G_i$ in the corresponding shade of red; it suffices to show that also Painter produces no blue $G_s$.

Suppose that Painter has produced a blue copy $H$ of $G_s$. Choose an edge $uv$ in $H$ maximizing $\min\{d_H(u), d_H(v)\}$. Since Painter colored $uv$ blue, one of its endpoints, say $u$, lies on $\hat{R}_\Delta(G_1, \ldots, G_{s-1}) - 1$ red edges. Since $u$ also lies on at least $d_H(u)$ blue edges, it has global degree at least $(\hat{R}_\Delta(G_1, \ldots, G_{s-1}) - 1) + \max_{uv \in E(G_s)} \min\{d_{G_s}(u), d_{G_s}(v)\}$, a contradiction. \(\square\)

As an application of Proposition 2.2, we determine $\hat{R}_\Delta(P_n_1, \ldots, P_n_s)$; the proof of this result introduces techniques used later, in the proof of Theorem 2.9.

**Proposition 2.3.** $\hat{R}_\Delta(P_n_1, \ldots, P_n_s) = s + 1$ when $n_i \geq 4$ for all $i$ in $\{1, \ldots, s\}$.

**Proof.** Letting $n = \max\{n_1, \ldots, n_s\}$, it suffices to prove that $\hat{R}_\Delta(P_n; s) = s + 1$ when $n \geq 4$.

The lower bound follows from Proposition 2.2 and the observation that $\hat{R}_\Delta(P_n; 1) = 2$.

For the upper bound, we provide a strategy for Builder. We use induction on $s$; the observation above establishes the case $s = 1$. Now suppose that Builder can force $P_n$ on $S_s$ when Painter has $s - 1$ colors available. Consider an $s$-color game played on $S_{s+1}$; let the colors used be blue and $s - 1$ shades of red. A consistent Painter uses the same color on all isolated edges; without loss of generality, he colors them blue. It suffices to show that for any $k$, Builder can force either a $P_n$ in some shade of red or a blue $P_{2^k}$ in which each endpoint has global degree 1.

We prove this claim by induction on $k$. The case $k = 1$ is immediate since Painter colors isolated edges blue. For the inductive step, Builder first forces many blue copies of $P_{2^{k-1}}$
having endpoints of global degree 1. Builder next selects one endpoint from each of these blue paths. On these endpoints, Builder plays a strategy that forces $P_n$ on $S_s$ against a Painter that uses $s - 1$ colors. (Since we require that each endpoint of each of the blue paths has global degree 1, Builder can do this while remaining in $S_{s+1}$.) Painter has $s$ colors available: blue and $s - 1$ shades of red. If Painter colors every edge red, then a $P_n$ arises in some shade of red. If instead Painter colors some edge blue, then Builder has connected two blue paths, yielding a blue $P_{2s}$. Since the endpoints of this new path were not touched in the course of the inductive strategy, each still has global degree 1. \hfill \Box

We next turn our attention to stars and double-stars.

**Definition 2.4.** Recall that a double-star is a tree with diameter 3. Such a tree has two central vertices; we denote by $S_{a,b}$ the double-star with central vertices of degrees $a$ and $b$.

**Theorem 2.5.** If $a_i \leq b_i$ for all $i$ in $\{1, \ldots, s\}$, then

$$b_1 - 1 + \sum_{i=2}^{s} (a_i - 1) + 1 \leq \hat{R}_{t}(S_{a_1,b_1}, \ldots, S_{a_s,b_s}) \leq \min_{X \subseteq \{1, \ldots, s\}} f_X(a_1, \ldots, a_s, b_1, \ldots, b_s),$$

generating the double-star with central vertices of degrees $a$ and $b$.

where

$$f_X(a_1, \ldots, a_s, b_1, \ldots, b_s) = \max \left\{ \sum_{i \in X} (b_i - 1) + \sum_{j \not\in X} (a_j - 1) + 1, \sum_{i \in X} (a_i - 1) + \sum_{j \not\in X} (b_j - 1) + 1 \right\}.$$

**Proof.** The lower bound follows by induction on $s$, using Proposition 2.2 and the observation that $\hat{R}_t(S_{a,b}; 1) = b$ when $a \leq b$.

To establish the upper bound, we provide a strategy for Builder. Builder first partitions the set of available colors into some sets $X$ and $Y$. Builder fixes two special vertices, $u$ and $v$; he aims to make them the central vertices of a monochromatic double-star. Let the *quota* of $u$ in color $i$ be $b_i - 1$ if $i \in X$ and $a_i - 1$ if $i \in Y$. For $v$, use the reverse values: let the quota in color $i$ be $a_i - 1$ if $i \in X$ and $b_i - 1$ if $i \in Y$. Whenever either $u$ or $v$ reaches its quota of incident edges in a color $c$, we say that the vertex is *saturated in color $c$*. Note that coloring $uv$ with a color in which both $u$ and $v$ are saturated produces a monochromatic double-star of the desired size in that color.

Initially, Builder takes $u$ and $v$ to be isolated vertices. He then repeats the following process for the remainder of the game. Let $G_u$ and $G_v$ denote the components of the presented graph that contain $u$ and $v$, respectively. Builder presents edge $uv$; suppose Painter colors it with color $c$. If $u$ was not already saturated in $c$, then Builder adds $uv$ and all of $G_v$ to $G_u$, creates new copies of $v$ and $G_v$, and repeats. If $u$ was saturated in $c$ but $v$ was not, then Builder adds $uv$ and all of $G_u$ to $G_v$, creates new copies of $u$ and $G_u$, and repeats. Finally, if both $u$ and $v$ were already saturated in $c$, then $u$ and $v$ are now the central vertices of a monochromatic $S_{a,b}$ in color $c$, so Builder has won.

Since Builder only ever presents edge $uv$, once $G_u$ and $G_v$ have been “recreated” either $u$ or $v$ is a vertex of maximum degree. Just before Builder presents an edge, $u$ has degree at
most $\sum_{i \in X} (b_i - 1) + \sum_{j \notin X} (a_j - 1)$ and $v$ has degree at most $\sum_{i \in X} (a_i - 1) + \sum_{j \notin X} (b_j - 1)$; thus the maximum degree used is at most one more than the maximum of these two quantities. Optimizing over the choice of $X$ yields the stated bound.

In the diagonal case, the minimum over $X$ in the upper bound in Theorem 2.5 occurs whenever $|X| = \lfloor s/2 \rfloor$. This yields a much simpler formula:

**Corollary 2.6.** If $a \leq b$, then $\hat{R}_\Delta(S_{a,b}; s) \leq \lfloor s/2 \rfloor (b - 1) + \lfloor s/2 \rfloor (a - 1) + 1$. \hfill \qed

On the other hand, when $a_i = b_i$ for all $i$, the bounds in Theorem 2.5 coincide:

**Corollary 2.7.** $\hat{R}_\Delta(S_{b_1,b_1}, \ldots, S_{b_s,b_s}) = \sum_{i=1}^s (b_i - 1) + 1$. In particular, $\hat{R}_\Delta(S_{b,b}; s) = s(b - 1) + 1$.

When each double-star is in fact a star, the upper bound in Theorem 2.5 holds with equality.

**Theorem 2.8.** $\hat{R}_\Delta(K_{1,k_1}, \ldots, K_{1,k_s}) = \min_{S \subseteq \{1, \ldots, s\}} \max \left\{ \sum_{i \in S} (k_i - 1), \sum_{i \notin S} (k_i - 1) \right\} + 1$. In particular, $\hat{R}_\Delta(K_{1,k}; s) = \lceil s/2 \rceil (k - 1) + 1$.

**Proof.** The upper bound follows from Theorem 2.5.

For the lower bound, we provide a strategy for Painter to win on $S_{d-1}$, where $d$ is the claimed bound. Call a vertex saturated in color $i$ when it lies on $k_i - 1$ edges of color $i$. Painter’s strategy is straightforward: when Builder presents an edge, Painter colors it with any color in which neither endpoint is already saturated. We claim this is always possible. Consider an edge $uv$ that Builder might present. If no color is available for Painter to use on $uv$, then for each $i$, either $u$ or $v$ must be saturated in color $i$. Let $S$ be the set of colors in which $u$ is saturated; $u$ has degree at least $\sum_{i \in S} (k_i - 1)$. Likewise, since $v$ is saturated in all colors in which $u$ is not, $v$ has degree at least $\sum_{i \notin S} (k_i - 1)$. Thus $u$ or $v$ already has degree at least $d - 1$, so Builder cannot present $uv$. \hfill \qed

The lower bound in Theorem 2.8 can be applied more generally, since $\hat{R}_\Delta$ is monotone: whenever $H_1, \ldots, H_s$ are subgraphs of $G_1, \ldots, G_s$, respectively, we have $\hat{R}_\Delta(H_1, \ldots, H_s) \leq \hat{R}_\Delta(G_1, \ldots, G_s)$. Consequently, always $R_\Delta(G_1, \ldots, G_s) \geq \hat{R}_\Delta(K_{1,\Delta(G_1)}, \ldots, K_{1,\Delta(G_s)})$.

We next turn to general trees. Corollary 2.7 shows that $\hat{R}_\Delta(S_{b,b}; s) = s(b - 1) + 1$; in fact, this is the maximum value of $\hat{R}_\Delta(T; s)$ over all trees with maximum degree $b$. This was shown in [3] for the case $s = 2$; we obtain the general result through a different approach.

**Theorem 2.9.** If $T_1, \ldots, T_s$ are trees, then $\hat{R}_\Delta(T_1, \ldots, T_s) \leq \sum_{i=1}^s (\Delta(T_i) - 1) + 1$. Moreover, this bound is sharp whenever all $T_i$ have adjacent vertices of maximum degree.

**Proof.** The sharpness follows from Corollary 2.7 and the monotonicity of $\hat{R}_\Delta$.

For the upper bound, we provide a strategy for Builder. To simplify notation, let $d$ be the claimed bound, let $k_i = \Delta(T_i)$ and let $h_i = |V(T_i)|$. If each $k_i$ is 1, then Builder wins by presenting a single edge. We proceed by induction on $\sum_i k_i$. If any $k_i$ is 1, then $T_i$ is a single edge, so color $i$ may be ignored: if Painter ever uses that color, then Builder wins. Thus
Builder wins by following a strategy to win \((T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_s; S_d)\), the existence of which is guaranteed by the induction hypothesis. Hence we may suppose that each \(k_i\) is at least 2. Let \(T^{k, h}\) denote the rooted tree in which all non-leaves have degree \(k\) and all leaves lie at distance \(h\) from the root. Since \(T_i \subseteq T^{k_i, h_i}\) for each \(i\), by monotonicity it suffices to show that \(\tilde{R}_\Delta(T^{k_1, h_1}, \ldots, T^{k_s, h_s}) \leq \sum_i (k_i - 1) + 1\).

Suppose that all \(k_i\) are at least 3. (We return later to the case where some of the \(k_i\) are 2.) Builder aims to grow a tree containing \(T^{k_i, h_i}\) in color \(i\), for some \(i\). More precisely, let a \((k, h)\)-subtree be a monochromatic rooted tree with the property that all non-leaves within distance \(h\) of the root have degree \(k\) in the tree and all leaves within distance \(h\) of the root have global degree 1. We claim that if Builder can force a \((k_i, h_i)\)-subtree \(T\) in color \(i\), then he can either win or force a \((k_i, h_i)\)-subtree \(T'\) in color \(i\) having more vertices within distance \(h_i\) of the root. Builder starts the process by presenting \(d\) edges incident to a single vertex. For some \(i\), at least \(k_i\) of these edges have color \(i\); without loss of generality we may suppose this happens in the first color (henceforth “red”). Builder has forced a star with \(k_1\) leaves; call this star \(T\).

Let \(v\) be the root of \(T\). If \(T\) has no leaves with distance less than \(h_1\) from \(v\), then \(T\) contains \(T^{k_1, h_1}\), so Builder wins. Otherwise, let \(x\) be some such leaf. Builder forces many copies of \(T\). By the induction hypothesis, Builder has a strategy to win \((T^{k_1-1, h_1}, T^{k_2, h_2}, \ldots, T^{k_s, h_s}; S_{d-1})\); he plays this strategy on the copies of \(x\) within the copies of \(T\). (Since each copy of \(x\) had global degree 1 when its copy of \(T\) was created, Builder remains within \(S_d\).) Builder either wins the original game or forces a red \(T^{k_1-1, h_1}\). In the latter case, let \(T'\) be the red \(T^{k_1-1, h_1}\) produced, and let \(x'\) be its root. Let \(v'\) be the copy of \(v\) within the copy of \(T\) containing \(x'\), and let \(T'\) be the maximal red tree rooted at \(v'\). All non-leaves of \(T'\) within distance \(h_1\) of \(v'\) lie on \(k_1\) red edges: those that were leaves in their copies of \(T\) lie on \(k_1 - 1\) red edges from \(\hat{T}\) and one red edge from \(T\), while all others were non-leaves in their copies of \(T\). Since leaves of \(\hat{T}\) lie within distance \(h_1\) of \(x'\), their distances from \(v'\) exceed \(h_1\), so their degree in red is unimportant. Every leaf of \(T'\) within distance \(h_1\) of \(v'\) has global degree 1, because each corresponds to a leaf in its copy of \(T\).

Builder now continues with \(T'\) in place of \(T\). Since each iteration increases the number of vertices in \(T\) within distance \(h_1\) of its root, the process eventually stops, at which point Builder wins.

Finally, suppose \(k_i = 2\) for some \(i\). In this case, Builder uses the same strategy as before, but with one small change. Suppose Builder again forces a red star with \(k_1\) leaves. If \(k_1 = 2\), then Builder cannot proceed as before, because \(T^{1, h_1}\) may not be well-defined. Note that \(T_1\) is a path. Since \(k_1 = 2\), Builder has forced a red \(P_3\) in which both endpoints have global degree 1. Builder forces many such copies of \(P_3\); letting \(x\) be one endpoint, Builder plays a winning strategy for \((P_2, T^{k_2, h_2}, \ldots, T^{k_s, h_s}; S_{d-1})\) on copies of \(x\). If Painter uses no red edges, then Builder wins. Otherwise, Builder obtains a longer red path in which both endpoints have global degree 1; he then chooses \(x\) to be one of these endpoints and repeats the process as needed, eventually either winning or obtaining a red \(T_1\).

In the diagonal case, we have the simpler bound below.

**Corollary 2.10.** If \(T\) is a tree, then \(\tilde{R}_\Delta(T; s) \leq s(\Delta(T) - 1) + 1\).
For the sake of comparison between $\hat{R}_\Delta$ and $R_\Delta$, we remark that it was shown in [7] that $R_\Delta(T) \leq 2s(\Delta(T) - 1)$ for any tree $T$; it was shown in [10] that this bound is asymptotically tight. Thus the maximum value of the on-line degree Ramsey number over the class of trees is about half that of the “off-line” degree Ramsey number.

References


