Problem 1. Determine the interval of convergence and the function to which the power series \( \sum_{k=0}^{\infty} (3x + 1)^k \) converges.

Solution. To determine which function this series converges to, we notice that it is a geometric series with \( a = 1 \) and \( r = 3x + 1 \), so
\[
\sum_{k=0}^{\infty} (3x + 1)^k = \frac{1}{1 - (3x + 1)} = \frac{-1}{3x}
\]
Then, using the fact that the geometric series converges for \( |r| < 1 \), we have that the series converges when \( |3x + 1| < 1 \) or \( -1 < 3x + 1 < 1 \), \( -2 < 3x < 0 \), so the series converges when \( -\frac{2}{3} < x < 0 \).

We could also do the ratio test here,
\[
\lim_{k \to \infty} \left| \frac{(3x + 1)^{k+1}}{(3x + 1)^k} \right| = \lim_{k \to \infty} \left| 3x + 1 \right| < 1
\]
This will again give \( -\frac{2}{3} < x < 0 \), but now we have to check the endpoints. If \( x = 0 \), the series is \( \sum 1^k \) which diverges. If \( x = -\frac{2}{3} \), the series is \( \sum (-1)^k \) which diverges. So the interval of convergence is \( -\frac{2}{3} < x < 0 \).

Problem 2. Determine the interval and radius of convergence of \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k4^k} (x + 2)^k \)

Solution. This series is NOT a geometric series, so we have to use the Ratio Test.
\[
\lim_{k \to \infty} \left| \frac{\frac{(-1)^{k+2}}{(k+1)4^{k+1}} (x + 2)^{k+1}}{\frac{(-1)^{k+1}}{k4^k} (x + 2)^k} \right| = \lim_{k \to \infty} \left| \frac{x + 2}{4} \right| \frac{k}{k + 1} < 1
\]
So, $|x + 2| < 4$ or $-4 < x + 2 < 4$, $-6 < x < 2$. Now, we have to check the endpoints. If $x = 2$, the series is $\sum \frac{(-1)^{k+1}}{k}$ which converges. If $x = 6$, the series is $\sum \frac{1}{k}$ which diverges. So the interval of convergence is $(-6, 2]$ and the radius of convergence is 4.

\[ \square \]

**Problem 3.** Find a power series representation of $f(x) = 2 \ln |1 - x|$.

**Solution.** We have a nice power series for $\frac{1}{1-x}$ and $\int \frac{1}{1-x} dx = -\ln |1-x| + C$. So, $-2 \int \frac{1}{1-x} dx = 2 \ln |1 - x| + C$.

\[
2 \ln |1-x| + C = -2 \int \frac{1}{1-x} dx = -2 \int \sum_{n=0}^{\infty} x^n dx
\]
\[
= -2 \sum_{n=0}^{\infty} \int x^n dx
\]
\[
= -2 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}
\]

\[ \square \]