Trigonometric Identities
Math 220 Spring ’09

There are several important trigonometric identities that are useful in calculus.

Pythagorean Identities

The classic Pythagorean Identity on a right triangle says $a^2 + b^2 = c^2$ where $a$ and $b$ are the legs of a right triangle and $c$ is the hypotenuse. If we think of a right triangle on the unit circle with $a = \sin \theta$, $b = \cos \theta$ and $c = 1$, we have

$$\sin^2 \theta + \cos^2 \theta = 1.$$

If we divide both sides of (1) by $\sin^2 \theta$, we get one of the other Pythagorean Identities.

$$1 + \cot^2 \theta = \csc^2 \theta.$$

Similarly, if we divide (1) by $\cos^2 \theta$, we get

$$\tan^2 \theta + 1 = \sec^2 \theta.$$

These three formulas are collectively referred to as the “Pythagorean Identities.”

Angle Addition Formulas

Using Euler’s Formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta),$$

and properties of exponentials, we can derive the angle addition formulas.

$$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$$

$$= [\cos(\alpha) + i \sin(\alpha)] [\cos(\beta) + i \sin(\beta)]$$

$$= [\cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)] + i[\sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)]$$

(2)

Since

$$e^{i(\alpha+\beta)} = \cos(\alpha + \beta) + i \sin(\alpha + \beta),$$

(3)
equating the real and imaginary parts of (2) and (3), we have the angle addition formulas.

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)$$
Half and Double Angle Formulas

We can obtain the Double Angle Formulas from taking \( \alpha = \beta \) in the angle addition formula.

\[
\begin{align*}
\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) \\
\sin(2\alpha) &= 2 \cos(\alpha) \sin(\alpha)
\end{align*}
\]

To derive the half-angle formula, let \( \alpha = \frac{\beta}{2} \), and consider the following.

\[
\begin{align*}
\cos(\beta) &= \cos^2\left(\frac{\beta}{2}\right) - \sin^2\left(\frac{\beta}{2}\right) \\
&= \cos^2\left(\frac{\beta}{2}\right) - \left[1 - \cos^2\left(\frac{\beta}{2}\right)\right]
\end{align*}
\]

Solving for cosine, we have

\[
\cos^2\left(\frac{\beta}{2}\right) = \frac{1}{2} + \frac{1}{2} \cos(\beta).
\]

Similarly, we could have replaced the cosine squared and arrived at

\[
\sin^2\left(\frac{\beta}{2}\right) = \frac{1}{2} - \frac{1}{2} \sin(\beta).
\]