Finding Inverses of Matrices

The motivating question we want to answer is for matrices $A$, $B$, and $C$, when do we have that $AB = AC$ implies that $B = C$. The answer is that this will hold when the matrix $A$ is invertible. This works because

$$B = A^{-1}AB = A^{-1}AC = C$$

Unfortunately, there are matrices that have no inverses. Our general procedure for finding the inverse of a matrix is described as follows using methods of row operations.

Given an $n \times n$ matrix $A$,

1. Form the augmented matrix $[A|I]$.
2. Row reduce $[A|I]$ to reduced row echelon form.
   - If we get a row of zeroes to the left of the vertical line, $A$ is not invertible.
   - Else we have transformed $[A|I] \rightarrow [I|A^{-1}]$.

This works for any size square matrix.

**Formula for $2 \times 2$ Matrices**

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we first have to check that $ad - bc \neq 0$. If $ad - bc \neq 0$, we can find the inverse of $A$. In this case:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Using Matrices to Solve Linear Systems**

For a system of $n$ linear equations in $n$ variables

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_n$$

$$\vdots$$

$$\vdots$$

$$a_{nn}x_1 + \cdots + a_{nn}x_n = b_n$$

We can form the coefficient matrix $A = [a_{ij}]$ and the vectors $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$,
\[ B = \begin{bmatrix}
    b_1 \\
    \vdots \\
    b_n
\end{bmatrix}. \] We can see that the linear system is equivalent to the matrix equation \( AX = B \). So, if \( A \) is invertible, we can see that \( X = A^{-1}B \).