Fine Structure of a Summatory Error Function

12153 [2020, 85]. Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. For a real number $x$ whose fractional part is not 1/2, let $(x)$ denote the nearest integer to $x$. For a positive integer $n$, let

$$a_n = \left( \sum_{k=1}^{n} \frac{1}{\langle \sqrt{k} \rangle} \right) - 2\sqrt{n}.$$ 

(a) Prove that the sequence $a_1, a_2, \ldots$ is convergent, and find its limit $L$.

(b) Prove that the set $\{\sqrt{n}(a_n - L): n \geq 1\}$ is a dense subset of $[0, 1/4]$.

Composite solution by Jean-Pierre Grivaux, Paris, France, Eugene A. Herman, Grinnell College, Grinnell, IA, and the editors. For (a), the limit is $-1$. The density assertion of (b) is true but weak. We prove that in fact the positive real line can be partitioned into successively adjacent intervals $I_1, I_2, \ldots$, with lengths growing only arithmetically, such that the values of the function defined in (b), at the integer points of an individual interval $I_j$ become arbitrarily dense in $[0, 1/4]$ as $j \to \infty$.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), R. Chapman (UK), H. Chau, H. Chen, C. Chiser (Romania), A. Deeb & H. Al-Assad (Syria), A. Dixit, G. Fera & G. Tescaro (Italy), O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, N. Hodges (UK), W. Janous (Austria), P. Lalonde (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), M. A. Prasad (India), C. Schacht, E. Schmeichel, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), T. Wiandt, T. Wilde (UK), Florida State University Problem Solving Group, and the proposer.