Extremal problems on saturation for the family of $k$-edge-connected graphs

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Abstract

Let $F$ be a family of graphs. A graph $G$ is $F$-saturated if $G$ contains no member of $F$ as a subgraph but $G + e$ contains some member of $F$ whenever $e \in E(G)$. The saturation number and extremal number of $F$, denoted sat($n, F$) and ex($n, F$) respectively, are the minimum and maximum numbers of edges among $n$-vertex $F$-saturated graphs. For $k \in \mathbb{N}$, let $F_k$ and $F'_k$ be the families of $k$-connected and $k$-edge-connected graphs, respectively. Wenger proved \text{sat}($n, F_k$) = $(k - 1)n - \binom{k}{2}$; we prove \text{sat}($n, F'_k$) = $(k - 1)(n - 1) - \left\lfloor \frac{n}{k+1} \right\rfloor \binom{k-1}{2}$. We also prove \text{ex}($n, F'_k$) = $(k - 1)n - \binom{k}{2}$ and characterize when equality holds. Finally, we give a lower bound on the spectral radius for $F_k$-saturated and $F'_k$-saturated graphs.

Keywords: saturation number, extremal number, $k$-edge-connected, spectral radius

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1 Introduction

Given a family $F$ of graphs, a graph $G$ is $F$-saturated if (1) no subgraph of $G$ belongs to $F$, and (2) adding to $G$ any edge of its complement $\overline{G}$ completes a subgraph that belongs to $F$. The saturation number of $F$, denoted sat($n, F$), is the least number of edges in an $n$-vertex $F$-saturated graph. The extremal number ex($n, F$) is the maximum number of edges in an

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n-vertex $F$-saturated graph. When $F$ has only one graph $F$, we simply write sat($n,F$) and ex($n,F$), such as when $F$ is $K_t$, the complete graph with $t$ vertices.

Initiating the study of extremal graph theory, Turán [6] determined the extremal number ex($n,K_{r+1}$); the unique extremal graph is the $n$-vertex complete $r$-partite graph whose parts sizes differ by at most 1. Saturation numbers were first studied by Erdős, Hajnal, and Moon [2]; they proved sat($n,K_{k+1}$) = $(k-1)n - \binom{k}{2}$. They also proved that equality holds only for the graph formed from a copy of $K_{k-1}$ with vertex set $S$ by adding $n-k+1$ vertices that each have neighborhood $S$. We call this the complete split graph $S_n,k$; note that $S_n,k$ has clique number $k$ and no $k$-connected subgraph, and $S_{n,2}$ is a star. For an excellent survey on saturation numbers, we refer the reader to Faudree, Faudree, and Schmitt [3].

In this paper, we study the relationship between saturation and edge-connectivity. For a given positive integer $k$, let $F_k$ be the family of $k$-connected graphs, and let $F'_k$ be the family of $k$-edge-connected graphs. Wenger [7] determined sat($n,F_k$). Since $K_{k+1}$ is a minimal $k$-connected graph, it is not surprising that $S_{n,k}$ is also a smallest $F_k$-saturated graph, but in fact the family of extremal graphs is much larger. A $k$-tree is any graph obtained from $K_k$ by iteratively introducing a new vertex whose neighborhood in the previous graph consists of $k$ pairwise adjacent vertices. Note that $S_{n,k}$ is a $(k-1)$-tree.

**Theorem 1.1** (Wenger [7]). sat($n,F_k$) = $(k-1)n - \binom{k}{2}$ when $n \geq k$. Furthermore, every $(k-1)$-tree with $n$ vertices has this many edges and is $F_k$-saturated.

For $n \geq k+1$, we determine sat($F'_k$) and ex($F'_k$). An $F'_k$-saturated graph has no edges, so henceforth we may assume $k \geq 2$. Let $\rho_k(n) = (k-1)(n-1) - \lfloor \frac{n}{k+1} \rfloor \binom{k-1}{2}$. In Section 2, we construct for $n \geq k + 1$ an $F'_k$-saturated graph with $n$ vertices having $\rho_k(n)$ edges, proving sat($n,F'_k$) ≤ $\rho_k(n)$.

Using induction on $n$, in Section 3 we prove that if $G$ is $F'_k$-saturated, then $\rho_k(n) \leq |E(G)| \leq (k-1)n - \binom{k}{2}$, where $E(G)$ denotes the edge set of a graph $G$. Since $S_{n,k}$ is also $F'_k$-saturated, the upper bound is sharp. Thus sat($n,F'_k$) = $\rho_k(n)$ and ex($n,F'_k$) = $(k-1)n - \binom{k}{2}$.

The spectral radius of a graph is the largest eigenvalue of its adjacency matrix. In Section 4, we give a lower bound on the spectral radius for $F_k$-saturated and $F'_k$-saturated graphs.

Additional notation is as follows. For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the degree and the neighborhood of $v$ in $G$, respectively. For $A, B \subseteq V(G)$, let $\overline{A} = V(G) - A$, let $[A,B]$ be the set of edges with endpoints in $A$ and $B$, and let $G[A]$ to denote the subgraph of $G$ induced by $A$. Let $[k] = \{1,2,\ldots,k\}$. Let $K_{k+1}$ denote the graph obtained from $K_{k+1}$ by deleting one edge; this graph is the unique smallest $k$-tree that is not a complete graph.
2 Construction

Recall that \( \rho_k(n) = (k - 1)(n - 1) - \left\lfloor \frac{n}{k+1} \right\rfloor \binom{k-1}{2} \) and that we restrict to \( k \geq 2 \) since \( \mathcal{F}_1' \)-saturated graphs have no edges. In this section, for \( n \geq k + 1 \), we construct an \( n \)-vertex \( \mathcal{F}_k' \)-saturated graph with \( \rho_k(n) \) edges. Since every \( \mathcal{F}_2' \)-saturated graph is a tree (and \( \rho_2(n) = n - 1 \)), we need only consider \( k \geq 3 \).

**Definition 2.1.** Given \( n, k \in \mathbb{N} \) with \( n > k \geq 3 \), let \( t = \left\lfloor \frac{n}{k+1} \right\rfloor \) and \( r = n - t(k + 1) \). Let \( H_{k,i} \) be a copy of \( K_{k+1}^- \) using vertices \( u_{i,1}, \ldots, u_{i,k+1} \), with \( u_{i,1} \) and \( u_{i,k+1} \) nonadjacent. Let \( F_{k,t} \) be the graph obtained from the disjoint union \( H_{k,1} + \cdots + H_{k,t} \) by adding the edge \( u_{i,j}u_{i+1,j} \) for all \( i \) and \( j \) such that \( i \in [t-1] \) and \( j \in [k+1] \setminus \{2, k\} \). Let \( G_{k,n} \) be the graph obtained from \( F_{k,t} \) by adding \( r \) new vertices, each having neighborhood \( V(H_{k,t}) \setminus \{u_{t,1}, u_{t,k+1}\} \).

![Figure 1: The graph \( G_{k,n} \).](image)

\[ \text{Figure 1: The graph } G_{k,n}. \]

**Proposition 2.2.** For \( n > k \geq 3 \), the graph \( G_{k,n} \) is \( \mathcal{F}_k' \)-saturated and has \( n \) vertices and \( \rho_k(n) \) edges.

*Proof.\* Since \( n = t(k + 1) + r \), the graph \( G_{k,n} \) has \( n \) vertices.

In \( G_{k,n} \), the vertices \( w_1, \ldots, w_r \) have degree \( k - 1 \) and hence cannot lie in a \( k \)-edge-connected subgraph. In \( F_{k,t} \), the edges joining \( V(H_{k,i}) \) and \( V(H_{k,i+1}) \) form a cut of size \( k - 1 \), so any \( k \)-edge-connected subgraph of \( G_{k,n} \) is contained in just one copy of \( K_{k+1}^- \). However, \( K_{k+1}^- \) has two vertices of degree \( k - 1 \), leaving only \( k - 1 \) other vertices. Hence \( G_{k,n} \) has no \( k \)-edge-connected subgraph.

In \( F_{k,t} \), there are \( t \left\lfloor \binom{k+1}{2} - 1 \right\rfloor + (k - 1)(t - 1) \) edges. The added vertices \( w_1, \ldots, w_r \)
contribute \( r(k - 1) \) more edges. Since \( n = t(k + 1) + r \), we compute

\[
|E(G_{k,n})| = t \left[ \left( \frac{k+1}{2} \right) - 1 \right] + (k-1)(t+r-1) = \frac{t}{2}k^2 + 3k - 4 + (k-1)(r-1)
\]

\[
= t \left( \frac{k-1}{2} \right)(k+4) + (k-1)(r-1) = (k-1)[t(k+1)+r-1] - t \left( \frac{k-1}{2} \right)
\]

\[
= (k-1)(n-1) - t \left( \frac{k-1}{2} \right) = \rho_k(n).
\]

Let \( xy \) be an edge in the complement of \( G_{k,n} \). It remains to show that the graph \( G' \) obtained by adding \( xy \) to \( G_{k,n} \) has a \( k \)-edge-connected subgraph. Note that the subgraph of \( G_{k,n} \) induced by \( V(H_{k,i}) \cup \{w_1, \ldots, w_r\} \) is the \( K_{k+1} \)-saturated graph \( S_{n,k} \) of [2], so \( G' \) contains \( K_{k+1} \) when \( x \) and \( y \) lie in this set. Similarly, if \( xy \) is the one missing edge of \( H_{k,i} \), then \( G' \) again contains \( K_{k+1} \). This leaves two nontrivial cases, by symmetry.

**Case 1:** \( x \in V(H_{k,i}) \) and \( y \in V(H_{k,j}) \) with \( 1 \leq i < j \leq t \). Since for each \( i \in [t] \), \( \kappa'(H_{k,i}) = k-1 \) and there are exactly \( k-1 \) edges between \( H_{k,j} \) and \( H_{k,j+1} \) for every \( j \in [t-1] \), the graph induced by \( \bigcup_{i=1}^{j} V(H_{k,i}) \) in \( G' \) is \( k \)-edge-connected. ***It is not clear that the reasons cited imply this conclusion. In any case, proof is needed.***

**Case 2:** \( x \in V(H_{k,i}) \) and \( y = w_j \) with \( i \in [t-1] \) and \( j \in [r] \). The subgraph of \( G' \) induced by \( \bigcup_{i=1}^{j} V(H_{k,i}) \cup \{y\} \) is \( k \)-edge-connected. ***No proof of this has been given.***

By Proposition 2.2, \( \text{sat}(n, F'_k) \leq \rho_k(n) \). Thus \( \text{sat}(n, F'_k) \) is much smaller than \( \text{sat}(n, F_k) \) when \( n \) is much larger than \( k \). Indeed, \( G_{k,n} \) is not \( F \)-saturated. In particular, adding the edge \( xy \) does not create a \( k \)-edge-connected subgraph.

3. **Saturation and extremal number of \( F'_k \)**

In this section, we show that if \( G \) is an \( F'_k \)-saturated \( n \)-vertex graph with \( n \geq k + 1 \), then \(|E(G)| \geq \rho_k(n)|. First, we investigate the properties of an \( F'_k \)-saturated graph.

**Lemma 3.1.** If \( G \) is \( F'_k \)-saturated and has more than \( k \) vertices, then \( \kappa'(G) = k-1 \).

**Proof.** Since \( G \) has no \( k \)-edge-connected subgraph, \( \kappa'(G) \leq k-1 \). If \( \kappa'(G) < k-1 \), then \( G \) has an edge cut \([S, \overline{S}]\) of size less than \( k-1 \). Since \(|V(G)| > k \), there are at least \( k \) pairs \((x, y)\) with \( x \in S \) and \( y \in \overline{S} \). Hence there is such a pair \((x, y)\) with \( xy \notin E(G) \). Let \( G' \) be the graph obtained by adding the edge \( xy \) to \( G \).

Since \( G \) has no \( k \)-edge-connected subgraph, any such subgraph of \( G' \) must contain the edge \( xy \). Hence it contains \( k \) edge-disjoint paths with endpoints \( x \) and \( y \), by Menger’s Theorem. Besides the edge \( xy \), there must be at least \( k-1 \) with endpoints \( x \) and \( y \) that
use edges of $[S, \overline{S}]$. This contradicts $|[S, \overline{S}]| < k - 1$. Hence $G'$ has no $k$-edge-connected subgraph, and $G$ cannot be $\mathcal{F}_k'$-saturated. \hfill \square

**Lemma 3.2.** Assume $k \geq 3$, and let $G$ be a $\mathcal{F}_k'$-saturated graph with at least $k + 2$ vertices. If $S$ is a vertex subset in $V(G)$ such that $|[S, \overline{S}]| = k - 1$ and $|S| \geq |\overline{S}|$, then $G[S]$ is a $\mathcal{F}_k'$-saturated graph with at least $k + 1$ vertices, and $G[\overline{S}]$ is $K_1$ or is a $\mathcal{F}_k'$-saturated graph with at least $k + 1$ vertices.

**Proof.** First, we prove for $T \in \{S, \overline{S}\}$ that the induced subgraph $G[T]$ is a complete subgraph or is $\mathcal{F}_k'$-saturated with at least $k + 1$ vertices. If $G[T]$ is not complete, then take $e \in E(G[T])$, and let $G'$ be the graph obtained from $G$ by adding $e$. Since $G$ is $\mathcal{F}_k'$-saturated, $G'$ contains a $k$-edge-connected subgraph $H$, and $e \in E(H)$. Since $|[T, \overline{T}]| = k - 1$, no vertex of $H$ lies in $\overline{T}$. Hence $H \subseteq G[T]$, which implies that $G[T]$ is $\mathcal{F}_k'$-saturated. Since $G[T]$ is not complete, that requires $|T| \geq k + 1$.

By the preceding paragraph and the fact that $K_{k+1}$ is $k$-edge-connected, it now suffices to show that $G[S]$ and $G[\overline{S}]$ cannot both be complete graphs. Suppose that they are. Since $G$ has no $k$-edge-connected subgraph and $|V(G)| \geq k + 2$, we have $|S| \leq k$ and thus $|\overline{S}| \geq 2$.

A $k$-edge-connected subgraph $H$ in the graph formed by adding an edge $e$ to $G$ must contain $e$. Hence $\delta(G) \geq k - 1$. The vertex of $\overline{S}$ incident to the fewest edges of $[S, \overline{S}]$ has degree at most $\left\lfloor \frac{k-1}{j} \right\rfloor + j - 1$, where $j = |\overline{S}|$. Since $j \geq 2$, we thus have $j \geq k - 1$.

If $j = k - 1$, then $\delta(G) \geq k - 1$ requires each vertex of $\overline{S}$ to be incident to exactly one edge of the cut. Adding an edge across the cut then increases the degree of only one vertex of $\overline{S}$ to $k$. Hence only that vertex can lie in $H$, which restricts its degree in $H$ to $1$.

We may therefore assume $|\overline{S}| = |S| = k$. Since $|[S, \overline{S}]| = k - 1$, some $v \in \overline{S}$ has degree $k - 1$, and every vertex of $\overline{S}$ has a nonneighbor in $S$. Choose $y \in \overline{S}$ with $y \neq v$, and choose $x \in S$ with $xy \notin E(G)$. The new $k$-edge-connected subgraph $H$ cannot contain $v$. If $H$ has $j$ vertices in $\overline{S} - \{v, y\}$, then the vertex among these with least degree in $H$ has degree at most $\left\lfloor \frac{k-1}{j} \right\rfloor + j$ in $H$. Since $j \leq k - 2$ and $\delta(H) \geq k$, we have $j \in \{0, 1\}$.

If $j = 0$, then $V(H) \cap \overline{S} = \{y\}$, and all edges of $[S, \overline{S}]$ are incident to $y$. If $j = 1$, then with $z$ being the vertex of $\overline{S}$ other than $y$ in $H$, all $k - 1$ edges of $[S, \overline{S}]$ are incident to $z$. Hence in either case we have a single vertex of $\overline{S}$ incident to all edges of the cut.

Since $|\overline{S}| = |S| = k$, the same argument applies to $S$. With only one vertex on each side incident to edges of the cut, we have $k - 1 \leq 1$, which contradicts $k \geq 3$. \hfill \square

**Lemma 3.3.** If $G$ is an $n$-vertex $\mathcal{F}_k'$-saturated graph with $n \geq k + 1$, then $G$ contains $K_{k+1}^-$. 

**Proof.** We use induction on $n$, the number of vertices. The claim holds when $n = k + 1$, since $K_{k+1}^-$ is the only $\mathcal{F}_k'$-saturated graph with $k + 1$-vertices.
Now consider \( n \geq k + 2 \). Since \( \kappa'(G) = k - 1 \) by Lemma 3.1, there exists \( S \subseteq V(G) \) such that \( |[S, \overline{S}]| = k - 1 \) and \( |S| \geq |\overline{S}| \). By Lemma 3.2, \( |S| \geq k + 1 \) and \( G[S] \) is \( F'_k \)-saturated. By the induction hypothesis, \( G[S] \) (and hence also \( G \)) contains \( K_{k+1}^- \).

The lemmas imply the main result of this section.

**Theorem 3.4.** For \( n \in \mathbb{N} \), with \( t = \left\lfloor \frac{n}{k+1} \right\rfloor \),

\[
\text{sat}(n, F'_k) = (k - 1)(n - 1) - t \binom{k - 1}{2},
\]

with equality achieved for \( k = 1 \) by \( K_n \), for \( k = 2 \) by trees, and for \( k \geq 3 \) by \( G_{k,n} \).

**Proof.** Since \( G_{k,t,r} \) is \( F'_k \)-saturated and \( |E(G_{k,t,r})| = \left( \frac{k^2 + 3k - 4}{2} \right) t + (r - 1)(k - 1) \), we have

\[
\text{sat}(n, F'_k) \leq \left( \frac{k^2 + 3k - 4}{2} \right) t + (r - 1)(k - 1).
\]

To prove that

\[
\text{sat}(n, F'_k) \geq \left( \frac{k^2 + 3k - 4}{2} \right) t + (r - 1)(k - 1),
\]

we use induction on \( n \). The bound holds for \( n = k + 1 \), as the only \( F'_k \)-saturated graph on \( k + 1 \)-vertices is \( K_{k+1} - e \). Let \( G \) be a \( F'_k \)-saturated graph with \( |V(G)| \geq k + 2 \). Since \( \kappa'(G) = k - 1 \) by Lemma 3.1, there exists \( S \subseteq V(G) \) such that \( |[S, \overline{S}]| = k - 1 \) with \( |S| \geq |\overline{S}| \). By Lemma 3.2, we have \( G[S] \) is \( F'_k \)-saturated and \( G[\overline{S}] \) is \( F'_k \)-saturated or is an isolated vertex. Let \( |S| \equiv r_1 \pmod{k + 1} \) and \( |\overline{S}| \equiv r_2 \pmod{k + 1} \). Since \( G[S] \) is \( F'_k \)-saturated, by the induction, we have

\[
E(G[S]) \geq \left( \frac{k^2 + 3k - 4}{2} \right) \left( \frac{|S| - r_1}{k + 1} \right) + (r_1 - 1)(k - 1).
\]

**Case 1.** Suppose \( G[\overline{S}] \) is an isolated vertex.

If \( r \geq 1 \), then we have \( r_1 = r - 1 \). Hence

\[
|E(G)| = |E(G[S])| + |E(G[\overline{S}])| + |[S, \overline{S}]| \geq \left( \frac{k^2 + 3k - 4}{2} \right) \left( \frac{|S| - r_1}{k + 1} \right) + (r_1 - 1)(k - 1) + (k - 1) = \left( \frac{k^2 + 3k - 4}{2} \right) t + (r - 1)(k - 1).
\]
If \( r = 0 \), then we have \( r_1 = r + k \). Hence
\[
|E(G)| = |E(G[S])| + |E(G[S])| + |S, S| \\
\geq \left( \frac{k^2 + 3k - 4}{2} \right) \left( \frac{|S| - r_1}{k + 1} \right) + (r_1 - 1)(k - 1) + (k - 1) \\
= \left( \frac{k^2 + 3k - 4}{2} \right) \left( \frac{t + (n - 1) - (r + k)}{k + 1} \right) + (r + k - 1)(k - 1) + (k - 1) \\
= \left( \frac{k^2 + 3k - 4}{2} \right) t + (r - 1)(k - 1) + \frac{k^2 - 3k + 2}{2} \\
\geq \left( \frac{k^2 + 3k - 4}{2} \right) t + (r - 1)(k - 1).
\]

**Case 2.** Suppose \( G[S] \) is \( \mathcal{F}_k^t \)-saturated.

By the induction, we have
\[
E(G[S]) \geq \left( \frac{k^2 + 3k - 4}{2} \right) \left( \frac{|S| - r_2}{k + 1} \right) + (r_2 - 1)(k - 1).
\]

If \( r_1 + r_2 \leq k \), then we have \( r_1 + r_2 = r \). Hence
\[
|E(G)| = |E(G[S])| + |E(G[S])| + |S, S| \\
\geq \left( \frac{k^2 + 3k - 4}{2} \right) \left( \frac{|S| - r_1}{k + 1} \right) + (r_1 - 1)(k - 1) \\
+ \left( \frac{k^2 + 3k - 4}{2} \right) \left( \frac{|S| - r_2}{k + 1} \right) + (r_2 - 1)(k - 1) + (k - 1) \\
= \left( \frac{k^2 + 3k - 4}{2} \right) t + (r - 1)(k - 1).
\]

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If \( r_1 + r_2 \geq k + 1 \), then we have \( r_1 + r_2 = r + k + 1 \). Hence

\[
|E(G)| = |E(G[S])| + |E(G[S])| + |[S, \overline{S}]| \\
\geq \left( \frac{k^2 + 3k - 4}{2} \right) \left( \frac{|S| - r_1}{k + 1} \right) + (r_1 - 1)(k - 1) \\
+ \left( \frac{k^2 + 3k - 4}{2} \right) \left( \frac{|\overline{S}| - r_2}{k + 1} \right) + (r_2 - 1)(k - 1) + (k - 1) \\
= \left( \frac{k^2 + 3k - 4}{2} \right) \left( \frac{n - (r + k + 1)}{k + 1} \right) + (r + k - 1)(k - 1) + (k - 1) \\
= \left( \frac{k^2 + 3k - 4}{2} \right) t + (r - 1)(k - 1) + \frac{k^2 - 3k + 2}{2} \\
\geq \left( \frac{k^2 + 3k - 4}{2} \right) t + (r - 1)(k - 1).
\]

This completes the proof of Theorem 3.4. \( \square \)

Now we prove that if \( G \) is \( \mathcal{F}_k' \)-saturated, then \( |E(G)| \leq (k - 1)|V(G)| - \binom{k}{2} \). Before proving it, note that \( \text{ex}(n, \mathcal{F}_1') = 0 \) and \( \text{ex}(n, \mathcal{F}_2') = n - 1 \), thus we assume that \( k \geq 3 \).

**Theorem 3.5.** For \( n \geq k + 1 \), we have

\[
\text{ex}(n, \mathcal{F}_k') = (k - 1)n - \binom{k}{2}.
\]

Furthermore, equality holds only when \( G \) is a graph obtained from a \( \mathcal{F}_k' \)-saturated graph \( H \) with at least \( k + 1 \) vertices by adding \( k - 1 \) edges from a single vertex not in \( V(H) \) to \( H \).

**Proof.** We prove induction on \( |V(G)| \). When \( |V(G)| = k + 1 \), there is the unique \( \mathcal{F}_k' \)-saturated graph.

Now, assume that \( |V(G)| > k + 1 \). By Lemma 3.2, there exists a vertex subset \( S \) such that \( G[S] \) is \( \mathcal{F}_k' \)-saturated such that \( |G[S]| \geq k + 1 \) and \( G[\overline{S}] \) is \( \mathcal{F}_k' \)-saturated such that \( G[\overline{S}] \geq k + 1 \) or is an isolated vertex. If \( G[S] \) and \( G[\overline{S}] \) are both \( \mathcal{F}_k' \)-saturated graphs with at least \( k + 1 \) vertices, by the induction, we have

\[
|E(G)| = |E(G[S])| + |E(G[\overline{S}]|) + k - 1 \\
\leq (k - 1)s - \binom{k}{2} + (k - 1)(n - s) - \binom{k}{2} + k - 1 < (k - 1)n - \binom{k}{2},
\]

since \( k \geq 3 \).
If $G[S]$ is a $\mathcal{F}_k'$-saturated graphs with at least $k + 1$ vertices, and if $G[S]$ is a single vertex, by the induction, we have

$$|E(G)| = |E(G[S])| + k - 1$$

$$\leq (k - 1)(n - 1) - \binom{k}{2} + k - 1 = (k - 1)n - \binom{k}{2}.$$

Equality holds only when $G$ is a graph obtained from a $\mathcal{F}_k'$-saturated graph $H$ with at least $k + 1$ vertices by adding $k - 1$ edges from a single vertex not in $V(H)$ to $H$. \qed

## 4 Spectral radius and $\mathcal{F}_k'$-saturated graphs

In this section, we give a necessary condition related to the spectral radius for $\mathcal{F}_k'$-saturated graphs.

Let $\lambda_1(G)$ be the spectral radius of $G$. The following two lemmas are well-known in spectral graph theory.

**Lemma 4.1** ([4]). If $H$ is a subgraph of $G$, then $\lambda_1(H) \leq \lambda_1(G)$.

**Lemma 4.2** ([1]). For any graph $G$,

$$\frac{2|E(G)|}{|V(G)|} \leq \lambda_1(G) \leq \Delta(G)$$

with equality if and only if $G$ is regular.

Given a graph $G$, let $P = \{V_1, V_2, \ldots, V_t\}$ be a vertex partition of $V(G)$. The quotient matrix $Q$ corresponding to the vertex partition $P$ is defined as $Q_{ij} = \frac{|[V_i, V_j]|}{|V_i|}$, if $i \neq j$; $Q_{ii} = \frac{2|E(G[V_i])|}{|V_i|}$. The vertex partition $P$ is an equitable partition if for any $1 \leq i, j \leq k$ and $v \in V_i$, we have $|N_G(v) \cap E_G(V_i, V_j)| = Q_{ij}$.

**Lemma 4.3** ([4]). Let $P = \{V_1, V_2, \ldots, V_t\}$ be an equitable partition of $V(G)$. If $Q$ is the quotient matrix corresponding to the partition $P$, then $\lambda_1(Q) = \lambda_1(G)$.

**Theorem 4.4.** If $G$ is $\mathcal{F}_k'$-saturated, then $\lambda_1(G) \geq \frac{k - 2 + \sqrt{k^2 + 4k - 4}}{2}$.

**Proof.** First we prove $\lambda_1(K_{k+1} - e) = \frac{k - 2 + \sqrt{k^2 + 4k - 4}}{2}$. Suppose that $V(K_{k+1} - e) = \{x_1, \ldots, x_{k+1}\}$, $\deg(x_1) = \deg(x_{k+1}) = k - 1$ and $\deg(x_i) = k$ for all $i \in \{2, \ldots, k\}$. Let $V_1 = \{x_1, x_{k+1}\}$ and $V_2 = \{x_2, \ldots, x_k\}$ be an equitable partition of $K_{k+1} - e$. Thus the quotient matrix $Q$ corresponding to the partition $P = \{V_1, V_2\}$ is

$$Q = \begin{pmatrix}
0 & 2 \\
2 & k - 2
\end{pmatrix}. \quad (1)$$
Thus we obtain
\[ \lambda_1(Q) = \frac{k - 2 + \sqrt{k^2 + 4k - 4}}{2}. \]

By Lemma 4.3, we have
\[ \lambda_1(K_{k+1} - e) = \lambda_1(Q) = \frac{k - 2 + \sqrt{k^2 + 4k - 4}}{2}. \]

Since \( G \) is \( F_k \)-saturated, by Lemma 3.3, \( K_{k+1} - e \) is a subgraph of \( G \). Thus, by Lemma 4.1, we have
\[ \lambda_1(G) \geq \lambda_1(K_{k+1} + e) = \frac{k - 2 + \sqrt{k^2 + 4k - 4}}{2}. \]
\[ \square \]

**Theorem 4.5.** If \( G \) is \( F_k \)-saturated with \( n \) vertices, then
\[ \lambda_1(G) \geq \frac{k - 2 + \sqrt{k^2 + 4k - 4}}{2}. \]

*Proof.* Suppose that \( G \) is \( F_k \)-saturated. By Theorem 1.1, we have
\[ |E(G)| \geq (k - 1)n - \binom{k}{2}. \]

By Lemma 4.2,
\[ \lambda_1(G) \geq \frac{2|E(G)|}{n} \geq \frac{2(k - 1)n - 2\binom{k}{2}}{n} = 2(k - 1) - \frac{k(k - 1)}{n}. \]

Suppose \( n = k + 1 \), then \( K_{k+1} - e \) is the only \( F_k \)-saturated graph. By Theorem 4.4, we have \( \lambda_1(K_{k+1} - e) = \frac{k - 2 + \sqrt{k^2 + 4k - 4}}{2} \). Suppose \( n \geq k + 2 \). For \( k = 1 \), \( \frac{k - 2 + \sqrt{(k+2)^2 - 8}}{2} = 0 \). For \( k = 2, 3 \), we obtain
\[ 2(k - 1) - \frac{k(k - 1)}{n} \geq \frac{k - 2 + \sqrt{k^2 + 4k - 4}}{2}. \]

For \( k \geq 4 \), since
\[ k > \frac{k - 2 + \sqrt{k^2 + 4k - 4}}{2}. \]

It is sufficient to prove
\[ 2(k - 1) - \frac{k(k - 1)}{n} \geq k. \]

Thus, we have
\[ 2(k - 1) - \frac{k(k - 1)}{n} - k = k - 2 - \frac{k(k - 1)}{n} \geq k - 2 - \frac{k(k - 1)}{k + 2} = \frac{k - 4}{k + 2} \geq 0. \]

This completes the proof. \( \square \)
For $t \geq 3$, let $\mathcal{F}_{d,t}$ be the family of $d$-regular simple graphs $H$ with $\kappa'(H) \leq t$. Recently, with Hyun, Park, Park, and Yu, the second author [5] proved that the minimum of the second largest eigenvalue over $\mathcal{F}_{d,t}$ is the second largest one of a smallest graph in $\mathcal{F}_{d,t}$. Theorem 4.4 and 4.5 also say that the minimum of the spectral radius over $\mathcal{F}$-saturated graphs and $\mathcal{F}'$-saturated graphs are the spectral radius of the smallest graph in the families of $\mathcal{F}$-saturated graphs and $\mathcal{F}'$-saturated graphs, respectively.

References


