On the bar visibility number of complete bipartite graphs

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Abstract

A $t$-bar visibility representation of a graph assigns each vertex up to $t$ horizontal bars in the plane so that two vertices are adjacent if and only if some bar for one vertex can see some bar for the other via an unobstructed vertical channel of positive width. The least $t$ such that $G$ has a $t$-bar visibility representation is the bar visibility number of $G$, denoted by $b(G)$. For the complete bipartite graph $K_{m,n}$, the lower bound $b(K_{m,n}) \geq \lceil \frac{mn+4}{2m+2n} \rceil$ from Euler’s Formula is well known. We prove that equality holds.

Keywords: bar visibility number; bar visibility graph; planar graph; thickness; complete bipartite graph.
MSC Codes: 05C62, 05C10

1 Introduction

In computational geometry, graphs are used to model visibility relations in the plane. For example, we may say that two vertices of a polygon “see” each other if the segment joining them lies inside the polygon. In the visibility graph on the vertex set, vertices are adjacent if they see each other. More complicated notions of visibility have been defined for families of rectangles and other geometric objects. Dozens of papers have been written concerning the computation and the recognition of visibility graphs and discussing applications to search

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problems and motion planning. For a textbook on algorithms for visibility problems, see Ghosh [6].

We consider visibility among horizontal segments in the plane. A graph \(G\) is a bar visibility graph if each vertex can be assigned a horizontal line segment in the plane (called a bar) so that vertices are adjacent if and only if the corresponding bars can see each other along an unobstructed vertical channel with positive width. The assignment of bars is a bar visibility representation of \(G\). The condition on positive width allows bars \([(a, y), (x, y)]\) and \([(x, z), (c, z)]\) to block visibility at \(x\) without seeing each other.


**Theorem 1** ([9, 11]). A graph \(G\) has a bar visibility representation if and only if for some planar embedding of \(G\) all cut-vertices appear on the boundary of one face.

Theorem 1 is quite restrictive. Nevertheless, assigning multiple bars to vertices permits representations of all graphs and leads to a complexity parameter measuring how many bars are needed per vertex, introduced by Chang, Hutchinson, Jacobson, Lehel, and West [4].

**Definition 2** ([4]). A \(t\)-bar visibility representation of a graph assigns to each vertex at most \(t\) horizontal bars in the plane so that vertices are adjacent if and only if some bar assigned to one sees some bar assigned to the other via an unobstructed vertical channel of positive width. The bar visibility number of a graph \(G\), denoted by \(b(G)\), is the least integer \(t\) such that \(G\) has a \(t\)-bar visibility representation.

Results in [4] include the determination of visibility number for planar graphs (always at most 2), plus \(b(K_n) = \lceil n/6 \rceil\) for \(n \geq 7\), the determination of \(b(K_{m,n})\) within 1, and \(b(G) \leq \lceil n/6 \rceil + 2\) for every \(n\)-vertex graph \(G\). Results on the visibility numbers for hypercubes [10] and an analogue for directed graphs [1] have also been obtained. For complete bipartite graphs, the result was as follows.

**Lemma 3** ([4]). \(r \leq b(K_{m,n}) \leq r + 1\), where \(r = \lceil mn + 4 \over 2m + 2n \rceil\).

The lower bound is simple. Given a \(t\)-bar representation, add edges to encode visibilities that produce edges of \(K_{m,n}\), and then shrink bars to single points. The result is a bipartite plane graph \(G\) with at most \(t(m+n)\) vertices and at least \(mn\) edges. Hence \(mn \leq 2t(m+n) - 4\) by Euler’s Formula, so \(b(K_{m,n}) \geq r\). Achieving equality requires almost all faces in \(G\) to have length 4.

In this paper, we prove \(b(K_{m,n}) = r\). Section 2 contains a short proof valid for \(K_{n,n}\). For this case, it suffices to decompose the graph into \(r\) bar visibility graphs, where a decomposition of \(G\) is a set of edge-disjoint subgraphs whose union is \(G\). The subgraphs can then be represented with disjoint projections on the horizontal axis. In Section 3, we present a different approach that solves the problem for all complete bipartite graphs.
2 The bar visibility number of $K_{n,n}$

The thickness $\theta(G)$ of a graph $G$ is the minimum number of planar subgraphs needed to decompose $G$. Beineke, Harary, and Moon [3] determined $t(K_{m,n})$ for most $m$ and $n$.

Lemma 4 ([3]). $\theta(K_{n,n}) = \lceil \frac{n+2}{4} \rceil$.

When $\theta(G)$ is the desired value for $b(G)$, we aim to decompose $G$ into that number of bar visibility graphs. The difficult case is when $b(G) < \theta(G)$.

Theorem 5. $b(K_{n,n}) = \lceil \frac{n+1}{4} \rceil$, except for $b(K_{3,3}) = 2$.

Proof. It is immediate that $K_{1,1}$ and $K_{2,2}$ are bar visibility graphs. Since $K_{3,3}$ is not planar, $b(K_{3,3}) \geq 2$; equality holds because $K_{3,3}$ decomposes into a 6-cycle and a matching of size 3, both of which are bar visibility graphs. Hence we may assume $n \geq 4$.

Let $r = \lceil (n+1)/4 \rceil$. When $\theta(K_{n,n}) = r$, we will decompose $K_{n,n}$ into $r$ bar visibility graphs. This will leave the case where $n \equiv 3 \mod 4$, in which case $r < \theta(K_{n,n})$ and $K_{n,n}$ cannot decompose into $r$ bar visibility graphs. Let $U$ and $V$ be the parts of $K_{n,n}$, with $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. Let $p = \lfloor n/4 \rfloor$.

For $n \equiv 0 \mod 4$, Chen and Yin [5] provided a decomposition of $K_{n,n}$ into $p + 1$ planar subgraphs $\{G_1, \ldots, G_{p+1}\}$. Let $[p] = \{1, \ldots, p\}$. For $1 \leq j \leq p$, let $U_i^j = \bigcup_{i \in [p]-\{j\}} \{u_{4i-3}, u_{4i-2}\}$, let $U_2^j = \bigcup_{i \in [p]-\{j\}} \{u_{4i-1}, u_{4i}\}$, let $V_i^j = \bigcup_{i \in [p]-\{j\}} \{u_{4i-3}, u_{4i-1}\}$, and let $V_2^j = \bigcup_{i \in [p]-\{j\}} \{u_{4i-2}, u_{4i}\}$.

Figure 1 shows the subgraph $G_j$, for $1 \leq j \leq p$. Being a 2-connected planar graph, it is a bar visibility graph. The subgraph induced by the eight special vertices $u_{4j-3}, \ldots, u_{4j}$ and $v_{4j-3}, \ldots, v_{4j}$ is $K_{4,4}$ minus the edges of the form $u_iv_i$. The remaining graph $G_{p+1}$ is the matching consisting of $u_iv_i$ for $1 \leq i \leq 4p$. Again this is a bar visibility graph.

![Figure 1: The graph $G_j$ in a planar decomposition of $K_{4p,4p}$](image)

For $n = 4p + 1$, we add two vertices $u_{4p+1}$ and $v_{4p+1}$, with $u_{4p+1}$ adjacent to $V$ and $v_{4p+1}$ adjacent to $U$. The edges incident to $u_{4p+1}$ and $v_{4p+1}$ can be added to the graph $G_{p+1}$ of the
previous case, as shown in Figure 2. Again this graph is planar and 2-connected, so again we have a decomposition \( \tilde{G}_1, \ldots, \tilde{G}_{p+1} \) into \( p + 1 \) bar visibility graphs.

\[
\begin{array}{c}
\text{Figure 2: The subgraph } \tilde{G}_{p+1} \text{ in the planar decomposition of } K_{4p+1,4p+1}
\end{array}
\]

For \( n = 4p + 2 \), we modify the decomposition given for \( K_{4p,4p} \) to accommodate the edges incident to \( \{u_{4p+1}, u_{4p+2}, v_{4p+1}, v_{4p+2}\} \). First form \( \tilde{G}_{p+1} \) by adding to the matching \( G_{p+1} \) the edges joining \( u_{4p+1} \) to \( \bigcup_{i \in [p]} \{v_{4i-2}, v_{4i}\} \), joining \( u_{4p+2} \) to \( \bigcup_{i \in [p]} \{v_{4i-3}, v_{4i-1}\} \), joining \( v_{4p+1} \) to \( \bigcup_{i \in [p]} \{v_{4i-2}, v_{4i-3}\} \), and joining \( v_{4p+2} \) to \( \bigcup_{i \in [p]} \{v_{4i}, v_{4i-1}\} \), as shown in Figure 3. To include the remaining edges involving the four added vertices, for \( 1 \leq j \leq p \) obtain \( \tilde{G}_j \) from \( G_j \) by adding \( u_{4p+i} \) to \( U_j \) and \( v_{4p+i} \) to \( V_j \), for \( i \in \{1,2\} \). Each of these four vertices gains the two neighbors in \( G_j \) that are shared by the vertices of the set to which it was added. Over the resulting \( \tilde{G}_1, \ldots, \tilde{G}_p \), it gains precisely the neighbors in the other part that it does not have in \( \tilde{G}_{p+1} \). We again have \( r \) 2-connected planar graphs decomposing \( K_{n,n} \).

\[
\begin{array}{c}
\text{Figure 3: The subgraph } \tilde{G}_{p+1} \text{ in the planar decomposition of } K_{4p+2,4p+2}
\end{array}
\]
The remaining case is $n = 4p + 3$. A graph $G$ is thickness $t$-minimal if $\theta(G) = t$ and every proper subgraph of $G$ has thickness less than $t$. When $n = 4p + 3$, the graph $K_{4p+3,4p+3}$ is a thickness $(p + 2)$-minimal graph. Hobbs and Grossman [7] and Bouwer and Broere [2] independently gave two different decompositions of $K_{4p+3,4p+3}$ into planar subgraphs $H_1, \ldots, H_{p+2}$. In each case, each $H_i$ for $1 \leq i \leq p + 1$ is a 2-connected maximal planar bipartite graph (hence a bar visibility graph), and the graph $H_{p+2}$ contains only one edge. Let this edge be $u_iv_j$ (it is $u_1v_1$ in [7] and $u_{4p+3}v_{4p-1}$ in [2]).

The bar visibility representation algorithm of [9] uses “$s,t$-numberings”, allowing one to choose any vertex of a bar visibility graph to be the unique lowest or highest bar in the representation. Since we have reduced to the case $n \geq 4$, we have $p + 1 \geq 2$. Choose a representation of $H_1$ in which $u_i$ is the lowest bar and a representation of $H_2$ in which $v_j$ is the highest bar. Place the representation of $H_1$ above the representation of $H_2$ to incorporate the edge $u_iv_j$ without using an extra bar for $u_i$ or $v_j$.

We must also show that the bars for $u_i$ in $H_1$ and $v_j$ in $H_2$ can prevent unwanted visibilities between bars for vertices above and below them. Since the graph is bipartite, we may assume that bars for the two parts occur on horizontal lines with those for $U$ having odd vertical coordinates and those for $V$ having even coordinates. In addition, the bars on one horizontal line can extend to meet at endpoints to block visibility between higher and lower bars for the other part (using both the requirement of positive width for visibility and the fact that we are representing the complete bipartite graph). The bars can extend so that on each horizontal line the leftmost occupied point is the same and the rightmost occupied point is the same. Now the two representations can combine as described above.

\[ \square \]

### 3 The bar visibility number of $K_{m,n}$

In this section we constructively determine the bar visibility number of $K_{m,n}$ for all $m, n \in \mathbb{N}$. The proof here is independent of the shorter proof for $m = 2$ given in the previous section, which relied on thickness results from earlier papers. This proof is self-contained.

A $p$-split of a graph $G$ is a graph $G'$ obtained from $G$ by replacing each vertex $u$ of $G$ by a $p$-set $S_u$ of independent vertices such that two vertices $u$ and $v$ in $G$ are adjacent if and only if some vertex of $S_u$ and some vertex of $S_v$ are adjacent in $G'$. The split number $\sigma(G)$ of $G$ is the least $p$ such that $G$ has a $p$-split that is a planar graph.

By contracting each bar in a $p$-bar visibility representation of graph $G$ into a single vertex and keeping each “line of sight” as an edge, we obtain a bar visibility graph $G'$, which must be planar. Thus $b(G) \geq \sigma(G)$. Equality holds if some $\sigma(G)$-split is a bar visibility graph, which holds when it is 2-connected.
Theorem 6. $b(K_{m,n}) = \lceil \frac{mn+4}{2m+2n} \rceil$.

Proof. Let $r = \lceil \frac{mn+4}{2m+2n} \rceil$. Lemma 3 yields $b(K_{m,n}) \geq r$, so it suffices to prove the upper bound. We may assume $m \geq n$. Let the two parts of $K_{m,n}$ be $X$ and $Y$ with $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$.

When $n$ is even and $m > \frac{1}{2}(n^2 - 2n - 4)$, or when $n$ is odd and $m > n^2 - n - 4$, we compute $r = \lceil \frac{n}{2} \rceil$. In this case let $G_i$ be the subgraph induced by $X \cup \{y_{2i-1}, y_{2i}\}$, except that $G_{(n+1)/2}$ is the subgraph induced by $X \cup \{y_n\}$ when $n$ is odd. Since $K_{m,2}$ and $K_{m,1}$ are bar visibility graphs, this decomposes $K_{m,n}$ into $r$ bar visibility graphs.

Henceforth we may assume $r \leq \lfloor \frac{n-1}{2} \rfloor$. Let $s = \lfloor \frac{n}{2} \rfloor - r$. For fixed $n$, the value of $r$ increases with $m$. Since $m \geq n$, we have $r \geq \lceil \frac{n^2+1}{4m} \rceil \geq \lceil \frac{n+1}{2} \rceil$. Thus $s \leq r$. The case $s = r$ requires $s = r = \frac{n+1}{4}$ and hence $n \equiv 3 \mod 4$. Computing $r$ when $n = 4p + 3$ and $m \in \{4p, 4, 4p + 5\}$ shows that $r = \frac{n+1}{4}$ occurs if and only if $n \equiv 3 \mod 4$ and $m \in \{n, n + 1\}$. Otherwise, $s < n/4 < r$. Note that when $n = 3$ the case of equality does not occur, since already $m > 3^2 - 3 - 4$ when $m = 3$, so we may assume $n \geq 4$.

We will construct a bar visibility graph $G$ that is an $r$-split of $K_{m,n}$. In $G$, each vertex will have a label in $X \cup Y$, with each label used at most $r$ times. When no vertices labeled $x_i$ and $y_j$ are yet adjacent, we say that $x_i$ misses $y_j$; otherwise we say that $x_i$ hits $y_j$. We place vertices in the coordinate plane, with vertices having labels in $X$ at integer points on the horizontal axis and vertices having labels in $Y$ at integer points on the vertical axis. To facilitate understanding, we first exhibit in Figure 4 the bar visibility graph $G$ that results from our process when $(m, n) = (8, 7)$. For clarity, we record only the subscripts of the labels on the vertices; the labels are from $X$ on the horizontal axis and from $Y$ on the vertical axis.

We first construct subgraphs separately in each half-plane bounded by the vertical axis. Combining the two subgraphs along the vertical axis will yield a connected plane graph $\hat{G}$ with $rn + sm$ vertices such that labels in $X$ occur $s$ times and labels in $Y$ occur $r$ times. Furthermore, each $x_i \in X$ will hit $n - 2(r - s)$ different vertices of $Y$, and the vertices of $Y$ that $x_i$ misses will form $r - s$ pairs such that each pair lies on a face of length 4 and each face is used at most once in this way. In Figure 4, where $r = s$, the last step is not needed.

When $s < r$, we then put one copy of $x_i$ in each such face, so that $x_i$ now hits all $n$ vertices of $Y$. We may need to add edges joining $X$ and $Y$ to ensure that the resulting graph $G$ is 2-connected. It is then a bar visibility graph, completing the proof.

We use the first $\lfloor rn/2 \rfloor$ integer points on the positive vertical axis and the first $\lfloor rn/2 \rfloor$ integer points on the negative vertical axis. Call these sets of points $B^+$ and $B^-$, respectively. Starting from the origin, label the $\lfloor rn/2 \rfloor$ points of $B^+$ in order using $y_1, \ldots, y_{n/2}$, through increasing indices cyclically modulo $n$. Similarly, label the $\lfloor rn/2 \rfloor$ points of $B^-$ cyclically from the origin, but start with $y_{(n/2)+1}$ and continue through increasing indices modulo $n$ (see Figure 4). If $r$ is even, then the labels on the last points of $B^+$ and $B^-$ are $y_n$ and $y_{(n/2)}$.\]
respectively; if \( r \) is odd, then the last labels are \( y_{\lceil n/2 \rceil} \) and \( y_n \), respectively. Each label \( y_i \) is used exactly \( r \) times.

![Figure 4: A bar visibility graph that is a 2-split of \( K_{8,7} \)](image)

Next we label horizontal points using vertices of \( X \) and add edges. For easy illustration and understanding, we squeeze each half-plane into a strip, by drawing the positive vertical axis, the horizontal axis, and the negative vertical axis as successive parallel horizontal lines (see Figure 5). As shown in Figure 4, most vertices on the horizontal axis will be made adjacent to two consecutive vertices on the positive vertical axis and two consecutive vertices on the negative horizontal axis. Figure 5 is based on this idea, with vertices on the middle row for the horizontal axis receiving two neighbors above and below.

We consider three cases, depending on the parities of \( m \) and \( n \).

**Case 1. \( n \) is even.** Let \( A^+ \) and \( A^- \) be the sets consisting of the first \( \frac{rm}{2} - 1 \) integer points on the positive and negative horizontal axes, respectively.

Step 1. When \( m \) is even, this step does nothing. When \( m \) is odd, we put \( s \) copies of \( x_m \) into \( A^+ \) (see Figure 5, where \( m = 13 \)). The first copy of \( x_m \) is at the first point of \( A^+ \) and...
sees copies of all of \(y_1, y_2, y_{n/2+1}, y_{n/2+2}\). The indices on corresponding points from \(B^+\) and \(B^-\) always differ by \(n/2\). Hence we can proceed making the \(i\)th copy of \(x_m\) adjacent to the \(i\)th copies of each of \(y_{2i-1}, y_{2i}, y_{n/2+2i-1}, y_{n/2+2i}\) in \(B^+ \cup B^-\). Since \(s < r\) and \(B^+ \cup B^-\) has \(r\) copies of each label in \(Y\), this step succeeds.

Figure 5: Pattern used in Case 1, shown for \(K_{13,10}\) with \((r, s) = (3, 2)\).

Step 2. We assign labels from the remaining vertices of \(X\) to points in \(A^+ \cup A^-\) not already occupied by \(x_m\). The first \(2s\) available points go to \(x_1\) and \(x_2\), in \(s\) successive pairs. Continuing, the \(i\)th set of the next \(2s\) available points go to \(x_{2i-1}\) and \(x_{2i}\) in \(s\) successive pairs, where “available” means that points assigned previously to \(x_m\) when \(m\) is odd are skipped. Once \(A^+\) is exhausted, the assignment continues with \(A^-\), again moving outward from the origin. The last \(2s\) points assigned go to \(x_{2\lfloor m/2 \rfloor - 1}\) and \(x_{2\lfloor m/2 \rfloor}\).

We need \(sm\) points in \(A^+ \cup A^-\). Since \(|A^+| + |A^-| = 2(\frac{rn}{2} - 1)\), this requires \(sm \leq 2(\frac{rn}{2} - 1)\). Since \(s = n/2 - r\), the condition is equivalent to \(r \geq \frac{mn + 4}{2m + 2n}\), which is given.

Note that a label \(x_i\) now hits a label \(y_j\) if and only if it also hits \(y_{j+n/2}\). Also, the first label in \(B^-\) is \(y_{n/2+1}\), immediately following the last label \(y_{n/2}\) in \(B^+\) or \(B^-\) (depending on whether \(r\) is odd or even, respectively), and the first label in \(B^+\) is \(y_1\), immediately following the last label \(y_{n}\) in \(B^-\) or \(B^+\) (again depending on whether \(r\) is odd or even, respectively). Thus each label \(x_j \in X\) now hits two intervals of \(2s\) successive labels in \(Y\) (cyclically), except that in each interval one label may be skipped due to the presence of \(x_m\) or to the transition from \(A^+\) to \(A^-\). The two intervals of labels are displaced cyclically by \(n/2\) from each other. Since \(2s < n/2\), we conclude that in all cases \(x_j\) hits \(4s\) distinct labels.

Step 3. The graph constructed in steps 1 and 2 is a plane graph. Furthermore, a label \(x_i\) misses \(y_j\) if and only if it also misses \(y_{n/2+j}\). Vertices with labels \(y_j\) and \(y_{n/2+j}\) lie together
on a 4-face having a diagonal along the horizontal axis. Since $4s + 2(r - s) = 2r + 2s = n$
when $n$ is even, it suffices to break the vertices missed by $x_i$ into $r - s$ such pairs of labels
on the same face. For each such pair $\{y_j, y_{j+n/2}\}$, we put a new vertex labeled $x_i$ along
the horizontal axis in a face whose vertices on the vertical axis are $y_j$ and $y_{j+n/2}$, making it
adjacent to those vertices. Now $x_i$ hits all labels in $Y$.

The resulting graph may have some cut-vertices on the unbounded face if copies of vertices
in $X$ did not exhaust $A^+ \cup A^-$. It can be made 2-connected by adding edges joining copies
of vertices in $X$ and $Y$. We now have a bar visibility graph that is an $r$-split of $K_{m,n}$. Figure
6 illustrates the result of applying this process to $K_{12,10}$, where $r = 3$ and $s = 2$.

**Figure 6**: Pattern used in Step 3 of Case 1, shown for $K_{12,10}$ with $(r, s) = (3, 2)$.

**Case 2.** $m$ is even and $n$ is odd. In this case, $s = \frac{n+1}{2} - r$, so $4(s - 1) + 3 + 2(r - s) = 2(r + s) - 1 = 2(n + 1)/2 - 1 = n$. We will assign each label $x_i \in X$ to $s - 1$ vertices that
hit four labels in $Y$, one vertex that hits three labels, and $r - s$ vertices that hit two labels,
again hitting all $n$ labels with $r$ copies of $x_i$.

**Subcase 2a.** $r$ is even. Here $|B^+| = |B^-| = \frac{rn}{2}$. Let $t = \lfloor (\frac{rn}{2} - 1)/(2s - 1) \rfloor$ and
$k = \frac{rn}{2} - 1 - t(2s - 1)$. Let $A^+$ consist of the first $p$ integer points on the positive horizontal
axis, where $p = 2st + 2\lceil k/2 \rceil$ (thus $p$ is even). Let $A^-$ consist of the first $sm - p$ integer
points on the negative horizontal axis.

Step 1. For $1 \leq i \leq t$, successively, we put $s$ pairs $\{x_{2i-1}, x_{2i}\}$ as labels onto the next $2s$
points of $A^+$, starting from the origin. As shown in Figure 7, these $s$ copies of $x_{2i-1}$ together
hit $2s$ consecutive labels in $B^+$ and $2s - 1$ consecutive labels in $B^-$ (the first copy hits only
three labels). Similarly, the $s$ copies of $x_{2i}$ together hit $2s - 1$ consecutive labels in $B^+$ and
2s consecutive labels in $B^-$ (the last copy hits only three labels). The last labels in $B^+$ and $B^-$ hit by $x_{2i}$ are the same as the first labels hit by $x_{2i+1}$.

The number of labels we have hit in $B^+$ is $1 + t(2s - 1)$, which equals $\frac{n}{2} - k$ and is at most $|B^+|$. The number of points in $A^+$ to which we have assigned labels is $2st$, which is at most $|A^+|$. Hence Step 1 succeeds.

As we have noted, the labels in $Y$ hit by each label in $\{x_1, \ldots, x_{2t}\}$ form two cyclically consecutive intervals of labels. For each such label, the $2(r - s)$ missed labels group into $r - s$ pairs $(y_b, y'_b)$ such that 

$$b + \lfloor \frac{n}{2} \rfloor \equiv b' \mod n.$$

Step 2. Put $s$ pairs $\{x_{2(t+1)−1}, x_{2(t+1)}\}$ as labels onto $A^+ \cup A^-$. Here we use the last $2\lfloor k/2 \rfloor$ points of $A^+$ and the first $2s - 2\lfloor k/2 \rfloor$ points of $A^-$. This succeeds, because after Step 1 there are exactly $2\lfloor k/2 \rfloor$ points remaining in $A^+$. Since $0 \leq k \leq 2s - 2$, there are not enough points to put all the pairs into $A^+$; some will go into $A^-$.

In both $B^+$ and $B^-$ there remain $k$ points that have not been hit at all. As was done for the earlier pairs in Step 1, we want to ensure that the labels in $Y$ hit by these two labels in $X$ are distinct and have the property that the missed labels again group into $r - s$ pairs with cyclic displacement $\lfloor n/2 \rfloor$. Since $r$ is even, both $B^+$ and $B^-$ have size $\frac{n}{2}$, with $B^+$ having first label $y_1$ and last label $y_n$, and $B^-$ having first label $y_{(n+3)/2}$ and last label $y_{(n+1)/2}$.

***Stuck here. In trying to build examples to show why we need different patterns for even and odd nonzero $k$, I have not been able to get an example to work out with either pattern. For example, consider $(m, n) = (20, 15)$. Here $r = 4$, $s = 4$, and $t = 4$, with $k = 1$. We want one pair $(x_9, x_{10})$ in $A^+$ and three pairs in $A^-$. However, $B^-$ starts with $y_9$ and ends with $y_8$. With $4s - 1 = 15$, we need each of $x_9$ and $x_{10}$ to hit 15 different labels in $Y$. Under the second pattern, it seems that $x_9$ hits $y_{14}$ twice, while $x_{10}$ hits $y_7$ and $y_{15}$ twice. Please send me the correct explicit pattern for Step 2 when $(m, n) = (20, 15)$ and in a case where $k = 2$.***

Step 2. Put $s$ pairs $\{x_{2(t+1)−1}, x_{2(t+1)}\}$ into $A^+$ and $A^-$. 

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Figure 7: Pattern used in Subcase 2a, shown for $K_{18,11}$ with $(r, s) = (4, 2)$.
There are $k$ or $k+1$ points left in $A^+$ corresponding to $k$ is even or odd. And there are $k$ points in both $B^+$ and $B^-$ which haven’t been hit by any $x_j$. Because $0 \leq k \leq 2s-2$, we can not put $s$ pairs $\{x_{2(t+1)-1}, x_{2(t+1)}\}$ into $A^+$ completely. Depending on the parity of $k$, we do the following.

- If $k$ is even, we put $k/2$ pairs $\{x_{2(t+1)-1}, x_{2(t+1)}\}$ into the last $k$ point of $A^+$ successively, use pattern as shown in Figure 8 (left) to join these points to points in $B^+$ and $B^-$. Then we put $s - k/2$ pairs $\{x_{2(t+1)-1}, x_{2(t+1)}\}$ into the first $2s - k$ points of $A^-$ successively, use pattern as shown in Figure 8 (right) to join these points to points in $B^+$ and $B^-$ (in Figure 8, $s = k = 4$). Notice that, when $k = 0$, the pattern in Figure 8 (right) is exactly the same as the pattern in Figure 7, the $s$ pairs $\{x_{2(t+1)-1}, x_{2(t+1)}\}$ will be put into $A^-$ completely.

- If $k$ is odd, we put $\lceil k/2 \rceil$ pairs $\{x_{2(t+1)-1}, x_{2(t+1)}\}$ into the last $k+1$ point of $A^+$ successively, use pattern as shown in Figure 9 (left) to join these point to points in $B^+$ and $B^-$. Then we put $s - \lceil k/2 \rceil$ pairs $\{x_{2(t+1)-1}, x_{2(t+1)}\}$ into the first $2s - k - 1$ points of $A^-$ successively, use pattern as shown in Figure 9 (right) to join these point to points in $B^+$ and $B^-$ (in Figure 9, $k = 3$ and $s = 4$).

![Figure 8](image1.png)  
**Figure 8:** Patterns in Subcase 2a for $\{x_{2(t+1)-1}, x_{2(t+1)}\}$ in $A^+$ and $A^-$ when $k$ is even.

![Figure 9](image2.png)  
**Figure 9:** Patterns in Subcase 2a for $\{x_{2(t+1)-1}, x_{2(t+1)}\}$ in $A^+$ and $A^-$ when $k$ is odd.

Step 3. Put $\frac{m}{2} - t - 1$ complete $s$ pairs $\{x_{2i-1}, x_{2i}\}(t + 2 \leq i \leq \frac{m}{2})$ into $A^-$. We continue putting $s$ pairs $\{x_{2i-1}, x_{2i}\}$, for $i = t + 2, \ldots, m/2$ into $A^-$ successively, and use pattern as shown in Figure 7 to join these points to points in $B^+$ and $B^-$. To complete this step, we need $(m/2 - t - 1)(2s - 1) + 1$ points in $B^+$ (and $B^-$). And after step 2, $2s - k$
points in $B^+$ (and $B^-$) have been hit by $x_{2(t+1)-1}$ or $x_{2(t+1)}$, but the last points in $B^+$ (and $B^-$) are still available, so there are $\frac{rm}{2} - (2s - k) + 1$ points in $B^+$ (and $B^-$) still available. This requires $(m/2 - t - 1)(2s - 1) + 1 \leq \frac{rm}{2} - (2s - k) + 1$, this condition is equivalent to $r \geq \frac{mn+1}{2m+2n}$, which is given.

The graph we just obtained is a plane bipartite graph $\hat{G}$. With a similar argument to that in Case 1, for $1 \leq i \leq m$, $x_i$ hits $4(s - 1) + 3$ different vertices of $Y$, the vertices $y_j$ that $x_i$ misses form pairs that lie on common 4-faces. The rest of the proof is exactly the same as the step 3 in Case 1. Figure 10 illustrates the plane graph $\hat{G}$ corresponding to $K_{8,7}$, in this case, $r = s = t = 2$ and $k = 0$.

**Subcase 2.2.** $r$ is odd.

In this subcase, $|B^+| = \frac{m+1}{2}$ and $|B^-| = \frac{m-1}{2}$. Suppose that $\frac{m-1}{2} - 1 \equiv k \mod (2s - 1)$ and $\frac{m-1}{2s-1-k} = t$. Let $A^+$ be the set consisting of the first $2st + k + 1$ integer points on the positive horizontal axe. Let $A^-$ be the set consisting of the first $2s(m/2 - t - 1) + 2s - k - 1$ integer points on the negative horizontal axe.

With a similar process to that in Subcase 2.1, we put vertices from $X$ in $A^+$, and join these vertices to vertices from $Y$ in $B^+$ and $B^-$ using the pattern as shown in Figure 7, until we finish the $t$ complete $s$ pairs $\{x_{2i-1}, x_{2i}\}$ $(1 \leq i \leq t)$. Then we continue putting $s$ pairs $\{x_{2(t+1)-1}, x_{2(t+1)}\}$ into $A^+$, when $A^+$ is exhausted, the assignment continues with $A^-$, again moving outward from the origin. We join vertices $x_{2(t+1)-1}$ and $x_{2(t+1)}$ in $A^+$ to vertices from $Y$ in $B^+$ and $B^-$ using the pattern as shown in Figure 11 (left), join vertices $x_{2(t+1)-1}$ and $x_{2(t+1)}$ in $A^-$ to vertices from $Y$ in $B^+$ and $B^-$ using the pattern as shown in Figure 11 (middle) (in Figure 11, $s = 4$).

Next we continue putting $s$ pairs $\{x_{2i-1}, x_{2i}\}$, for $i = t + 2, \ldots, m/2$ into $A^-$ successively, and use pattern as shown in Figure 11 (right) to join these vertices to vertices from $Y$ in $B^+$ and $B^-$. To complete this step, we need $(m/2 - t - 1)(2s - 1) + 1$ points in $B^+$ (and $B^-$), and there are $\frac{m-1}{2} - (2s - k - 1) + 1$ points in $B^+$ (and $B^-$) still available. This requires $(m/2 - t - 1)(2s - 1) + 1 \leq \frac{m-1}{2} - (2s - k - 1) + 1$, this condition is equivalent to $r \geq \frac{mn+1}{2m+2n}$, which is given.

![Figure 10: Example for $K_{8,7}$](image-url)
The rest of the proof is exactly the same as that in Case 1. Figure 12 illustrates the plane graph $\tilde{G}$ corresponding to $K_{14,9}$, in this case $r = 3$, $s = 2$, $k = 0$ and $t = 4$. All three vertices $x_1$, $x_7$ and $x_{10}$ miss pair $y_5$, $y_8$, both $x_2$ and $x_8$ miss pair $y_1$, $y_5$, both $x_3$ and $x_{12}$ miss pair $y_3$, $y_8$, $x_4$ misses pair $y_1$, $y_8$, both $x_5$ and $x_{14}$ miss pair $y_2$, $y_6$, $x_6$ misses pair $y_2$, $y_7$. By adding all these missing edges to $\tilde{G}$, we can get the plane graph $G$ which is a bar visibility graph and a 3-split of $K_{14,9}$.

**Case 3.** $m$ is odd, $n$ is odd. In this case, $s = \frac{n+1}{2} - r$. Similar to Case 2, except for $x_m$ in Subcase 3.2, we will assign each label $x_i \in X$ to $s - 1$ vertices that hit four labels in $Y$, one vertex that hits three labels, and again $r - s$ vertices that hit two labels, again hitting all $n$ labels with $r$ copies of $x_i$. Since $m$ is odd, $x_m$ can not be paired with any other labels, we have to process it first.

**Subcase 3.1.** $s < r$ and $r$ is even.
In this subcase, \(|B^+| = |B^-| = \frac{m-1}{2} + t\). Suppose that \(\frac{m-1}{2} - s - 1 \equiv k \mod (2s-1)\) and \(\frac{m-s-1-k}{2s-1} = t\). Let \(A^+\) be the set consisting of the first \(2st + s + k + 1\) integer points on the positive horizontal axis. Let \(A^-\) be the set consisting of the first \(2s(\frac{m-1}{2} - t - 1) + 2s - k - 1\) integer points on the negative horizontal axis.

Step 1. Put \(s\) copies of \(x_m\) into \(A^+\)

The first copy of \(x_m\) is at the first point of \(A^+\) and join to the first copies of each \(y_1, y_2\) and \(y_{(n+1)/2+1}\). Next we make the \(i\)th copy of \(x_m\) to be adjacent to \(i\)th copies of \(y_{2i-1}, y_{2i}, y_{(n+1)/2+2i-2}, y_{(n+1)/2+2i-1}\), for \(i = 2, 3, \ldots, s\). Since \(s < r\) and \(B^+ \cup B^-\) has \(r\) copies of each label in \(Y\), this step succeeds.

Step 2. Put \(t\) complete \(s\) pairs \(\{x_{2i-1}, x_{2i}\}(1 \leq i \leq t)\) into \(A^+\).

Starting from the second point of \(A^+\), for \(1 \leq i \leq t\), we put \(s\) pairs \(\{x_{2i-1}, x_{2i}\}\) into available points in \(A^+\) successively. Then join these points to points of \(B^+\) and \(B^-\) using pattern as shown in Figure 13 (in Figure 13, \(s = 4\)), in which the first \(y_j\) which the first \(x_{2i-1}\) hits in \(B^+\) (and \(B^-\)) is common to the last \(y_j\), which the last \(x_{2(i-1)}\) hits in \(B^+\) (and \(B^-\)).

Step 3. Put \(s\) pairs \(\{x_{2(t-1)+1}, x_{2(t+1)}\}\) into \(A^+\) and \(A^-\).

We continue putting the \(s\) pairs \(\{x_{2(t-1)+1}, x_{2(t+1)}\}\) into \(A^+\), when \(A^+\) is exhausted, the assignment continues with \(A^-\), again moving outward from the origin. We join vertices \(x_{2(t-1)+1}\) and \(x_{2(t+1)}\) in \(A^+\) to vertices from \(Y\) in \(B^+\) and \(B^-\) using the pattern as shown in Figure 13 (right). We join vertices \(x_{2(t-1)+1}\) and \(x_{2(t+1)}\) in \(A^-\) to vertices from \(Y\) in \(B^+\) and \(B^-\) using the pattern as shown in Figure 14 (left). The last label used is \(x_{2(t+1)}\), it has degree 3. We let it be adjacent to one extra vertex at the lower right corner, which may bring extra coverage, it is shown as the dashed line in Figure 14 (left).

Step 4. Put \(\frac{m-1}{2} - t - 1\) complete \(s\) pairs \(\{x_{2i-1}, x_{2i}\}(t+2 \leq i \leq \frac{m-1}{2})\) into \(A^-\).

We continue putting \(s\) pairs \(\{x_{2i-1}, x_{2i}\}\), for \(i = t+2, \ldots, \frac{m-1}{2}\) into \(A^-\) successively, and use pattern in Figure 14 (right) to join these points to points in \(B^+\) and \(B^-\). To complete this step, we need \((\frac{m-1}{2} - t - 1)(2s - 1) + 1\) points in \(B^+\) (and \(B^-\)), and there are \(\frac{m}{2} - (2s - k - 1) + 1\) points in \(B^+\) (and \(B^-\)) still available. This requires \((\frac{m-1}{2} - t - 1)(2s - 1) + 1 \leq \frac{m}{2} - (2s - k - 1) + 1\), this condition is equivalent to \(r \geq \frac{mn+3}{2m+2n}\), which is given.

![Patterns for A^+](image)

**Figure 13**: Patterns used for \(A^+\) in Subcase 3.1.
Subcase 3.2. $s < r$ and $r$ is odd.

In this subcase, $|B^+| = \frac{rn-1}{2}$ and $|B^-| = \frac{rn-1}{2}$. Suppose that $\frac{rn-1}{2} - s - 1 \equiv k \mod (2s-1)$ and $\frac{rn-1}{2} - s - 1 - k = t$. Let $A^+$ be the set consisting of the first $2st+s+k+1$ integer points on the positive horizontal axis. Let $A^-$ be the set consisting of the first $2s\left(\frac{m-1}{2} - t - 1\right) + 2s - k - 1$ integer points on the negative horizontal axis. The following steps are similar to that in Subcase 3.1.

Step 1. We put $s$ copies of $x_m$ into $A^+$, each of which will have degree 4. The first copy of $x_m$ is put into the second point of $A^+$ and let it be adjacent to $y_2, y_3, y_{(n+1)/2+1}, y_{(n+1)/2+2}$. We continue to make the $i$'th copy of $x_m$ to be adjacent to $i$'th copies of $y_{2i}, y_{2i+1}, y_{(n+1)/2+2i-1}, y_{(n+1)/2+2i}$, for $i = 1, 2, \ldots, s$.

Step 2. Starting from the first point of $A^+$, for $1 \leq i \leq t$, we put $s$ pairs $\{x_{2i-1}, x_{2i}\}$ into available points in $A^+$ successively. Then join these points to points in $B^+$ and $B^-$ using pattern as shown in Figure 16 (left) (in Figure 16, $s = 4$), in which the first $y_j$ which the first $x_{2i-1}$ hits in $B^+$ (and $B^-$) is common to the last $y_j$ which the last $x_{2(i-1)}$ hits in $B^+$.

Figure 14: Patterns used for $A^-$ in Subcase 3.1.

Figure 15 illustrates the plane graph $\hat{G}$ corresponding to $K_{9,5}$, in this case $r = 2$, $s = 1$, $k = 0$ and $t = 3$, both $x_1$ and $x_6$ miss pair $y_1, y_3$, both $x_2$ and $x_7$ miss pair $y_1, y_4$, $x_3$ misses pair $y_2, y_4$, $x_4$ misses pair $y_3, y_5$, both $x_5$ and $x_9$ miss pair $y_3, y_5$, $x_8$ misses $y_3$. By adding all these missing edges to $\hat{G}$, we can get the plane graph $G$ which is a bar visibility graph and a 2-split of $K_{9,5}$.

Figure 15: Example for $K_{9,5}$. 
(and \(B^-\)).

Step 3. Then we continue putting the \(s\) pairs \(\{x_{2(t+1)-1}, x_{2(t+1)}\}\) into \(A^+\), when \(A^+\) is exhausted, the assignment continues with \(A^-\), again moving outward from the origin. We join vertices \(x_{2(t+1)-1}\) and \(x_{2(t+1)}\) in \(A^+\) to vertices from \(Y\) in \(B^+\) and \(B^-\) using the pattern as shown in Figure 16 (right), join vertices \(x_{2(t+1)-1}\) and \(x_{2(t+1)}\) in \(A^-\) to vertices from \(Y\) in \(B^+\) and \(B^-\) using the pattern as shown in Figure 17 (left).

Step 4. We continue putting \(s\) pairs \(\{x_{2i-1}, x_{2i}\}\), for \(i = t+2, \ldots, \frac{m-1}{2}\) into \(A^-\) successively, and use pattern in Figure 17 (right) to join these points to points in \(B^+\) and \(B^-\). To complete this step, we need \((\frac{m-1}{2} - t - 1)(2s - 1) + 1\) points in \(B^+\) (and \(B^-\)), and there are \(\frac{m-1}{2} - (2s - k - 1) + 1\) points in \(B^+\) (and \(B^-\)) still available. This requires \((\frac{m-1}{2} - t - 1)(2s - 1) + 1 \leq \frac{m-1}{2} - (2s - k - 1) + 1\), this condition is equivalent to \(r \geq \frac{mn+5}{2m+2n}\), which is given, for both \(m\) and \(n\) are odd.

![Figure 16: Patterns used for \(A^+\) in Subcase 3.2.](image)

![Figure 17: Patterns used for \(A^-\) in Subcase 3.2.](image)

Figure 18 illustrates the plane graph \(\hat{G}\) corresponding to \(K_{11,9}\), in this case \(r = t = 3\), \(s = 2\) and \(k = 1\), \(x_1\) misses pair \(y_3, y_7\), both \(x_2\) and \(x_9\) miss pair \(y_2, y_6\), both \(x_3\) and \(x_6\) miss pair \(y_4, y_9\), both \(x_4\) and \(x_7\) miss pair \(y_5, y_9\), \(x_5\) misses pair \(y_4, y_8\), \(x_8\) misses pair \(y_1, y_6\), \(x_{10}\) misses pair \(y_2, y_7\), \(x_{11}\) misses \(y_1\). By adding all these missing edges to \(\hat{G}\), we can get the plane graph \(G\) which is a bar visibility graph and a 3-split of \(K_{11,9}\).

**Subcase 3.3.** \(s = r\).
In this subcase, \( m = n = 4p + 3 \) (\( p \geq 1 \)) and \( r = p + 1 \). And when \( m = 4p + 4, n = 4p + 3, \)
\[
r = \left\lceil \frac{(4p+4)(4p+3) + 4}{2(4p+4) + 2(4p+3)} \right\rceil = p + 1.
\]
In Case 2, we have obtained a \( r \)-split of \( K_{4p+4,4p+3} \), which corresponding to a \( r \)-bar visibility representation of it. We delete the vertices labeled \( x_{4p+4} \) in the \( r \)-split of \( K_{4p+4,4p+3} \), and if necessary, add all other possible edges between \( x_i \) and \( y_j \) for any \( i, j \) to make the result graph 2-connected, then we obtain a \( r \)-split of \( K_{4p+3,4p+3} \), which corresponding to a \( r \)-bar visibility representation of it.

\[
\square
\]

References


