Matchings and The Chinese Postman Problem in Odd-Regular Graphs

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Joint work with
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The Matching Number

**Def.** A **matching** in a graph $G$ is a set of disjoint edges. The **matching number** $\alpha'(G)$ is the maximum size of a matching in $G$. 

![Diagram of a graph with matched edges and unmatched edges.](image-url)
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![Diagram of a graph showing matching and cut-edge examples](image)

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**Thm.** (O–West [2011]) Determined $\min \alpha'(G)$ over $k$-conn. $l$-regular $n$-vertex graphs and found those where equality holds. For example, if $G \in \mathcal{F}_{n,r}$ is $(2t+1)$-conn. with $t > 0$, then $\alpha'(G) \geq \frac{n}{2} - \frac{r-t}{2(r+1)^2+t} \frac{n}{2}$. 

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We seek $(2r + 1)$-regular graphs with many cut-edges.
Regular Graphs with Many Cut-Edges

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$B_r$ is the smallest such graph.
Cut-edges in Graphs in $\mathcal{H}_r$

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To enlarge a graph $G$ in $\mathcal{H}_r$: 

![Diagram of graph enlargement process]
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How do $p(G)$ and the number of cut-edges change?
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$\hat{\mathcal{H}}_r$ from $\hat{\mathcal{T}}_r$: trees with all leaves on same side.

**Thm.** In $\mathcal{F}_{n,1}$, $\rho(G)$ maximized $\iff G \in \mathcal{H}_r$. $\frac{2n-5}{3}$
Results on Parity Number

**Thm.** (Kostochka–Tulai [1996]; special case) If $G$ is a $(2r + 1)$-regular $n$-vertex graph (with $n \geq 4r + 6$), then

$$p(G) \leq \frac{n}{2} + \left\lfloor \frac{n-(4r+6)}{2r+4} \right\rfloor.$$
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Example achieving equality for $r = 2$ (parity subgraph includes all cut-edges plus a matching on $n-2$ vertices):
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Pf. If no balloons, or $n = 10$, then $\exists$ perfect matching and $p(G) = \frac{n}{2} \leq \frac{2n-5}{3}$. So, $n > 10$ and $\exists$ cut-edge $e$. 

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![Diagram](G_1 \text{ e } G_2)
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![Graph diagram showing $G_1$, $G_2$, and $G'_1$ connected by an edge $e$.]
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![Diagram](image)

- $n + 10 = n_1 + n_2$
- $p(G) = p(G'_1) + p(G'_2) - 5$
- $p(G) \leq \frac{2n_1-5}{3} + \frac{2n_2-5}{3} - 5 = \frac{2n-5}{3}$

Valid if $G_1, G_2 \neq B_1$. 
The Hard Case: Every cut-edge hits a copy of $B_1$
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\[ \cdots \xymatrix{ & *+<10pt>{*},p+<1pt>{red}+<1pt>{blue} & *+<10pt>{*},p+<1pt>{red}+<1pt>{blue} & \ar@{-}[ll] & *+<10pt>{*},p+<1pt>{red}+<1pt>{blue} & \ar@{-}[ll] & *+<10pt>{*},p+<1pt>{red}+<1pt>{blue} & \ar@{-}[ll] & \cdots } \]
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Form $G''$ from $G'$: replace each thread (through 2-verts) by one edge weighted by the length of the thread.
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Form $G''$ from $G'$: replace each thread (through 2-verts) by one edge weighted by the length of the thread.

Idea: Combine a parity subgraph of $G''$ with the red edges for balloons in $G$. 
Claim: $G''$ has a perf. matching with $\leq \frac{1}{3}$ of the weight.
Appealing to Edmonds

**Claim:** $G''$ has a perf. matching with $\leq \frac{1}{3}$ of the weight.

If so: Let $n = |V(G)|$; $m = |E(G)| = \frac{3n}{2}$; $b = \#\text{balloons}$,
Appealing to Edmonds

**Claim:** $G''$ has a perf. matching with $\leq \frac{1}{3}$ of the weight.

**If so:** Let $n = |V(G)|$; $m = |E(G)| = \frac{3n}{2}$; $b = \#$ balloons, so $|E(G')| = m - 8b = \text{wt}(G'')$. Using $b \leq \frac{n+2}{6}$ and $n > 16,$
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**Claim:** \( G'' \) has a perf. matching with \( \leq \frac{1}{3} \) of the weight.

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p(G) \leq p(G') + 3b \leq \frac{m-8b}{3} + 3b = \frac{3n-16b}{6} + 3b = \frac{n}{2} + \frac{b}{3} \leq \frac{n}{2} + \frac{1}{3} \left( \frac{n+2}{6} \right) < \frac{2n-5}{3}.
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Thm. (Edmonds [1965]) A $k$-regular multigraph of even order such that every edge cut $[S, \bar{S}]$ has size at least $k$ when $|S|$ is odd has a family $\mathcal{M}$ of perfect matchings that covers each edge equally often.
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**Cor.** A $(2r + 1)$-regular, $2r$-edge-connected multigraph with total wt. $W$ has a perfect matching with wt. $\leq \frac{W}{2r+1}$. 
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Pf. To cover each edge $p$ times, $|\mathcal{M}| = p(2r + 1)$. The total weight over all the matchings is $pW$; pigeonhole.
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In the inductive case, equality requires that $G'_1$ and $G'_2$ both satisfy equality, so $G'_1, G'_2 \in \mathcal{H}_1$. 

$G'_1$ $G_1$ $B_1$ $n_1$ vertices

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Now $G$ also is constructed by attaching copies of $B_1$ to the leaves of a tree in $T_1$, so $G \in \mathcal{H}_1$. 
Upper Bound on Number of Balloons

**Lem.** If $G \in \mathcal{F}_{n,r}$, then $b(G) \leq \frac{(2r-1)n+2}{4r^2+4r-2}$, with equality if and only if $G \in \mathcal{H}^r_r$. ($\frac{n+2}{6}$ when $r = 1$)
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**Pf.** Form $G'$ by shrinking each balloon to a vertex, which will have degree 1.
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\[ G' \]

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By vertex degrees, \( (2r+1)n' - 2rb(G) = 2m' \geq 2n' - 2 \).

\( \therefore 2rb(G) \leq (2r - 1)n' + 2 \).
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Equality at each step forces $G' \in \mathcal{T}_r$ and $G \in \mathcal{H}_r$. ■
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**Thm.** (Berge–Tutte Formula) \( \alpha'(G) = \min_{S \subseteq V(G)} \frac{1}{2} (n - \text{def}(S)), \)

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**Pf.** More work for cuts with size between 3 and \( 2r - 1 \).
Cut-edges and Even Factors in Odd-degree Regular Graphs

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slides available on DBW preprint page

Joint work with
Bjarne Toft