Acyclic graphs with at least \(2\ell + 3\) vertices are \(\ell\)-recognizable

Alexandr V. Kostochka\*, Mina Nahvi\†, Douglas B. West\‡, Dara Zirlin\§

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Abstract

The \((n - \ell)\)-deck of an \(n\)-vertex graph is the multiset of subgraphs obtained from it by deleting \(\ell\) vertices. A family of \(n\)-vertex graphs is \(\ell\)-recognizable if every graph having the same \((n - \ell)\)-deck as a graph in the family is also in the family. We prove that the family of \(n\)-vertex graphs having no cycles is \(\ell\)-recognizable when \(n \geq 2\ell + 3\).

1 Introduction

The \(k\)-deck of a graph is the multiset of \(k\)-vertex induced subgraphs. We also write this as the \((n - \ell)\)-deck when the graph has \(n\) vertices and \(\ell\) vertices are to be deleted. An \(n\)-vertex graph is \(\ell\)-reconstructible if it is determined by its \((n - \ell)\)-deck. It is an elementary observation, via a counting argument, that the \(k\)-deck of a graph always determines its \((k - 1)\)-deck. Therefore, an enhancement of the Reconstruction Problem is to find for each graph the maximum \(\ell\) such that it is \(\ell\)-reconstructible.

Often a reconstruction argument has two main parts. First, one proves that the deck determines that the graph is in a particular class or has a particular property. If the \((n - \ell)\)-deck determines this, then the class or property is \(\ell\)-recognizable. Then, using the knowledge that every reconstruction from the deck is in that class or has that property, one determines that only one such graph has that deck. This makes the graphs in the family \(\ell\)-reconstructible.

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\*University of Illinois at Urbana–Champaign, Urbana IL, and Sobolev Institute of Mathematics, Novosibirsk, Russia: kostochk@math.uiuc.edu. Research supported in part by NSF grant DMS-1600592 and grants 18-01-00353A and 19-01-00682 of the Russian Foundation for Basic Research.

\†University of Illinois at Urbana–Champaign, Urbana IL: mnahvi2@illinois.edu.

\‡Zhejiang Normal Univ., Jinhua, China and Univ. of Illinois at Urbana–Champaign, Urbana IL: dwest@illinois.edu. Supported by National Natural Science Foundation of China grants NSFC 11871439 and 11971439.

\§University of Illinois at Urbana–Champaign, Urbana IL 61801: zirlin2@illinois.edu. Research supported in part by Arnold O. Beckman Campus Research Board Award RB20003 of the University of Illinois at Urbana-Champaign.
Here, as a step in studying $\ell$-reconstructibility of trees, we consider recognition of acyclic graphs. We prove that when $n \geq 2\ell + 3$, the $(n - \ell)$-deck of an $n$-vertex graph determines whether the graph contains a cycle. Since the $(n - \ell)$-deck determines the 2-deck, we also know the number of edges. Thus we can recognize whether all graphs having a given $(n - \ell)$-deck are trees, since an $n$-vertex acyclic graph is a tree if and only if it has $n - 1$ edges.

It is conjectured that the correct threshold is $n \geq 2\ell + 1$. It is known that when $n = 2\ell$ the $n$-vertex path $P_n$ and the graph $C_{\ell+1} + P_{\ell-1}$ (disjoint union of a cycle with $\ell + 1$ vertices and a path with $\ell - 1$ vertices) have the same $(n - \ell)$-deck.

2 The Proof

Let $D$ be the $(n - \ell)$-deck of an $n$-vertex graph (we henceforth just call it the “deck”). We will assume $n > 2\ell$. The members of $D$ are the “cards” in the deck. We begin with a notion generalizing the degree list.

Definition 1. Given a vertex $v$ in a graph $G$, the $k$-ball at $v$, written $U_k(v)$, is the subgraph induced by all vertices within distance $k$ of $v$ in $G$. A $k$-butterfly is a tree with diameter $2k$. A $k$-center in a graph $G$ is a vertex $v$ that is the center of a $k$-butterfly.

When $G$ is a forest, the maximal $k$-butterfly at a $k$-center $v$ is the $k$-ball at $v$. If the $k$-ball at $v$ does not contain a path of length $2k$, then $v$ is not a $k$-center. Our approach to the proof is to consider an acyclic and a non-acyclic graph having the same $(n - \ell)$-deck, show that they have the same number of $k$-centers for an appropriate value of $k$, and obtain a contradiction by showing that they cannot have the same number of $k$-centers.

In order to count the $k$-centers using the $(n - \ell)$-deck, we will count the $k$-centers whose $k$-balls have each size. The special case $k = 1$ gives us the degree list, except for vertices with degree at most 1. The key point is uniqueness of the maximal $k$-butterfly containing a particular $k$-butterfly.

Lemma 2. In a graph $G$ with girth at least $2k + 2$, every $k$-butterfly is contained in a unique maximal $k$-butterfly.

Proof. Since $G$ has girth at least $2k + 2$, every $k$-butterfly is an induced subgraph of $G$. Since a $k$-butterfly $B$ contains a path $P$ of length $2k$, its center $v$ is uniquely determined. No $k$-butterfly with a center $w$ other than $v$ contains $P$, because the distance from $w$ to one of the ends of $P$ would exceed $k$. Hence no $k$-butterfly with center $w$ contains $B$. Hence the maximal butterfly containing $B$ can only be $U_k(v)$. □

Lemma 3. Let $G$ be an $n$-vertex graph. If $G$ has girth at least $2k + 2$, and every connected $(n - \ell)$-card of $G$ has radius greater than $k$, then the number of $k$-centers in $G$ whose $k$-balls have each size (and the total number of $k$-centers) is determined by the $(n - \ell)$-deck of $G$. 

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Proof. Since every \((n - \ell)\)-card has radius greater than \(k\), every \(k\)-butterfly is a proper subgraph of some card. Hence by examining the deck we see all the \(k\)-butterflies and determine the maximum number of vertices in a \(k\)-butterfly; call it \(m\). We have \(m < n - \ell\).

Let \(D_i\) be the set of full \(k\)-balls with \(m - i\) vertices. We determine \(|D_i|\), iteratively for \(i\) from 0 through \(m - 2\ell - 1\) (the full \(k\)-ball at a \(k\)-center must have at least \(2k + 1\) vertices). We already have \(|D_0|\).

For \(i > 0\), obtain from the \((m - i)\)-deck the multiset \(C_i\) of \((m - i)\)-cards that are \(k\)-butterflies. Obtain also the multiset \(C'_i\) of \(k\)-butterflies with \(m - i\) vertices that are contained in full \(k\)-balls with more than \(m - i\) vertices. That is, for each member \(B\) of each \(D_j\) with \(j < i\), include in \(C'_i\) the multiset of connected subgraphs of \(B\) that retain a path of length \(2k\) and have \(m - i\) vertices. By Lemma 2, each \(k\)-butterfly \(B\) with \(m - i\) vertices that is not a maximal \(k\)-butterfly appears exactly once in the multiset \(C'_i\). Thus \(D_i = C_i - C'_i\).

Finally, since the \(k\)-centers correspond bijectively to full \(k\)-balls, the number of \(k\)-centers in \(G\) is \(\sum_{i=0}^{m-2\ell-1} |D_i|\).

Henceforth we will assume that there are \(n\)-vertex graphs \(F\) and \(H\) both having deck \(D\), where \(F\) contains no cycle and \(H\) contains a cycle.

**Lemma 4.** All cycles in \(H\) appear in the same component of \(H\) and have more than \(n - \ell\) vertices. The deck contains more than \(n - \ell\) cards that are paths. Thus \(F\) contains more than \(n - \ell\) paths with \(n - \ell\) vertices.

**Proof.** Since every subgraph of \(F\) is acyclic, every card in \(D\) is acyclic. Hence no cycle in \(H\) appears on a card, so every cycle in \(H\) has more than \(n - \ell\) vertices. Since \(n > 2\ell\), there is not room for disjoint cycles in \(H\). Along a shortest cycle, we obtain more than \(n - \ell\) cards that are paths and must also occur as subgraphs of \(F\). \(\square\)

**Definition 5.** Let \(\hat{k}\) denote the minimum radius over all connected cards in \(D\). Henceforth \(k\) will always be \(\hat{k} - 1\). Let \(a\) denote the number of vertices outside a longest path in \(F\); thus a longest path \(P\) in \(F\) has \(n - a\) vertices, and \(0 \leq a \leq \ell\).

**Lemma 6.** For \(k = \hat{k} - 1\), the number of \(k\)-centers in \(F\) and \(H\) is the same and is at least \(n - \ell + 1\).

**Proof.** Since every connected card of \(F\) has radius at least \(k + 1\), we have \(2k + 2 \leq n - \ell\). Since all cards are acyclic, the girth of \(H\) is at least \(n - \ell + 1\). Thus Lemma 3 applies, and all reconstructions from the deck have the same number of \(k\)-centers. The number of vertices in any cycle in \(H\) is at least \(n - \ell + 1\), which is more than \(2k + 1\). Hence every vertex on a cycle in \(H\) is a \(k\)-center, so there are at least \(n - \ell + 1\). \(\square\)

**Lemma 7.** Letting \(a\) be the number of vertices in \(F\) outside a longest path \(P\), we have \(a \geq n - \ell - 1 - 2\hat{k}\).
Proof. In a card with smallest radius ($\hat{k}$) among the connected cards, we see at most $2\hat{k} + 1$ vertices of $P$. In this card, there remain at least $n - \ell - 2\hat{k} - 1$ vertices outside $P$. □

Lemma 8. For $k = \hat{k} - 1$, a longest path $P$ in forest $F$ contains at most $\ell + 3$ $k$-centers.

Proof. The $k$ vertices closest to each end of $P$ cannot be $k$-centers. Thus $P$ contains at most $n - a - 2k$ $k$-centers, where $a$ is the number of vertices outside $P$. By Lemma 7, $a \geq n - \ell - 2\hat{k} - 1$. Thus $n - a - 2k \leq n - 2(\hat{k} - 1) - (n - \ell - 2\hat{k} - 1) = \ell + 3$. □

Lemma 9. The minimum radius $\hat{k}$ among the connected cards of $H$ is at least $(n - 2\ell + 1)/2$.

Proof. Let $C$ be a connected card of radius $\hat{k}$. Graph $H$ has $\ell$ vertices outside $C$, including at least $n - \ell + 1 - (2\hat{k} + 1)$ vertices from a cycle in $H$ and at least one vertex outside the component containing the cycle. Hence $\ell \geq n - \ell - 2\hat{k} + 1$, implying the claim. □

In light of Lemma 6, our aim is to show that $F$ and $H$ cannot have the same deck by showing that they cannot have the same number of $k$-centers, where $k = \hat{k} - 1$.

Lemma 10. For $k = \hat{k} - 1$, all $k$-centers of $F$ lie in the component $F'$ of $F$ that contains a longest path $P$. Also, if $x$ and $y$ are $k$-centers in $F'$, then all vertices on the path joining $x$ and $y$ in $F$ are $k$-centers.

Proof. Let $C$ be a connected card of radius $\hat{k}$; it omits exactly $\ell$ vertices. If $C$ does not intersect $F'$, then $C$ omits at least $n - a$ vertices, but $n - a \geq n - \ell > \ell$. Since $C$ is connected and intersects $F'$, in fact $C \subseteq F'$. Now there are at most $\ell$ vertices outside $F'$. If there is a $k$-center outside $F'$, then $2k + 1 \leq \ell$. We were supposedly proving that there could not be a $k$-center outside $F'$, but I don’t see that now. Do we need it?

For the second statement, let $Q$ be the path joining $x$ and $y$ in $F'$. Since $x$ and $y$ are $k$-centers, there are paths of length $k$ leaving each that contain no edge of $Q$. The union of these paths is a path having at least $k$ vertices in both directions from each vertex of $Q$. Hence the vertices of $Q$ are $k$-centers. □

Example 11. The next lemma is complicated somewhat by the fact that there may be connected cards with smallest radius whose center is not on $P$. Consider for example a tree having two branch vertices $w$ and $z$ with a common neighbor $y$. grow two paths of length 12 from $w$ and three paths of length 8 from $z$. Let $\ell = 12$. The longest path $P$ has length 24, with center $w$.

A connected card $C$ with smallest radius arises by deleting six vertices from each end of $P$; it has center $z$ and radius 8. With $k = 7$, there are 11 $k$-centers on $P$ and five $k$-centers outside $P$ ($z$ and its four neighbors). No vertices of $F'$ outside $P$ are omitted by $C$.
Lemma 12. Let $s$ be the number of $k$-centers in $F$ lying outside $P$. Let $C$ be a connected card with smallest radius $(\hat{k})$, and let $z$ be the center of $C$. Let $d$ be the number of edge-disjoint paths of length $\hat{k}$ in $C$ sharing an endpoint at $z$. Under these conditions, $F'$ has at least $s + n - a - 2\hat{k} - d - 1$ vertices not in $C$. Also, $H$ has at least $n - \ell + d - 1$ $k$-centers.

Proof. We have noted that every connected card is contained in the component $F'$ of $F$ containing $P$. Let $z$ be a center of $C$ closest to $P$ ($C$ may have one or two centers), and let $Q$ be the path from $z$ to $P$. Since $C$ has radius $\hat{k}$, vertex $z$ is a $k$-center. Let $t$ be the distance from $z$ to $P$. After the $t$ steps from $z$ to $P$ along $Q$, only $\hat{k} - t$ vertices in each direction along $P$ can be included in $C$ (plus the vertex of $Q \cap P$), so $C$ omits at least $n - a - 2\hat{k} + 2t - 1$ vertices of $P$.

Let $x$ be a $k$-center of $F$ that is outside $P$ and is not on $Q$ (this includes the neighbors of $z$ outside $Q \cup P$). Since $x$ is a $k$-center of $F$, in $F'$ there is a path of $2\hat{k} + 1$ vertices containing $x$. Hence there is a vertex $x'$ in $F'$ at distance $k$ from $x$ that is reached by following a path from $z$ through $x$ and $k$ steps further. Since $C$ has radius $k + 1$, it can only include $k + 1$ vertices along any path leaving $z$. Hence if $x$ is not adjacent to $z$, then this vertex $x'$ must be omitted by $C$. Furthermore, since there is a unique path from each vertex to $z$, each vertex $x'$ can only be excluded by one vertex $x$ in this way.

Since $n - \ell \geq 2\hat{k}$, there are $k$-centers on $P$. Hence all vertices of $Q$ are $k$-centers. We have excluded one vertex of $F'$ from $C$ for each vertex of $C$ outside $P$ that is a $k$-center not on $Q$ and not adjacent to $z$. The number of these $k$-centers is $s - t - (d - 1)$ if $t \geq 1$; it is $s - d + 2$ if $t = 0$.

Together with the vertices on $P$ omitted by $C$, we have $s + n - a - 2\hat{k} - d + t$ vertices omitted if $t \geq 1$, and $s + n - a - 2\hat{k} - d + 1$ if $t = 0$. Hence in either case we have at least $s + n - a - 2\hat{k} - d - 1$, since $\hat{k} = k + 1$.

As we have noted, all $d$ neighbors of $z$ along paths that extend for $\hat{k}$ steps are $k$-centers in $F$. Thus we have $d$ $k$-centers with a common neighbor. Since $H$ has girth at least $n - \ell + 1$, the only way $k$-centers on a cycle in $H$ can have a common neighbor is if they are separated by two steps along the cycle. Hence $H$ has at least $d - 2$ $k$-centers in addition to the at least $n - \ell + 1$ $k$-centers on a cycle. □

Theorem 13. The family of acyclic graphs is $\ell$-recognizable when $n \geq 2\ell + 2$.

Proof. Let $F$ and $H$ be acyclic and non-acyclic graphs as discussed earlier, and let $C$ be a connected card with smallest radius $\hat{k}$. Let $d$ be the number of paths of length $\hat{k}$ in $C$ meeting at a center $z$. Let $b$ be the number of vertices in $V(F) - P - C$ and $b'$ be the number of vertices in $P - C$. By definition, $b + b' = \ell$.

Similarly, let $s$ be the number of $k$-centers outside of $P$ and $s'$ be the number of $k$-centers in $P$, where $k = \hat{k} - 1$. By Lemma 12, we have $b + b' \geq s + n - a - 2\hat{k} - d - 1$ and $s + s' \geq n - \ell + d - 1$. 


Note that $s' = n - a - 2k$. Thus

$$\ell = b + b' \geq s + s' - d \geq n - \ell - 2.$$ 

We conclude that $n \leq 2\ell + 2$ if it is possible to have both acyclic and non-acyclic $n$-vertex graphs with the same $(n - \ell)$-deck.

However, this requires equality in all the inequalities that produce this bound. A closer look needed. \qed