Chapter 5

Graph Coloring

This fragment is for the list coloring extension of Brooks’ Theorem, somewhere in the middle of the chapter.

We now prove the list coloring version of Brooks’ Theorem. Recall that proving that a graph is $k$-choosable is stronger than (and implies) that it is $k$-colorable. The proof here is an extension to the list coloring context (in Krivelevich [2022]) of the short proof by Zajac [2018] of Brooks’ original theorem.

When a connected graph $G$ has maximum degree 2, we already know that $G$ is both 2-colorable and 2-choosable if and only if $G$ is not an odd cycle. Hence it remains to consider graphs with maximum degree at least 3. Many arguments about list colorings use the simple notion that if we have used fewer than $|L(v)|$ colors on the neighbors of $v$ when the time comes to choose a color for $v$, then a color remains available in $L(v)$ to use on $v$ and extend the proper coloring. We use this idea repeatedly in the proof; the trick is to choose colors along the way so that this will remain true when we color each vertex $v$.

5.0.1. Theorem. (List Extension of Brooks’ Theorem; Vizing [1976]) When $k \geq 3$, every connected graph with maximum degree $k$ other than $K_{k+1}$ is $k$-choosable.

Proof: Let $G$ be such a graph, and let $n = |V(G)|$. When $G$ is not $k$-regular, we can choose a vertex of degree less than $k$ as $v_n$. Since $G$ is connected, we can grow a spanning tree of $G$ from $v_n$, assigning indices in decreasing order as we reach vertices. Each vertex other than $v_n$ in the resulting ordering $v_1, \ldots, v_n$ has a higher-indexed neighbor along the path to $v_n$ in the tree. We choose colors for vertices in this order. When we color $v_i$, at most $k - 1$ colors have been used on lower-indexed neighbors, so a color in $L(v_i)$ remains available to use as $f(v_i)$, including when $i = n$. Hence we may assume that $G$ is $k$-regular.

Claim: If $(u, v, w)$ is an induced 3-vertex path such that a color $f(u)$ has been chosen from $L(u)$, then it is possible to choose $f(w)$ from $L(w)$ so that at most one color from $\{f(u), f(w)\}$ appears in $L(v)$. To do this, we set $f(w) = f(u)$ if $f(u) \in L(w)$, choose $f(w) \in L(w)$ arbitrarily if $f(u) \not\in L(v)$, and choose $f(w) \in L(w) - L(v)$.
if $f(u) \in L(v) - L(w)$. We can do this in the last case because $L(v)$ and $L(w)$ are different but have the same size.

We first apply the claim in the special case where $G$ has a non-spanning cycle $C$ such that some vertex of $C$ has no neighbor outside $C$. Since $G$ is connected, we can then find two consecutive vertices $w$ and $v$ along $C$ such that all neighbors of $w$ lie on $C$ but $v$ has a neighbor $u$ outside $C$. By the induction hypothesis, $G - V(C)$ has an $L$-coloring $f$. We will extend $f$ to $V(C)$ to obtain an $L$-coloring of $G$. Using the Claim, we choose $f(w) \in L(w)$ so that at most one color from $\{f(u), f(w)\}$ appears in $f(v)$.

Index the vertices along $C$ in order as $x_1, \ldots, x_r$ with $x_1 = w$ and $x_r = v$. Having chosen $f(w)$ as above, we next choose colors for $x_2, \ldots, x_r$ in order. For $i \leq k - 1$, when we reach $x_i$ at least one of its $k$ neighbors is uncolored, and hence a color in its list is available to use as $f(x_i)$. When we reach $v$, we have used at most one color from $L(v)$ on its neighbors $u$ and $w$ and hence at most $k - 1$ colors on its $k$ neighbors, so again a color in $L(v)$ remains available to use as $f(v)$.

Now consider the general case for $G$. Since $G$ is connected and not complete, $G$ has $P_3$ as an induced subgraph, say with vertices $u, v, w$ in order. Let $P$ be a longest path in $G$ starting with $u, v, w$, say $P = (x_1, \ldots, x_\ell)$ with $(x_1, x_2, x_3) = (u, v, w)$. By the choice of $P$, all neighbors of $x_\ell$ lie on $P$. Let $x_i$ be the neighbor of $x_\ell$ with smallest index. The cycle $[x_i, \ldots, x_\ell]$ contains all neighbors of $x_\ell$; if it does not include all the vertices of $G$, then the preceding special case applies.

Hence we may assume $\ell = n$ and $i = 1$. Now, since $k \geq 3$, vertex $v$ has a neighbor $x_j$ outside $\{u, w\}$. With $x_3 = w$, we choose colors for the vertices in the order $u, x_3, \ldots, x_j, v, \ldots, x_f, v$. Every vertex in this order has a later neighbor, except for $v$. However, after choosing any $f(u) \in L(u)$, the Claim allows us to pick $f(w)$ from $L(w)$ so that at most one color from $\{f(u), f(w)\}$ appears in $f(v)$. Hence for each subsequent vertex, when we reach it we have used at most $k - 1$ colors from its list on its neighbors, so a color remains available for it.