Cut-edges and Regular Subgraphs in Odd-degree Regular Graphs

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slides available on DBW preprint page

Joint work with
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**Background**

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What about \( 2k \)-factors for \( k > 1 \)? Harder to guarantee, since every \( 2k \)-factor contains a 2-factor.
Our Results

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All the results also apply to multigraphs.
Our tool: Belck’s Theorem for \( l \)-factors

A special case of the \( f \)-factor Theorem of Tutte [1952].
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**Notation:** For $T \subseteq V(G)$, let $d_G(T) = \sum_{v \in T} d_G(v)$. 
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**Thm.** (Belck [1950]) A multigraph $G$ has a $\ell$-factor iff

$$q(S, T) \leq \ell(|S| - |T|) + d_{G-S}(T)$$

for all disjoint $S, T \subset V(G)$, where $q(S, T)$ counts the components $Q$ of $G-S-T$ with $\|T, V(Q)\| + \ell|V(Q)|$ odd.
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**Thm.** (Belck [1950]) A multigraph \( G \) has a \( l \)-factor iff

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- For \( l = 1 \), setting \( T = \emptyset \) reduces to the 1-factor condition \( o(G - S) \leq |S| \).
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If $l = 2k$, then $Q$ is “$T$-odd” iff $\|T, V(Q)\|$ is odd.
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If $l = 2k$, then $Q$ is “$T$-odd” iff $\|T, V(Q)\|$ is odd.

Hence $q(S, T) \equiv \|T, R\| \mod 2$, where $R = V(G) - S - T$. 
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**Parity Lemma:** No \( 2k \)-factor \( \Rightarrow \exists \) disjoint \( S, T \) with

\[
q(S, T) \geq 2k(|S| − |T|) + d_{G−S}(T) + 2.
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Sufficient Condition for $2k$-factor

**Thm.** For $k \leq (2r+1)/3$, every $(2r+1)$-regular multigraph with at most $2r - 3(k - 1)$ cut-edges has a $2k$-factor.
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**Pf.** Suppose no $2k$-factor. We prove $c > 2r - 3(k - 1)$. 
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Recall $q(S, T) = \#Q$ in $G - S - T$ so that $\|T, V(Q)\|$ is odd.

Let $q(S, T) = q_1 + q_2 + q_3$, counting three types of $Q$. 
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Let $q(S, T) = q_1 + q_2 + q_3$, counting three types of $Q$.

$q_1 : \quad S \quad Q \quad T \quad q_1 \leq c$
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& |S| \quad \quad \quad \quad |Q| \quad \quad \quad |T| \\
& q_1 \leq c
\end{align*}
$$

$q_2$:

$$
\begin{align*}
& q_2 : \\
& S \quad \quad \quad \quad Q \quad \quad \quad T \\
& |S| \quad \quad \quad \quad |Q| \quad \quad \quad |T| \\
& q_2 \leq \|R, S\|
\end{align*}
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$q_1 : \quad S \quad \quad Q \quad \quad T \quad \quad q_1 \leq c$

$q_2 : \quad S \quad \quad \geq \quad Q \quad \quad T \quad \quad q_2 \leq ||R, S||$

$q_3 : \quad S \quad \quad ? \quad Q \quad \quad T \quad \quad q_1 + q_2 + 3q_3 \leq d_{G-S}(T)$
Sufficient Condition for $2k$-factor

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Recall $q(S, T) = \#Q$ in $G - S - T$ so that $||T, V(Q)||$ is odd.

Let $q(S, T) = q_1 + q_2 + q_3$, counting three types of $Q$.

- $q_1$:
  - $q_1 \leq c$

- $q_2$:
  - $q_2 \leq ||R, S||$

- $q_3$:
  - $q_1 + q_2 + 3q_3 \leq d_{G-S}(T)$

$$3q(S, T) = 3(q_1 + q_2 + q_3) \leq 2c + 2 ||R, S|| + d_{G-S}(T)$$
Lower Bound on Number of Cut-Edges, $c$

The upper and lower bounds on $3q(S, T)$ yield

$$2c + 2 \|R, S\| + d_{G-S}(T) \geq 3d_{G-S}(T) + 6k(|S| - |T|) + 6,$$
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simplifying to

$$c \geq d_{G-S}(T) + 3k(|S| - |T|) - \|R, S\| + 3.$$
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$$c \geq d_{G-S}(T) + 3k(|S| - |T|) - \|R, S\| + 3.$$

Since $G$ is $(2r + 1)$-regular, $d_{G-S}(T) = (2r + 1)|T| - \|T, S\| \geq (2r + 1)|T| - [(2r + 1)|S| - \|R, S\|].$
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The upper and lower bounds on $3q(S, T)$ yield
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\]

simplifying to $c \geq d_{G-S}(T) + 3k(|S| - |T|) - \|R, S\| + 3$.

Since $G$ is $(2r + 1)$-regular, $d_{G-S}(T) = (2r + 1)|T| - \|T, S\| \geq (2r + 1)|T| - [(2r + 1)|S| - \|R, S\|]$.

Hence $c \geq (2r + 1 - 3k)(|T| - |S|) + 3$. 
Lower Bound on Number of Cut-Edges, $c$

The upper and lower bounds on $3q(S, T)$ yield

$$2c + 2 \|R, S\| + d_{G-S}(T) \geq 3d_{G-S}(T) + 6k(|S| - |T|) + 6,$$

simplifying to

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Since $G$ is $(2r + 1)$-regular, $d_{G-S}(T) = (2r + 1)|T| - \|T, S\| \geq (2r + 1)|T| - [(2r + 1)|S| - \|R, S\|].$

Hence

$$c \geq (2r + 1 - 3k)(|T| - |S|) + 3.$$ 

Finally, every $Q$ counted by $q(S, T)$ adds at least 1 to $d_{G-S}(T)$, since $\|T, V(Q)\|$ is odd.
Lower Bound on Number of Cut-Edges, $c$

The upper and lower bounds on $3q(S, T)$ yield
$$2c + 2 \|R, S\| + d_{G-S}(T) \geq 3d_{G-S}(T) + 6k(|S| - |T|) + 6,$$
simplifying to
$$c \geq d_{G-S}(T) + 3k(|S| - |T|) - \|R, S\| + 3.$$

Since $G$ is $(2r + 1)$-regular, $d_{G-S}(T) = (2r + 1)|T| - \|T, S\| \geq (2r + 1)|T| - [(2r + 1)|S| - \|R, S\|].$

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When does equality hold?
Characterization

For $k \leq (2r + 1)/3$, a $(2r + 1)$-regular multigraph with $c = 2r + 4 - 3k$ and no $2k$-factor must satisfy equality in all the inequalities producing $c \geq 2r + 1 - 3(k - 1)$. 
Characterization

For \( k \leq \frac{2r+1}{3} \), a \((2r+1)\)-regular multigraph with \( c = 2r + 4 - 3k \) and no \( 2k \)-factor must satisfy equality in all the inequalities producing \( c \geq 2r + 1 - 3(k - 1) \). Thus \( q_1 = c \), \( q_2 = \|R, S\| \), \( q_1 + q_2 + 3q_3 = d_{G-S}(T) \),

\[
(2r + 1)|S| = \|T, S\| + \|R, S\|,
\]

and \( |T| - |S| \geq 1 \) (with equality when \( k < \frac{(2r+1)}{3} \)).
Characterization

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Thm. For $k \leq (2r + 1)/3$, a $(2r + 1)$-regular $G$ with $c = 2r + 4 − 3k$ has no $2k$-factor iff $V(G)$ splits to $R, S, T$ so
(a) $S$ and $T$ are independent sets with $|T| > |S|$,  
(b) all cut-edges join $T$ to distinct components of $G[R]$,  
(c) all edges at $S$ lead to $T$ (maybe via “blisters”),  
(d) exactly $k(|T| − |S|) − 1$ components of $G[R]$ are joined to $T$ by exactly three edges each,  
(e) other comps. of $R$ are $(2r + 1)$-regular, w/o cut-edge,  
(f) if $k < (2r + 1)/3$, then $|T| − |S| = 1$. 
Fewest Cut-Edges with No $2k$-factor
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**Def.** Blistering an edge $uv$ in a $(2r+1)$-regular multigraph

= insert a 2-edge-connected $(2r+1)$-regular multigraph

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Necessity of \( k \leq (2r + 1)/3 \)

**Prop.** For \( r \in \mathbb{N} \), when \( k > (2r + 1)/3 \) there are \((2r + 1)\)-regular graphs that are 3-edge-connected but have no \( 2k \)-factor.
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**Pf.** Let \( B_r = C_3 + rP_2 \) (\( 2r + 3 \) vertices, three with degree \( 2r \) and the rest with degree \( 2r + 1 \)).
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Pf. Let \( B_r = C_3 + rP_2 \) (2r + 3 vertices, three with degree 2r and the rest with degree 2r + 1).

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\( G \) has no \(2k\)-factor if \( q(S, T) > d_{G-S}(T) + 2k(|S| - |T|) \), which is equivalent to \( 6k > 4r + 2 \), or \( k > (2r + 1)/3 \).
2-Regular Subgraphs \((k = 1)\)

When \(c > 2r\), how large must \(f_2(G)\) be, where \(f_d(G) = \max \#\) verts in a \(d\)-regular subgraph?
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Examples

\[ d + c > 0 \implies \text{2-regular subgr. with } \geq n - \left\lfloor \frac{d+c-1}{2} \right\rfloor \text{ verts.} \]
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Explode any vertex \( y \in Y \) using 2-connected cubic \( F \)
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The Result

Let $\mathcal{F}$ be the family of all multigraphs obtained from cubic bipartite multigraphs in this way. (Delete one vertex, explode some subset of vertices in that part.)
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**Pf.** Use induction on $c$.

The induction step $c > 0$ is easy and reduces the problem to the base case $c = 0$ for subcubic multigraphs w/o cut-edges.
Induction Step \((c > 0)\)

\(G\) has \(c\) cut-edges and deficit \(d\).
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Some 2-regular subgraph \(H_i\) omits \(\leq \frac{d_i+c_i-1}{2}\) vertices.
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Since \(c = c_1 + c_2 + 1\) and \(d = d_1 + d_2 - 2\),
Induction Step ($c > 0$)

$G$ has $c$ cut-edges and deficit $d$.

Let $G_i$ have $c_i$ cut-edges and deficit $d_i$.

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Since $c = c_1 + c_2 + 1$ and $d = d_1 + d_2 - 2$,

Use $H_1 + H_2$; it omits at most $\frac{d + c - 1}{2}$ vertices.
Induction Step \((c > 0)\)

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Use \(H_1 + H_2\); it omits at most \(\frac{d + c - 1}{2}\) vertices.

Furthermore, equality in the bound for \(G\) requires equality for both \(G_1\) and \(G_2\).
Base Case \((c = 0)\)

Reduce to \(n \geq 2\), mindegree 2, maxdegree 3.
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From \(G\), get weighted cubic \(G'\) – ‘suppress’ 2-vertices:

\[ \begin{array}{c}
\bullet & \bullet & \bullet \\
\end{array} \quad \rightarrow \quad \begin{array}{c}
\bullet & 3 \\
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\(G'\) is 3-regular, no cut-edge, has a perfect matching \(M\)
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\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{diagram.png}} \quad \rightarrow \quad \text{\includegraphics[width=0.1\textwidth]{diagram.png}}
\end{array}
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\(G'\) is 3-regular, no cut-edge, has a perfect matching \(M\) whose \textbf{weight} is at most \(1/3\) of the total weight.
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\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \rightarrow & \bullet & \bullet
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Omits \(n - |V(H)| \leq n - 2m/3\)
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Reduce to \(n \geq 2\), mindegree 2, maxdegree 3.

From \(G\), get weighted cubic \(G'\) – ‘suppress’ 2-vertices:

\[
\begin{array}{c}
\bullet \bullet \bullet \bullet \rightarrow \bullet \bullet \bullet \\
0 \quad 1 \quad 2 \quad 3
\end{array}
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If \(d = 3\) and no 2-factor (and \(c = 0\)), show that \(G \in \mathcal{F}\).
The Case $d = 3$ and $c = 0$ and no 2-factor

At each of the three 2-vertices of $G$, add a cut-edge and a balloon to form a 3-regular graph $G'$. 
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\( \therefore m = 3|S| \), and \( S \) is independent, and \( G' - S \) consists of the balloons plus odd components getting three edges.
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Open Problems

**Ques.** For a \((2r + 1)\)-regular graph with \(c\) cut-edges, when \(c > 2r\) how large a 2-regular subgraph is forced?
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