Trees with at least $6\ell + 10$ vertices are $\ell$-reconstructible

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Abstract

The $(n - \ell)$-deck of an $n$-vertex graph is the multiset of (unlabeled) subgraphs obtained from it by deleting $\ell$ vertices. An $n$-vertex graph is $\ell$-reconstructible if it is determined by its $(n - \ell)$-deck, meaning that no other graph has the same deck. We prove that every tree with at least $6\ell + 10$ vertices is $\ell$-reconstructible.

1 Introduction

The $j$-deck of a graph is the multiset of its $j$-vertex induced subgraphs. We write this as the $(n - \ell)$-deck when the graph has $n$ vertices and the focus is on deleting $\ell$ vertices. An $n$-vertex graph is $\ell$-reconstructible if it is determined by its $(n - \ell)$-deck. Since every member of the $(j-1)$-deck arises $n-j+1$ times by deleting a vertex from a member of the $j$-deck, the $j$-deck of a graph determines its $(j-1)$-deck. Therefore, a natural reconstruction problem is to find for each graph the maximum $\ell$ such that it is $\ell$-reconstructible. For this problem, Manvel [11, 12] extended the classical Reconstruction Conjecture of Kelly [6] and Ulam [17].

Conjecture 1.1 (Manvel [11, 12]). For $\ell \in \mathbb{N}$, there exists a threshold $M_\ell$ such that every graph with at least $M_\ell$ vertices is $\ell$-reconstructible.

Manvel named this “Kelly’s Conjecture” in honor of the final sentence in Kelly [7], which suggested that one can study reconstruction from the $(n-2)$-deck. Manvel noted that Kelly may have expected the statement to be false.

The classical Reconstruction Conjecture is $M_1 = 3$. Lacking a proof of Conjecture 1.1 for any fixed $\ell$, we study threshold numbers of vertices for $\ell$-reconstructibility of graphs in

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special classes. The survey by Kostochka and West [8] describes prior such results. Here our aim is to reduce the threshold number of vertices to guarantee \( \ell \)-reconstructibility of trees.

Reconstruction arguments for special families have two parts, named (when \( \ell = 1 \)) by Bondy and Hemminger [1]. When the \((n-\ell)\)-deck guarantees that all reconstructions or no reconstructions lie in the specified family, the family is \( \ell \)-recognizable. Separately, using the knowledge that every reconstruction from the deck is in the family, one determines that only one graph in the family has that deck; this makes the family weakly \( \ell \)-reconstructible. Together, the two steps make graphs in the family \( \ell \)-reconstructible.

N\'{y}dl [14] conjectured that trees with at least \( 2\ell + 1 \) vertices are weakly \( \ell \)-reconstructible, having presented in [13] two trees with \( 2\ell \) vertices having the same \( \ell \)-deck, to make the conjecture sharp. The two trees arise from a path with \( 2\ell - 1 \) vertices by adding one leaf, adjacent either to the central vertex of the path or to one of its neighbors. Kostochka and West [8] used the results of Spinoza and West [15] to give a short proof of N\'{y}dl’s result.

When \( \ell = 2 \), the disjoint union of a 4-cycle and an isolated vertex has the same deck as these two trees, so 5-vertex trees are not a 2-recognizable family. However, Kostochka, Nahvi, West, and Zirlin [10] proved that \( n \)-vertex acyclic graphs are \( \ell \)-recognizable when \( n \geq 2\ell + 1 \), except for \( (n, \ell) = (5, 2) \). As noted earlier, the \((n-\ell)\)-deck guarantees the \( k \)-deck whenever \( k < n - \ell \), so the \((n-\ell)\)-deck also yields the 2-deck, which fixes the number of edges. An \( n \)-vertex graph is a tree if and only if it is acyclic and has \( n - 1 \) edges, so the family of \( n \)-vertex trees is \( \ell \)-recognizable when \( n \geq 2\ell + 1 \) (except for \( (n, \ell) = (5, 2) \)).

This suggests modifying N\'{y}dl’s conjecture to say that trees with at least \( 2\ell + 1 \) vertices are \( \ell \)-reconstructible (modifying to \( 2\ell + 2 \) when \( \ell = 2 \)). Indeed, Kelly [7] proved that trees with at least three vertices are 1-reconstructible, and Giles [2] proved that trees with at least six vertices are 2-reconstructible. Hunter [5] proved that caterpillars with at least \((2 + o(1))\ell \) vertices are \( \ell \)-reconstructible. Groenland, Johnston, Scott, and Tan [4] found one counterexample to N\'{y}dl’s conjecture for \( \ell = 6 \): two specific trees with 13 vertices having the same 7-deck. However, they proved that a threshold number of vertices does exist for trees:

**Theorem 1.2 ([4]).** When \( j \geq \frac{8}{9}n + \frac{4}{9}\sqrt{8n + 3} + 1 \), every \( n \)-vertex tree is determined by its \( j \)-deck. Thus \( n \)-vertex trees are \( \ell \)-reconstructible when \( n \geq 9\ell + 24\sqrt{2\ell} + o(\sqrt{\ell}) \).

For \( \ell = 3 \), this theorem applies when \( n \geq 194 \). Using reconstruction of rooted trees, Kostochka, Nahvi, West, and Zirlin [9] gave a lengthy proof of the threshold \( n \geq 25 \) when \( \ell = 3 \). Our aim in this paper is to lower the general threshold for \( \ell \)-reconstructibility of trees by proving the following theorem, which brings the threshold for \( \ell = 3 \) down to \( n \geq 28 \).

**Theorem 1.3.** When \( n \geq 6\ell + 10 \), all \( n \)-vertex trees are \( \ell \)-reconstructible.

As noted earlier, the family of such trees is already known to be \( \ell \)-recognizable, so we may assume that all reconstructions from the \((n-\ell)\)-decks of such trees are trees.
2 Vines and Diameter

In our study of $\ell$-reconstructibility of $n$-vertex trees, we use different methods depending on the existence of special subgraphs and the value of the diameter in relation to $n$ and $\ell$.

**Definition 2.1.** In a graph $G$, the *distance* between two vertices is the minimum length of a path containing them. The *eccentricity* of a vertex in a graph $G$ is the maximum of the distances from it to other vertices. A *center* of $G$ is a vertex of minimum eccentricity, and the minimum eccentricity is called the *radius* of $G$. The *diameter* of $G$ is the maximum eccentricity, which is the maximum distance between vertices.

It is an elementary exercise that a tree has one center or two adjacent centers, when the diameter is even or odd, respectively.

In the deck of an $n$-vertex tree, no cards are paths precisely when there are no paths with $n - \ell$ vertices. We then know the diameter, because we see all paths in the cards and know that there are no longer paths. Later in this section we will compute the diameter from the deck even when it is larger.

In order to identify important structures in a tree from the deck, we will want to count various subgraphs. We used these concepts in [10] in the more general situation of $n$-vertex graphs whose $(n - \ell)$-decks have no cards containing cycles.

**Definition 2.2.** A *$j$-vine* is a tree with diameter $2j$. A *$j$-evine* is a tree with diameter $2j + 1$. A *$j$-center* or *$j$-central edge* is the central vertex or edge in a $j$-vine or $j$-evine, respectively.

**Lemma 2.3.** In a tree $T$, every $j$-vine or $j$-evine $H$ lies in a unique maximal $j$-vine or $j$-evine, respectively.

**Proof.** The unique maximal such graph containing $H$ is the subgraph induced by the set of all vertices within distance $j$ of its central vertex or central edge. This is a $j$-vine or $j$-evine, respectively. Any other such subgraph containing it would have to have the same center or central edge, but then it cannot have any additional vertices. $\square$

For a family $\mathcal{F}$ of graphs, an *$\mathcal{F}$-subgraph* of a graph $G$ is an induced subgraph of $G$ belonging to $\mathcal{F}$. For $F \in \mathcal{F}$, let $m(F, G)$ be the number of copies of $F$ that are maximal $\mathcal{F}$-subgraphs in $G$. The case $\ell = 1$ of the next lemma is due to Greenwell and Hemminger [3]. Similar statements for general $\ell$ appear for example in [4]. We include a proof for completeness; it is slightly simpler than proofs in the literature involving inclusion chains.

**Lemma 2.4** ([10]). Let $\mathcal{F}$ be a family of graphs such that every subgraph of $G$ belonging to $\mathcal{F}$ lies in a unique maximal subgraph of $G$ belonging to $\mathcal{F}$. If for every $F \in \mathcal{F}$ with at least $n - \ell$ vertices the value of $m(F, G)$ is known from the $(n - \ell)$-deck of $G$, then for all $F \in \mathcal{F}$ the $(n - \ell)$-deck $\mathcal{D}$ determines $m(F, G)$.
Proof. Let $t = |V(G)| - |V(F)|$; we use induction on $t$. When $t \leq \ell$, the value $m(F,G)$ is given. When $t > \ell$, group the induced subgraphs of $G$ isomorphic to $F$ according to the unique maximal $F$-subgraph of $G$ containing them (as an induced subgraph). Counting all copies of $F$ then yields

$$s(F,G) = \sum_{H \in F} s(F,H)m(H,G).$$

Since $|V(F)| < n - \ell$, we know $s(F,G)$ from the deck, and we know $s(F,H)$ when $F$ and $H$ are known. By the induction hypothesis, we know all values of the form $m(H,G)$ when $F$ is an induced subgraph of $H$ except $m(F,G)$. Therefore, we can solve for $m(F,G)$. □

Next we establish consistent notation for our subsequent discussion.

**Definition 2.5.** Always $D$ denotes the $(n - \ell)$-deck of an $n$-vertex tree $T$; we call $D$ simply the deck of $T$. Also $r$ denotes always the maximum number of vertices in a path in $T$. A connected subcard or simply subcard of $T$ is a connected subgraph of $T$ with at most $n - \ell$ vertices; it appears in a connected card. We use “csc” for “connected subcard”.

**Definition 2.6.** Fix $k$ to be the largest integer $j$ such that $T$ contains a $j$-evine and every $j$-evine in $T$ has fewer than $n - \ell$ vertices. This fixes $k$ in terms of $T$ for the remainder of the paper.

**Lemma 2.7.** The value of $k$ is determined by the deck of $T$.

Proof. Each edge forms a 0-evine (which are the only 0-evines), and every $j$-evine with $j > 0$ contains a smaller $(j - 1)$-evine, so the value of $k$ is well-defined. It remains to compute $k$.

All subgraphs with at most $n - \ell$ vertices are visible from the deck. The deck yields a largest value $j'$ such that some csc is a $j'$-evine and all such cscs have fewer than $n - \ell$ vertices. Since $k$ is an integer having these properties, $j' \geq k$. Since some csc is a $j'$-evine with fewer than $n - \ell$ vertices, $2j' + 2 < n - \ell$.

By the choice of $j'$, no $j'$-evine in $T$ has exactly $n - \ell$ vertices, so a smallest $j'$-evine with more than $n - \ell$ vertices can only be a path, which would require $2j' + 2 > n - \ell$. Thus no such $j'$-evine exists, yielding $k \geq j'$. Hence $j' = k$. □

**Lemma 2.8.** In a reconstruction $T$ from $D$, there exist $k$-vines, and every $k$-vine in $T$ has fewer than $n - \ell$ vertices.

Proof. By the definition of $k$, there is a $k$-evine in $T$. A $k$-evine contains two $k$-vines whose centers form the central edge of the $k$-evine.

If some $k$-vine has at least $n - \ell$ vertices, then let $C$ with center $z$ be a largest $k$-vine in $T$. Since $T$ contains a $k$-evine, the diameter of $T$ is at least $2k + 1$, and hence the radius
of $T$ is at least $k + 1$. Thus $T$ has a vertex $x$ at distance $k + 1$ from $z$. Note that $x$ is not contained in $C$, but $x$ has a neighbor $y$ in $C$ at distance $k$ from $z$. Adding $x$ and the edge $xy$ to $C$ creates a $k$-evine with more than $n - \ell$ vertices, contradicting the definition of $k$. □

**Corollary 2.9.** For $j \leq k$, the deck $D$ determines the numbers of maximal $j$-evines and $j$-vines with each isomorphism type. All reconstructions from $D$ have the same numbers of $j$-centers and $j$-central edges.

*Proof.* Fix $j$, and let $F$ be the family of $j$-vines or the family of $j$-evines. By the definition of $k$ and Lemmas 2.7 and 2.8, $m(F, T) = 0$ for all $F \in F$ having at least $n - \ell$ vertices. Hence Lemma 2.4 applies to compute $m(F, T)$ for all $F \in F$. By Lemma 2.3, there is a one-to-one correspondence between the maximal $j$-vines and the $j$-centers, and similarly for $j$-evines and $j$-central edges. □

Since $1$-vines are stars, setting $j = 1$ in Corollary 2.9 provides the degree list if $k \geq 1$. Groenland et al. [4] proved that the degree list is $\ell$-reconstructible for all $n$-vertex graphs whenever $n - \ell > \sqrt{2n \log(2n)}$. Taylor [16] had shown that asymptotically $n > \ell e$ is enough. For trees we obtain a simpler intermediate threshold that suffices for our needs.

**Corollary 2.10.** For $n \geq 2\ell + 3$, the degree list of any $n$-vertex tree is determined by its $(n - \ell)$-deck.

*Proof.* All $1$-vines are stars. Each vertex with degree at least $2$ is the center of exactly one maximal $1$-vine. Suppose first that no star has at least $n - \ell$ vertices. For $t \geq 3$, by the counting argument (Lemma 2.4) the deck determines the number of maximal $1$-vines having $t$ vertices. This is the number of vertices with degree $t - 1$ in any reconstruction.

Now suppose that some star has at least $n - \ell$ vertices. Since $n - \ell \geq 4$, we see in the deck that there are no $3$-cycles or $4$-cycles, so two stars share at most a common leaf or an edge joining the centers. Having two stars with at least $n - \ell$ vertices thus requires $n \leq 2\ell + 2$.

Hence only one vertex has degree at least $n - \ell - 1$. Its degree is $d$ if and only if exactly $\binom{d}{n-\ell-1}$ cards are stars. Thus we have $m(S, T)$ for any reconstruction $T$ and every star $S$ with at least $n - \ell$ vertices. Again Lemma 2.4 applies and we obtain the number of vertices with degree $t - 1$ whenever $t \geq 3$.

Since every reconstruction is a tree, the remaining vertices have degree $1$. □

**Lemma 2.11.** Every connected card in $D$ has diameter at least $2k + 2$, and some connected card has diameter at most $2k + 3$. 

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Proof. A connected card with diameter at most \(2k + 1\) would be a \(j\)-evine or \(j\)-vine with \(j \leq k\) having \(n - \ell\) vertices, contradicting the definition of \(k\) or Lemma 2.8.

For the second claim, let \(C\) be a connected card. If \(C\) has diameter at least \(2k + 3\), then \(C\) contains a path with \(2k + 4\) vertices. Hence \(n - \ell \geq 2k + 4\) and \(T\) contains a \((k + 1)\)-evine. By the definition of \(k\), some \((k + 1)\)-evine has at least \(n - \ell\) vertices. Since \(n - \ell \geq 2k + 4\), we can iteratively delete leaves of such a \((k + 1)\)-evine outside a path with \(2k + 4\) vertices to trim the subtree to \(n - \ell\) vertices. We thus obtain a card that is a \((k + 1)\)-evine and has diameter \(2k + 3\). Hence some card has diameter at most \(2k + 3\). □

In a tree \(T\), the number \(r\) of vertices in a longest path is the diameter plus 1.

**Lemma 2.12.** \((r - 3)/2 \geq k \geq (r - \ell - 4)/2\). In particular, if \(r \geq 3\ell + 6\), then \(k \geq \ell + 1\).

**Proof.** Lemma 2.11 guarantees a connected card with diameter at least \(2k + 2\). It contains a path with at least \(2k + 3\) vertices, so \(r \geq 2k + 3\).

Let \(P\) be a longest path in \(T\). By Lemma 2.11, some connected card \(C\) has diameter at most \(2k + 3\). Since \(T\) has no cycle and \(C\) has no path with more than \(2k + 4\) vertices, at most \(2k + 4\) vertices of \(P\) appear in \(C\). In addition, \(T\) has only \(\ell\) vertices outside \(C\). Hence \(r \leq 2k + 4 + \ell\), which yields the claimed lower bound on \(k\).

When \(r \geq 3\ell + 6\), the lower bound on \(k\) simplifies to \(k \geq \ell + 1\). □

When \(r < n - \ell\), the value of \(r\) is the maximum number of vertices in a path contained in a card. Next we show that the deck also determines \(r\) when \(r \geq n - \ell\).

**Lemma 2.13.** If \(n \geq 4\ell + 8\) and \(r \geq n - \ell\), then \(k \geq \ell + 2\), all paths in \(T\) with more than \(n - r\) vertices intersect any longest path \(P\), all \(k\)-centers and \((k - 1)\)-centers in \(T\) lie on \(P\), and the value of \(r\) is determined by the deck.

**Proof.** With \(r \geq n - \ell\) and \(n \geq 4\ell + 8\), Lemma 2.12 yields \(k \geq (n - 2\ell - 4)/2 \geq \ell + 2\). All paths with more than \(n - r\) vertices intersect \(P\), since only \(n - r\) vertices exist outside \(P\).

Since \(\ell \geq n - r\), all paths with at least \(\ell + 1\) vertices intersect \(P\). Since also \(k \geq \ell + 2\), all paths with at least \(k - 1\) vertices intersect \(P\). From a vertex outside \(P\), only one path leads to \(P\); hence all \(k\)-centers and \((k - 1)\)-centers lie on \(P\).

Since \(T\) has a \(k\)-vine, \(r \geq 2k + 1\), so \(k\)-centers (and \((k - 1)\)-centers) do exist on \(P\). When \(2j + 1 \leq r\), all the vertices of \(P\) are \(j\)-centers except the \(j\) vertices closest to each end. By Corollary 2.9, the deck determines the number \(s\) of \(k\)-centers. Hence \(r = s + 2k\). □

When \(k \geq \ell + 1\), it follows from Corollary 2.9 that we know the number of \((\ell + 1)\)-centers in \(T\). Furthermore, we know the multiset of maximal \((\ell + 1)\)-vines in \(T\). Among these, there is a particular type of \((\ell + 1)\)-vine that figures heavily in our analysis.
Definition 2.14. A branch vertex in a tree is a vertex having degree at least 3. A leg in a non-path tree is a path from a leaf to the nearest branch vertex. A spider is a tree with at most one branch vertex; its degree is the degree of the branch vertex. The spider $S_{j_1, \ldots, j_d}$ with $1 + \sum_{i=1}^{d} j_i$ vertices is the union of legs with lengths $j_1, \ldots, j_d$ having a common endpoint. A spi-center is an $(\ell+1)$-center that is the branch vertex in a copy of $S_{\ell+1,\ell+1,\ell+1}$ in $T$. We will often discuss 3-legged spiders whose three legs have the same length. For this special situation with legs of length $j$ we write $S^j$; that is, $S_{\ell+1,\ell+1,\ell+1}^j = S_{\ell+1,\ell+1,\ell+1}^j$.

Corollary 2.15. If $k \geq \ell + 1$, then the deck determines the number of spi-centers in $T$.

Proof. When $j \leq k$, by Corollary 2.9 we know all the maximal $j$-vines. The spi-centers correspond bijectively to the maximal $(\ell + 1)$-vines that contain $S^j$. □

Definition 2.16. An $r$-path is an $r$-vertex path. In any subgraph of a tree whose longest paths have $r$ vertices, a full path is an $r$-path. A full card is a connected card containing an $r$-path. Similarly, a full csc or full subtree is a csc or subtree containing an $r$-path.

A sparse card is a connected card (that is, having $n - \ell$ vertices) containing a full path on which there is only one branch vertex of the card, and that branch vertex is the primary vertex of the card. The degree of a sparse card $C$ is the degree in $C$ of its primary vertex.

Lemma 2.17. Suppose $n \geq 4\ell + 1$ and $r < n - \ell$. If $D$ contains a sparse card $C$ that is a 3-legged spider with branch vertex $z$, then $T$ has no spi-center other than $z$.

Proof. Since $C$ has $n - \ell$ vertices, the leg of $C$ leaving the $r$-vertex path in $C$ has $n - \ell - r$ vertices. A copy of $S_{\ell+1}^j$ with branch vertex other than $z$ would have a leg completely outside $C$. Since $T$ has only $\ell$ vertices outside $C$, this cannot occur. □

Definition 2.18. In a tree $T$, an offshoot from a vertex set $S$ (or from the subgraph induced by $S$) is a component of $T - S$, rooted at its vertex having a neighbor in $S$. The length of an offshoot is the maximum number of vertices in a path in it that begins at its root.

Lemma 2.19. Let $P$ be a full path in $T$ with vertices $v_1, \ldots, v_r$. If $v_j$ with $j \leq (r+1)/2$ is a spi-center and no vertex between $v_j$ and $v_{r+1-j}$ is a spi-center, then all longest paths have $v_j, \ldots, v_{r+1-j}$ as their $r - 2j + 2$ central vertices.

Proof. Since $v_j$ is a spi-center, $T$ has an offshoot from $P$ at $v_j$ with length at least $\ell + 1$. Since $P$ is a longest path, $\ell + 1 \leq j - 1$.

All longest paths in a tree have the same central vertex (or vertex pair). If a longest path $P'$ diverges from $P$ at some vertex $w$ between $v_j$ and $v_{r+1-j}$, then $P'$ has at least $j$ vertices outside $P$. Now $w$ is a spi-center, contradicting the hypothesis. □
We will often consider offshoots from a longest path \( P \). The union of a vertex \( z \) on \( P \) together with the offshoots from \( P \) at \( z \) is a rooted tree with \( z \) as root. The next lemma describes somewhat abstractly an exclusion argument that we use in various situations to obtain offshoots at a vertex \( z \) of a tree being reconstructed.

**Lemma 2.20.** For a rooted tree \( R \) with root \( z \), the \( z \)-offshoots are the components of the graph \( R - z \), rooted at the neighbors of \( z \). If the largest \( z \)-offshoots of \( R \) are known (with multiplicities), then the complete list of \( z \)-offshoots is determined by the multiset of all rooted subtrees obtained from individual \( z \)-offshoots.

*Proof.* The \( z \)-offshoots are determined in nonincreasing order of number of vertices. The largest \( z \)-offshoots are given initially. Having determined all \( z \)-offshoots with more than \( p \) vertices, let \( C \) be a rooted tree with \( p \) vertices. Since we know all \( z \)-offshoots of \( R \) larger than \( C \), we know how many copies of \( C \) arise from larger \( z \)-offshoots by deleting some number of leaves. The number of copies of \( C \) as a \( z \)-offshoot is obtained from the number of copies of \( C \) in the given multiset by excluding those that arise from larger \( z \)-offshoots. \( \square \)

### 3 Sparse Cards and 3-Legged Spiders

We maintain the terminology and notation from Section 2. In this section we show that trees containing \( S^{\ell+1} \) (that is, having a spi-center) are \( \ell \)-reconstructible when their diameter and number of vertices are not extremely small. Note that \( r \geq 2\ell + 3 \) due to the presence of \( S^{\ell+1} \). We may need a stronger lower bound on \( r \) to guarantee \( k \geq \ell + 1 \). By Lemma 2.12, \( r \geq 3\ell + 6 \) implies \( k \geq \ell + 1 \). This lower bound on \( r \) is also implied by \( n \geq 6\ell + 6 \) and \( r \geq n - 3\ell \).

**Lemma 3.1.** Suppose \( n \geq 6\ell + 5 \) and \( r \geq 3\ell + 6 \). If \( T \) has a sparse card and exactly one spi-center, then \( T \) is \( \ell \)-reconstructible.

*Proof.* As noted above, \( r \geq 3\ell + 6 \) implies \( k \geq \ell + 1 \). By Lemma 2.7, we know \( k \), and then Corollary 2.15 gives us the number of spi-centers. A full path omits at least one leg in any copy of \( S^{\ell+1} \), so \( S^{\ell+1} \subseteq T \) implies \( r < n - \ell \). We can then see whether there is a sparse card in the deck. Hence this case is recognizable.

Let \( C_1 \) be a sparse card with full path \( P \) and primary vertex \( z \). We claim that \( z \) is the spi-center in \( T \). Otherwise, let \( w \) be the vertex on \( P \) closest to the spi-center. Since \( P \) is a longest path, the two portions of \( P \) leaving \( w \) are as long as the leg of \( S^{\ell+1} \) not on \( P \), making \( w \) the spi-center. Also, since \( C_1 \) is a sparse card, the offshoots in \( T \) from \( P \) at \( z \) together have at least \( n - \ell - r \) vertices, so at most \( \ell \) vertices lie in offshoots from \( P \) at vertices other than \( z \). Thus no vertex of \( P \) other than \( z \) can be a spi-center. See Figure 1.
Letting the vertices of $P$ be $v_1, \ldots, v_r$ in order, we may specify $j$ with $j \leq (r + 1)/2$ by $z \in \{v_j, v_{r+1-j}\}$, since we see $z$ in $C_1$. We cannot yet tell which of $v_j$ and $v_{r+1-j}$ is $z$ (unless $j = (r + 1)/2$). By Lemma 2.19, every full path in $T$ has $v_j, \ldots, v_{r+1-j}$ as its central portion.

![Figure 1: Sparse card $C_1$ and large csc $C_2$](image)

Let $C_2$ be a largest csc that has a full path $P'$ whose $j$th and $(r + 1 - j)$th vertices have degree 2 in $C_2$. Since the $r - 2j + 2$ central vertices of $P'$ are the same as in $P$, one end of the central portion of $P'$ is $z$, but we do not know which end. Since $z$ is a spi-center, we omit at least $\ell + 1$ vertices in the offshoots from $P'$ at $z$, so $C_2$ has fewer than $n - \ell$ vertices. By its maximality, $C_2$ contains in full all offshoots from $P'$ except those at $v_j$ and $v_{r+1-j}$. By fixing $C_2$ and the choice of $P'$ within it, we may assume $P' = P$. An offshoot at $z$ may have length $j - 1$, permitting flexibility in forming full paths, but our choice of $C_2$ and designation of $v_1, \ldots, v_r$ fixes $P$ within $C_2$. See Figure 1.

By Lemma 2.7, every $k$-vine in $T$ has fewer than $n - \ell$ vertices and hence appears as a csc. Since $\ell + 1 \leq k$, this also holds for every $(\ell + 1)$-vine. Let $\hat{C}$ be the unique largest $(\ell + 1)$-vine containing $S^{\ell+1}$; it has center $z$. In $\hat{C}$ we see all vertices at distance $\ell + 1$ from $z$ in $T$; let $n'$ be the number of them. We have indexed $V(P)$ as $v_1, \ldots, v_r$, with $v_j$ being the spi-center $z$. Let $T_1$ and $T_2$ be the components of $T - z$ containing $v_{j-1}$ and $v_{j+1}$, respectively; we do not yet know these subtrees. Similarly, we do not yet know $W$ and $W'$, defined to be the unions of the offshoots from $P$ in $T$ at $z$ and at the vertex other than $z$ in $\{v_j, v_{r+1-j}\}$, respectively.

**Claim:** If we know $z = v_j$, and the offshoots from $C_2$ at all vertices other than $z$ have been determined, then $T$ can be reconstructed from the deck. These offshoots include $W'$, so under this hypothesis we also know $T_1$ and $T_2$, and it remains to determine $W$.

Let $\mathcal{C}$ be the family of cscs consisting of a path with $2\ell + 3$ vertices and offshoots from the center of the path, at least one of which has length at least $\ell + 1$. All such cscs have $z$ as the center of that path, since they contain $S^{\ell+1}$ and $T$ has only one spi-center.

Since $r \geq 3\ell + 3$, any csc in $\mathcal{C}$ in which two path legs of length $\ell + 1$ come from $T_1$ and $T_2$ omits at least $\ell + 1$ vertices of $P$ and hence is properly contained in a card. We know their number of vertices, since knowing $n$, $r$, $T_1$, and $T_2$ tells us $|V(W)|$.

Among the cscs in $\mathcal{C}$ having $2\ell + 3 + |V(W)|$ vertices, we seek those that contain $W$. Let $n_0$, $n_1$, and $n_2$ be the numbers of vertices at distance $\ell + 1$ from $z$ in $W$, $T_1$, and $T_2$, respectively. We know $n_1$ and $n_2$ from $T_1$ and $T_2$, and we compute $n_0 = n' - n_1 - n_2$. The number of cscs with $2\ell + 3 + |V(W)|$ vertices having path legs in $W$ and $T_2$ is $n_0n_2$ times
the number of rooted subtrees of $T_1$ with $|V(W)|$ vertices, and we know those subtrees. We similarly eliminate the cscs having legs in $W$ and $T_1$. The remaining $n_1n_2$ members of $C$ with $2\ell + 3 + |V(W)|$ vertices show us $W$. This completes the proof of the claim.

It therefore suffices to determine $T_1$ and $T_2$ and which of $v_j$ and $v_{r+1-j}$ is $z$.

**Case 1:** $j < (r + 1)/2$ and $C_2$ is asymmetric. By asymmetric, we mean that no automorphism of $C_2$ reverses the indexing of $P$. That is, in $C_2$ the vertices $v_j$ and $v_{r+1-j}$ are distinguishable, although we do not yet know which is $z$.

Recall that $C_2$ has fewer than $n - \ell$ vertices. Consider augmentations of $C_2$ obtained by adding a path leaving one of $\{v_j, v_{r+1-j}\}$. Let $m$ be the maximum $i$ such that we can obtain cscs by adding to $C_2$ a path of $i$ vertices from either $v_j$ or $v_{r+1-j}$. Since $C_1$ has offshoots with at least $n - r - \ell$ vertices from $P$ at $z$, there are at most $\ell$ vertices in offshoots from $P$ at other vertices, and hence $m \leq \ell$. Since $T$ has a path of $\ell + 1$ vertices grown from $P$ at $z$, we can tell which of $\{v_j, v_{r+1-j}\}$ is $z$ by considering the trees obtained from $C_2$ by growing a path of $m + 1$ vertices from $v_j$ or $v_{r+1-j}$. The one that exists as a csc fixes $z$, since $C_2$ is asymmetric. By reversing the indexing of $C_2$ along $P$ if needed, we may assume $z = v_j$.

Now let $C_3$ be a largest subtree of $T$ containing $C_2$ in which $v_j$ has degree 2. Since $C_3$ omits at least $\ell + 1$ vertices in an offshoot at $v_j$, we see all of $C_3$ in a card, and $C_3$ contains all offshoots from $P$ at vertices other than $v_j$ in full. This determines $T_1$, $T_2$, and $W'$, and the Claim applies to complete the reconstruction of $T$.

**Case 2:** $j < (r + 1)/2$ and $C_2$ is symmetric. By symmetry we may assume $z = v_j$, but we do not yet know the offshoots at $v_j$ (that is, $W$) or at $v_{r+1-j}$ (that is, $W'$).

**Subcase 2a:** $r + 1 - 2j \leq \ell + 1$. Let $t = r - 2j + 1 + \ell + 1$. Let $C_4$ be a largest subtree containing $S_{\ell+1,\ell+1,t}$ such that offshoots from the spider occur only at the vertex $w$ in the long leg having distance $r + 1 - 2j$ from the branch vertex. Since $P$ is a longest path, $\ell + 1 \leq j - 1$, so $j + t \leq r$ and $T$ contains such a tree with the long leg of the spider along $P$.

The branch vertex of this spider in $C_4$ must be $z$, since $T$ has only one spi-center. By Lemma 2.19, the long leg of $S_{\ell+1,\ell+1,t}$ in $C_4$ follows $P$ past $v_{r+1-j}$, and $v_{r+1-j} = w$. Since the leg extends $\ell + 1$ vertices past $v_{r+1-j}$ and there are at most $\ell$ vertices in offshoots from $P$ not at $z$, the offshoots from the spider at $w$ must lie in $W'$. Since there are at most $\ell$ vertices in $W'$, and $r + 1 - 2j \leq \ell + 1$, in $C_4$ there are at most $5\ell + 5$ vertices. Since $n \geq 6\ell + 5$, the tree $C_4$ fits in a card, and we see it in the deck. By its maximality, $C_4$ shows us all of $W'$. Now we know $T_1$ and $T_2$, and the Claim applies to complete the reconstruction of $T$.

**Subcase 2b:** $r + 1 - 2j \geq \ell + 2$. Let $C_5$ be a largest subtree containing $S_{\ell+1,j-1,j}$ such that the end of the leg of length $j$ in the spider is a leaf of $C_5$. Since $\ell + 1 \leq j - 1$ and $T$ has only one spi-center, the branch vertex of the required spider in $C_5$ must be $z$. All paths of length at least $j$ from $z$ lie in $T_2$. Since $r - j \geq \ell + j + 1$, the subtree $C_5$ omits more than $\ell$ vertices from the end of $P$ and hence appears in a card. Whether the long leg of the spider in $C_5$ extends into $W'$ or not, by its maximality $C_5$ shows us $T_1$ and all offshoots in $W$ (it is
possible that \( W \) has length \( j - 1 \), making \( T_1 \) and \( W \) confusable, but we know them both).

From \( C_2 \), we know all of \( T_2 \) except \( W' \). Knowing \( W \), we know \( |V(W)| \). Since we also know \( C_2 \), we know how many vertices remain for \( W' \); let \( m \) denote this value. Now consider subtrees of \( T \) containing a full path \( u_1, \ldots, u_r \) such that \( u_j \) has degree 2, the offshoots from \( u_{r+1-j} \) total \( m \) vertices, and the offshoots from other vertices along the path total \( n - m - |V(W)| \) vertices. Since such subtrees with \( u_j = v_j \) omit at least a path of \( \ell + 1 \) vertices from \( W \), they fit in cards, so we see them. Since we know \( W \), we know all such subtrees in which \( u_{r+1-j} = z \). Discarding them leaves a csc that shows us \( W' \), completing the reconstruction of \( T \).

**Case 3:** \( j = (r + 1)/2 \). Here the primary vertex \( z \) of \( C_1 \) is the central vertex of \( P \). The csc \( C_2 \) defined earlier gives us all of \( T \) except the offshoots at \( z \); in particular, we know \( T_1 \) and \( T_2 \). Since there is no vertex \( v_{r+1-j} \) distinct from \( z \) we need only determine the offshoots from \( C_2 \) at \( z \). The Claim applies to complete the reconstruction of \( T \). \( \square \)

**Lemma 3.2.** Suppose \( n \geq 6 \ell + 5 \), and \( r \geq 3 \ell + 6 \), and \( S^{\ell+1} \subseteq T \). If also \( T \) has a sparse card, then \( T \) is \( \ell \)-reconstructible.

*Proof.* The hypotheses of this lemma are the same as in Lemma 3.1, except for dropping the restriction to having only one spi-center. Hence we have all the steps of Lemma 3.1 available until we first use the restriction on spi-centers. In particular, we have the sparse card \( C_1 \) containing \( r \)-vertex path \( P \) with primary branch vertex \( z \), recognize \( k \geq \ell + 1 \), and see \( S^{\ell+1} \) in \( T \). Again, \( z \) is a spi-center, and any other spi-center is in \( V(C_1) - V(P) \). Again, we define \( j \) with \( j \leq (r + 1)/2 \) by \( z = v_j \) with \( P \) indexed as \( v_1, \ldots, v_r \). By Lemma 2.19, all \( r \)-vertex paths have the same central vertices \( v_j, \ldots, v_{r+1-j} \).

Lemma 3.1 handles the case when \( T \) has only one spi-center. Since by Corollary 2.15 we know the number of spi-centers, we recognize that we are not in that case. Hence we may assume that \( C_1 \) has a vertex in an offshoot from \( P \) at \( z \) that is a spi-center. There may be more than one such vertex; let \( x \) be one such vertex. Because \( P \) is a longest path and some path from \( z \) extends at least \( \ell + 1 \) vertices beyond \( x \), we have \( j - 1 \geq \ell + 2 \).

Let \( C' \) be a largest subtree of the following sort: \( C' \) contains a vertex \( z' \) such that one offshoot from \( z' \) in \( C' \) has length \( r - j \) and all other offshoots from \( z' \) in \( C' \) contain spi-centers in \( C' \). There is such a subtree with \( z' = z \), consisting of \( z \) and all components of \( T - z \) containing \( v_{j+1}, \ldots, v_r \) or a spi-center other than \( z \). Indeed, since \( P \) is a longest path, this is the only way to form \( C' \). Thus \( C' \) omits \( v_1, \ldots, v_{j-1} \), so we see \( C' \) in a card, since \( j - 1 \geq \ell + 2 \). Hence we see in full in \( C' \) the components of \( T - z \) that contain \( v_r \) or spi-centers other than \( z \). Let \( Q_0 \) be the component of \( T - z \) containing \( v_r \) (we recognize it in \( C' \) because it does not contain a spi-center of \( C' \)), and let \( Q_1, \ldots, Q_t \) be the components containing spi-centers other than \( z \). (If \( j = (r + 1)/2 \), then \( C' \) gives us as \( Q_0 \) the bigger of
the components of $T - z$ containing $v_1$ or $v_r$; if they have the same size, then we have two choices for $Q_0$ and know both components.)

Let $R$ be the component of $T - z$ containing $v_1$. We still must reconstruct components of $T - z$ other than $Q_0, \ldots, Q_t$, including $R$. Call these “excess components”. Let $C''$ be the family of cscs having a vertex $z'$ of degree $t + 2$ such that $t + 1$ components obtained by deleting $z'$ are $Q_0, \ldots, Q_t$. In each such csc, $z' = z$. Let $C''$ be a largest member of $C''$. If $C''$ has fewer than $n - \ell$ vertices, then the component of $C'' - z$ other than $Q_0, \ldots, Q_t$ is an excess component of $T - z$. (If there is more than one such $C''$, then each gives an excess component.)

To find the other components of $T - z$, consider smaller members of $C''$, keeping track of the cscs of the next size generated by larger excess components. Each one that occurs more frequently than in this way yields another excess component.

Hence we may assume that $C''$ has $n - \ell$ vertices. Since $R$ has more than $\ell$ vertices, the component of $C'' - z$ other than $Q_0, \ldots, Q_t$ must be contained in $R$, but need not be all of $R$. Furthermore, the excess components of $T - z$ other than $R$ together have at most $\ell$ vertices. To find them, consider a largest csc $C_2$ having a vertex $z'$ such that $C_2 - z'$ has $t + 2$ components that are paths (one with $j - 1$ vertices, one with $r - j$ vertices, and $t$ others with $\ell + 1$ vertices). Such a csc exists with $z' = z$, deleting at least $\ell + 1$ vertices beyond a spi-center in each of $Q_1, \ldots, Q_t$. In addition, since $j - 1 > \ell$, we must have $z' = z$, and the specified paths come from $Q_0, \ldots, Q_t$ and $R$. Since $C_2$ has fewer than $n - \ell$ vertices, the remaining components of $C_2 - z$ are the excess components of $T - z$.

It remains only to reconstruct $R$. Let $m$ be the maximum of the lengths of the offshoots $Q_1, \ldots, Q_t$ in $C''$, with $Q_1$ having length $m$. Note that $m \leq j - 1$, since $P$ is a longest path, and $m \geq \ell + 2$. If $m < j - 1$, then let $C_3$ be a largest csc containing a path $P'$ with vertices $u_1, \ldots, u_r$ such that $d(u_j) = 3$ and the offshoot from $P'$ at $u_j$ is a path and has length $m$. The length of $P'$ forces $u_j = z$, and the offshoot must come from one of $Q_1, \ldots, Q_t$. Note that since $Q_1$ contains a spi-center, beyond which there are distinct paths with more than $\ell$ vertices, and $C_3$ keeps at most one path from $Q_1$, the csc $C_3$ omits more than $\ell$ vertices of $T$ (including when $t = 1$). Hence $C_3$ is seen in a card in the deck. Since $C_3$ is a largest csc of the type described, the components of $C_3 - z$ other than the path of $m$ vertices are the full offshoots $Q_0$ and $R$ from $z$ in $T$. We already know $Q_0$, so now we know also $R$.

Hence we may assume $m = j - 1$. Let $C_4$ be the family of subtrees containing a vertex $z'$ of degree 3 such that the three components when $z'$ is deleted are $Q_0, Q_1$, and a third offshoot $R'$ that has length $j - 1$. Note that $z'$ must be $z$. Also, $R'$ is contained in $R$ or in some $Q_i$ with $2 \leq i \leq t$. If $t \geq 2$, then since $j - 1 > \ell + 2$ and each offshoot containing a spi-center has at least $\ell + 1$ vertices beyond a spi-center, every member of $C_4$ has at most $n - \ell$ vertices and is seen in a card. Since we know $Q_1, \ldots, Q_t$, we know the members of $C_4$ of each size that arise when $R'$ is contained in some $Q_i$. In the largest member of $C_4$ that is not generated in this way, $R' = R$, completing the reconstruction.

In the remaining case, $m = j - 1$ and $t = 1$. If $Q_1$ has at two spi-centers $x$ and $x'$, then
we may assume that \( x' \) is not on the path from \( z \) to \( x \) in \( T \). Define \( \hat{Q}_1 \) by deleting from \( Q_1 \) a path of \( \ell + 1 \) vertices beyond \( x' \) (and its offshoots). Let \( C_4 \) be a largest subtree containing a vertex \( z' \) of degree 3 such that two of the components obtained by deleting \( z' \) are \( \hat{Q}_1 \) and \( Q_0 \). Note that \( z' \) must be \( z \). Also \( C_4 \) has a spi-center at \( x \) and has fewer than \( n - \ell \) vertices, since it lacks more than \( \ell \) vertices from \( Q_1 \). In making the third component of \( C_4 - z \) largest, it must be \( R \), since \( R \) has more than \( \ell \) vertices and all other components of \( T - z \) other than \( Q_1 \) and \( Q_0 \) together have fewer than \( \ell \) vertices.

Finally, we may assume also that \( Q_1 \) has only one spi-center \( x \), at distance \( d \) from \( z \). Let \( Q'_1 \) be a largest sub-offshoot from \( P \) at \( z \) that has length \( m \) but has degree 2 at \( x \). Since we know \( Q_1 \), we also know \( Q'_1 \) and its number of vertices. Let \( C_5 \) be a largest subtree of \( T \) containing a vertex \( z' \) of degree 3 such that the components of \( C_5 - z' \) are \( Q'_1 \), \( Q_0 \), and \( R' \), where \( R' \) has length \( m \) with the vertex at distance \( d \) from \( z \) having degree 2. Note that \( z' \) must be \( z \). Since we know \( Q_0 \) and \( Q_3 \) and all other components of \( T - z \) except \( R \), we know the number of vertices in \( R \). Since \( R \) has at least \( \ell + 1 \) vertices and the total number of vertices in components of \( T - z \) outside \( Q_0 \cup Q_1 \cup R \) is at most \( \ell \), the subtree \( R' \) we see in \( C_5 \) is contained in \( R \) unless the copy of \( Q'_1 \) in \( C_5 \) actually comes from \( R \) and \( R' \) comes from \( Q_1 \). The latter requires \(|R'| \leq |Q'_1| \). By the maximality of \(|V(C_5)|\) and the option to choose \( Q'_1 \) instead of \( R' \) from \( Q_1 \), in this case we have \(|Q'_1| \leq |R'| \). Hence \(|V(Q'_1)| = |V(R')|\). If \( Q'_1 \) and \( R' \) are not isomorphic, then we know which is which because we know \( Q'_1 \); if \( Q'_1 \cong R' \), then it does not matter which we call \( Q'_1 \). Hence we know \( R' \). (There may still be several choices for \( R' \) within \( R \) and hence several choices for \( C_5 \), but we just use any one of them.)

Let \( n_1 \) be the number of vertices of \( Q_1 \) not in \( Q'_1 \), and let \( n_2 \) be the number of vertices of \( R \) not in \( R' \). We now know \( n_1 \) and \( n_2 \). Recall that \( n_1 \geq \ell + 1 \). Let \( \mathcal{C}_6 \) be the family of subtrees containing \( C_5 \) in which the component of \( C_5 - z \) viewed as \( Q'_1 \) gains no additional vertices. Let \( T' \) denote the subtree of \( T \) whose components obtained by deleting a certain vertex \( z' \) of degree 3 are \( Q_1 \), \( Q_0 \), and \( R' \). Let \( n' = |V(T')| \). If \( n_2 > n_1 \) or \( Q'_1 \cong R' \), then a largest member of \( \mathcal{C}_6 \) shows us all of \( R \). If \( Q'_1 \cong R' \) and \( n_2 \leq n_1 \), then we know the members of \( \mathcal{C}_6 \) that arise from \( T' \) by deleting \( n_1 - n_2 \) vertices from \( Q_1 \) while keeping a copy of \( Q'_1 \) (possibly \( Q'_1 \) appears more than once in \( Q_1 \)). Deleting these members of \( \mathcal{C}_6 \) leaves one member of \( \mathcal{C}_6 \) with \( n' - n_1 + n_2 \) vertices; in this subtree we see \( R \).

\[ \square \]

**Lemma 3.3.** If \( n > 6\ell + 9 \), and \( r \geq n - 3\ell - 3 \), and \( S^{\ell+1} \subseteq T \), then \( T \) is \( \ell \)-reconstructible.

**Proof.** By Lemma 2.13, we know \( r \) from the deck. As noted in Lemma 2.12, \( r \geq 3\ell + 6 \) yields \( k \geq \ell + 1 \). By Corollary 2.15, we then know the number of spi-centers in \( T \), so we know \( S^{\ell+1} \subseteq T \) and recognize that we are in this case. Since \( r \geq 3\ell + 6 \), Lemma 3.2 implies that \( T \) is \( \ell \)-reconstructible if \( T \) has a sparse card. We may therefore restrict our attention to the case where \( T \) has no sparse card, which we recognize from the deck.
When \( r \geq n - 3\ell - 3 \) and \( T \) has no sparse card, every spi-center lies on every longest path, since otherwise the spi-center forces at least \( 2\ell + 3 \) vertices in one offshoot from a longest path, yielding a sparse card. Since \( S^{\ell+1} \subseteq T \), we have a spi-center on a longest path. Among subtrees having an \( r \)-vertex path \( P \) containing a spi-center on \( P \) as close to the center of \( P \) as possible and no other branch vertex on \( P \), choose \( C_1 \) to be one having a longest offshoot and within this the most vertices. Let \( z \) be the spi-center in \( C_1 \). There may be more than one largest such card; pick one to be \( C_1 \).

With the vertices of \( P \) indexed as \( v_1, \ldots, v_r \), define \( j \) with \( j \leq (r+1)/2 \) by \( z \in \{v_j, v_{r+1-j}\} \). Since \( \{v_{j+1}, \ldots, v_r\} \) contains no spi-center, all \( r \)-vertex paths contain \( v_j, \ldots, v_{r+1-j} \), by Lemma 2.19. Since \( T \) has no sparse card, \( C_1 \) has fewer than \( n - \ell \) vertices, and in \( C_1 \) we see all offshoots at \( z \) in full. Since \( C_1 \) has fewer than \( n - \ell \) vertices and \( r \geq n - 3\ell - 2 \), there is exactly one offshoot \( Q \) at \( z \) having length at least \( \ell + 1 \).

Let \( C_2 \) be a largest subtree containing an \( r \)-vertex path \( P' \) with vertices \( u_1, \ldots, u_r \) on which the vertices \( u_j \) and \( u_{r+1-j} \) have degree 2. We have noted that \( (u_j, \ldots, u_{r+1-j}) = (v_j, \ldots, v_{r+1-j}). \) Thus \( u_j \in \{v_j, v_{r+1-j}\} \). Since the omitted offshoots at \( z \) contain at least \( \ell + 1 \) vertices, \( C_2 \) fits in a card, and we see \( C_2 \) as a largest such csc. Hence in \( C_2 \) we may assume \( P' = P \), and we see all offshoots from \( P \) except at \( v_j \) and \( v_{r+1-j} \).

**Case 1:** The length of \( Q \) is less than \( j - 1 \). If \( j = r + 1 - j \), then \( C_1 \) tells us the offshoots from \( C_2 \) at \( v_j \), and we have reconstructed \( T \). If \( j < (r+1)/2 \), then we know the offshoots at \( v_j \) or \( v_{r+1-j} \), but we do not yet know which of the two vertices it is. Let \( y \) be the vertex of \( \{v_j, v_{r+1-j}\} \) other than \( z \).

Let \( W_1 \) be the union of the offshoots from \( P \) at \( z \), let \( W_2 \) be the union of the offshoots at \( y \), and let \( m_i = |V(W_i)| \). We already know \( W_1 \) and \( m_1 \). We also know \( m_2 \), since we know \( r \) and all offshoots from \( P \) outside \( W_2 \). We want to find \( W_2 \) and decide which of \( \{W_1, W_2\} \) is attached to \( v_j \). (Note: If \( j = (r+1)/2 \), then there is only one place to attach \( W_1 \), there is no \( W_2 \) or \( y \) to worry about, and \( C_2 \) tells us the rest of \( T \). Hence we may assume \( j < (r+1)/2 \).)

Suppose first that \( m_2 \geq m_1 \). (Note that \( y \) may or may not be a spi-center.) Let \( C_3 \) be a largest subtree containing an \( r \)-vertex path on which one of the vertices in positions \( j \) and \( r + 1 - j \) has degree 2. Since \( C_3 \) omits \( m_1 \) vertices, \( C_3 \) has fewer than \( n - \ell \) vertices, and we see \( C_3 \) in a card as a largest such csc. If \( m_2 > m_1 \), then we see \( W_2 \) in position along \( P \), together with all offshoots from \( P \) except those in \( W_1 \). We attach \( W_1 \) to the vertex in \( \{v_j, v_{r+1-j}\} \) having degree 2 in \( C_3 \) to complete the reconstruction of \( T \).

If \( m_2 = m_1 \), then there are two choices for \( C_3 \), one showing \( W_2 \) and the other showing \( W_1 \), each occurring in the right position among the offshoots from \( P \). We may have \( W_1 = W_2 \), and even the two choices for \( C_3 \) may be the same, but in all cases we complete the reconstruction.

Now suppose \( m_2 < m_1 \). In this case let \( C_3 \) be a largest subtree containing an \( r \)-vertex path \( u_1, \ldots, u_r \) such that \( u_j \) or \( u_{r+1-j} \) has degree 2, and such that offshoots with a total of \( m_2 + 1 \) vertices are grown from the other of \( \{u_j, u_{r+1-j}\} \). Since \( W_1 \cup W_2 \) has \( m_1 + m_2 \) vertices,
$C_3$ omits $m_1-1$ vertices from $W_1 \cup W_2$. Since $m_1 \geq \ell + 1$, $C_3$ fits in a card, and we see it as a largest such csc. Since $W_2$ has only $m_2$ vertices, $C_3$ determines which of $\{v_j, v_{r+1-j}\}$ is $z$.

We now know all of $T$ except the offshoots from $P$ at $y$. Let $C_4$ be the family of largest subtrees containing an $r$-vertex path $\langle u_1, \ldots, u_r \rangle$ such that one of $u_j$ and $u_{r+1-j}$ has degree 2 and the other has offshoots from the path with a total of $m_2$ vertices. Such subtrees omit $m_1$ vertices and hence are visible as cscs. They may arise from $T$ by deleting $m_1 - m_2$ vertices from $W_1$ and all of $W_2$, or by deleting all of $W_1$. Since we know $W_1$ and where in $C_2$ it is attached, we know all the members of $C_4$ that arise in the first way. The remaining member of $C_4$ shows us $W_2$, completing the reconstruction of $T$.

**Case 2:** $Q$ has length $j - 1$. We return to $C_1$. By the choice of $C_1$, the offshoot $Q$ is a largest offshoot from an endpoint of a path with $r + 1 - j$ vertices in $T$. This means that when we form $C_2$ as a largest subtree containing an $r$-vertex path (which always must go through $v_j, \ldots, v_{r+1-j}$) subject to $v_j$ and $v_{r+1-j}$ having degree 2, it must be that one of the components of $C_2 - \{v_j, v_{r+1-j}\}$ is $Q$ (in at least one of the choices for $C_2$ if there is more than one of the same size). It is possible that we see copies of $Q$ in $C_2$ containing both $v_j$ and $v_{r+1-j}$, in which case we do not know yet which is $z$.

Let $W_1$ consist of the other offshoots from $v_j, \ldots, v_{r+1-j}$ at $z$. Again there is exactly one such offshoot with length at least $\ell + 1$ (since there is no sparse card), and it is no bigger than $Q$. Call it $Q'$, with $m'$ vertices. Again let $W_2$ consist of the offshoots from $C_2$ at the other vertex in $\{v_j, v_{r+1-j}\}$, and let $m_i = |V(W_i)|$. We may assume that the path is indexed so that $z = v_j$ if $C_2$ is symmetric or if $C_2$ shows only one copy of $Q$ giving a candidate for $z$.

**Subcase 2a:** $C_2$ is not symmetric (under reversal of $v_1, \ldots, v_r$). In $C_2$ we see at least one copy of $Q$, rooted at $v_j$ or $v_{r+1-j}$. Let $C_3$ be a largest csc containing $C_2$ in which a root of $Q$ has degree 2. Since $C_2$ is not symmetric, then the vertices $v_j$ and $v_{r+1-j}$ are distinguished by what we see along $C_2$ (this is true whether or not they are both roots of copies of $Q$). Hence in addition to $C_3$ we can also obtain a largest $C'_3$ where the offshoots are grown from the other of these two vertices. If $m_2 \geq \ell$, then both $C_3$ and $C'_3$ have at most $n - \ell$ vertices and show us $W_2$ and $W_1$ with their roots along $C_2$. Hence we may assume $m_2 < \ell$, and we know at least $W_2$. Hence we also know $m_2$ and can compute $m_1$, though we do not yet know $m'$. We may assume the indexing so that $z = v_j$.

To determine $m'$, let $C_4$ be a largest subtree containing an $r$-vertex path plus offshoots from the $j$th vertex of the path, one of which is a path and has length $\ell + 1$. Since $m_2 < \ell$, this branch vertex is $z$. Since $T$ has no sparse card, $C_4$ fits in a card and we see it as a largest such csc. Since $W_1$ has only one offshoot with length at least $\ell + 1$, in $C_4$ we see all other offshoots at $z$. Subtracting their sizes from $m_1$ tells us $m'$, the number of vertices in $Q'$.

To find $Q'$, let $C_5$ be the family of subtrees consisting of an $r$-vertex path plus one offshoot of length $j - 1$ from the $j$th vertex, such that the offshoot has $m'$ vertices. Since $m' \leq |V(Q)|$, some of these arise by deleting vertices from $Q$; others show $Q'$. Since we know $Q$, we know
the number of ways to find in \(Q\) an offshoot of length \(j - 1\) with \(m'\) vertices. If we can determine the number of vertices of \(Q'\) at distance \(j - 1\) from \(z\) and the number of vertices at distance \(r - j\) from \(z\) (we see the latter in \(C_3\)), then we know all the members of \(C_5\) to delete. In the remaining members we see \(Q'\), completing the reconstruction of \(T\).

Consider the subtree \(C_6\) consisting of an \(r\)-vertex path \(u_1, \ldots, u_r\) plus one offshoot isomorphic to \(Q\) at \(u_j\). Since \(T\) has no sparse card, \(C_6\) has fewer than \(n - \ell\) vertices, and we can count its appearances in \(T\). By the asymmetry of \(C_2\), we have \(u_j = z\). Since we know \(W_2\), we know how many vertices can serve as \(u_r\). If \(Q' \neq Q\), then the number of vertices that can serve as \(u_1\) is the number of vertices of \(Q'\) at distance \(j - 1\) from \(z\). We can thus determine this quantity from the number of copies of \(C_6\).

If \(Q' = Q\), then we had more than one choice for our original subtree \(C_1\), and they were identical, so we knew \(Q'\) all along.

**Subcase 2b:** \(C_2\) is symmetric (under reversal of \(v_1, \ldots, v_r\)). In this case, we cannot distinguish \(v_j\) and \(v_{r+1-j}\) in \(C_2\). We still consider the family \(C_3\) of subtrees containing \(C_2\) in which \(v_j\) or \(v_{r+1-j}\) has degree 2. If no \(csc\) in \(C_3\) is a full card, then let \(C_3\) be a largest such \(csc\) among those having an offshoot of length \(j - 1\) from \(C_2\). We may assume that the offshoot is at \(v_j\), showing us \(W_1\). Since \(W_1\) tells us \(m_1\) and we know \(C_2\), we now also know \(m_2\). If \(m_2 > m_1\), then a largest member of \(C_3\) shows us \(W_2\). If \(m_2 \leq m_1\), then we know all the members of \(C_3\) having \(m_2\) vertices outside \(C_2\) that arise by deleting \(m_1 - m_2\) vertices from \(W_1\) in \(C_3\). The remaining member of \(C_3\) with \(m_2\) vertices outside \(C_2\) shows us \(W_2\).

Now suppose that \(C_2\) is symmetric and \(C_3\) contains a full card. Again we may assume by symmetry that the offshoots from \(C_2\) in \(cscs\) in \(C_3\) are attached at \(v_j\). Since \(C_3\) contains a full card, \(m_2 \leq l\). The case \(j = r+ 1 - j\) is easier; we postpone it and first consider \(j < (r+1)/2\).

Consider the copies of \(S_{j-1,j-1,r-j}\) in \(T\). Since \(z\) lies in all longest paths and only one offshoot from a longest path at \(z\) can have length at least \(\ell + 1\), the branch vertex in each copy is \(z\). Furthermore, with the components of \(T - z\) containing \(v_1\) and \(v_r\) being \(T_1\) and \(T_2\), respectively, each such spider has one leaf each in \(Q\), \(T_1\), and \(T_2\). The number of these spiders is thus \(t_0t_1t_2\), where these factors are the numbers of vertices at distance \(j - 1\) from \(z\) in \(Q\), at distance \(j - 1\) from \(z\) in \(T_1\), and at distance \(r - j\) from \(z\) in \(T_2\). Since \(T\) has no sparse cards, these spiders fit into cards, so we know the number of them. Since we know \(Q\) from \(C_1\), we know \(t_Q\). From knowing \(C_2\) and \(m_2 \leq \ell\), we know \(t_2\) (it is the same as the number of vertices of \(Q\) at distance \(j - 1\) from \(z\), since \(Q\) is isomorphic to two of the components of \(C_2 - \{v_j, v_{r+1-j}\}\)). Hence we also know \(t_1\).

Now consider the copies of \(S_{j-1,\ell+1,r-j}\) in \(T\). By the same reasoning as before, in each the branch vertex is \(z\) and the leaves are in \(Q\), \(T_1\), and \(T_2\); also we know the number of such subcards. This number is \(t_0s_1t_2 + s_0t_1t_2\), where \(s_q\) and \(s_1\) are the numbers of vertices at distance \(\ell + 1\) from \(z\) in \(Q\) and in \(T_1\), respectively. Since we know \(Q\), we know all these numbers except \(s_1\), so now we also know \(s_1\).

Now let \(C\) be the family of \(cscs\) containing \(S_{j-1,\ell+1,r-j}\) such that no offshoots are grown.
from the legs of lengths $\ell + 1$ or $r - j$ (note that $j - 1 = \ell + 1$ is possible). Again, the branch vertex of the spider must be $z$. Largest cscs in $C$ may or may not be full cards, but in either case we know all members of $C$ that arise by choosing the leg of length $\ell + 1$ without offshoots from $T_1$ and the leg of length $j - 1$ with offshoots from $Q$, since we know $s_1$ and $Q$. A largest cscs among the remaining members of $C$ contains the leg of length $\ell + 1$ from $Q$ and shows all of $T_1$.

Now we know all of $T$ except $W_2$, so we know $m_2$. We return to the family $C_3$. Since we know $T_1$, we know which members of $C_3$ that grow $m_2$ vertices beyond $C_2$ arise by growing vertices from $T_1$ and the leg of length $j - 1$ with offshoots from $Q$. A largest cscs among the remaining members of $C$ contains the leg of length $\ell + 1$ from $Q$ and shows all of $T_1$.

Finally, suppose $j = (r + 1)/2$ and $C_3$ contains a full card. There are three offshoots from $z$ with length $j - 1$: two copies of $Q$ and one of $T_1$, and we know $Q$ and the shorter offshoots. The copies of $S_{j-1}$ have branch vertex at $z$ and leaves in $T_1$ and the two copies of $Q$. We count them and know $t_Q$, so we obtain $t_1$ (as defined above). We can make use of copies of $S_{j-1,\ell+1,r-j}$ and cscs containing them as above (with more careful counting) to obtain $T_1$. □

### 4 The Vines at the Ends of $P$

In the previous section we proved the $\ell$-reconstructibility of $n$-vertex trees $T$ with $n \geq 6\ell + 10$ under the conditions $r \geq n - 3\ell$ and $S^{\ell+1} \subseteq T$. We next consider trees not containing $S^{\ell+1}$. As before, the same numerical hypotheses yield $k \geq \ell + 3$, using Lemma 2.12. Thus forbidding $S^{\ell+1}$ will also forbid $S^{k-1}$ and $S^k$, which in some cases is all we need.

**Lemma 4.1.** When $S^k \not\subseteq T$, in every reconstruction $T$ all longest paths have the same $r - 2k + 2$ central vertices. The central $r - 2k$ vertices are all the $k$-centers in $T$. The two vertices at distance $k - 1$ from the ends of every longest path are $(k - 1)$-centers but not $k$-centers. From the deck, we can determine the two maximal $(k - 1)$-vines $T_1$ and $T_2$ centered at these two vertices.

**Proof.** When $S^j \not\subseteq T$ and $P$ is any longest path in $T$, every $j$-vertex path in $T$ intersects $P$ at a vertex having distance at least $j$ from the ends of $P$. Hence $T$ has no $(j - 1)$-center outside the central $r - 2j + 2$ vertices of any longest path. Those central vertices are indeed $(j - 1)$-centers, so all longest paths have the same central $r - 2j + 2$ vertices.

Considering $j = k$ and $j = k - 1$, we find that the $k$-centers are precisely the vertices of $P$ except for the last $k$ vertices on each end, and the $(k - 1)$-centers are the vertices of $P$ except for the last $k - 1$ vertices on each end. In particular, the vertices at distance $k - 1$ from the ends of $P$ are $(k - 1)$-centers but not $k$-centers.

By Corollary 2.9, we know all the maximal $k$-vines and all the maximal $(k - 1)$-vines. Each maximal $k$-vine has a unique center and contains exactly one maximal $(k - 1)$-vine having the same center. Hence we can eliminate the maximal $(k - 1)$-vines contained in
Reconstructing $T$ requires assembling $T_1$ and $T_2$ and any part of $T$ omitted by them. Let $x_i$ be the center of $T_i$. Since $S^{k-1} \not\subseteq T$, exactly two edges incident to $x_i$ in $T_i$ start paths of length $k-1$ in $T_i$. Our next task is to determine which of these two edges incident to $x_i$ in $T_i$ starts the path to $x_{3-i}$. By “orienting” $T_i$, we mean determining which of these two edges in $T_i$ lies along the $x_1, x_2$-path in $P$ (here $P$ is any longest path). To facilitate this task we introduce definitions and notation for various objects in the tree.

**Definition 4.2. Structure of $T_i$; see Figure 2.** The edge of $T_i$ incident to $x_i$ that belongs to the $x_1, x_2$-path in $P$ is the trunk edge of $T_i$. The trunk edge of $T_i$ is the central edge in a maximal $(k-1)$-evine. The other edge of $P$ incident to $x_i$ in $T_i$ is not the central edge of a $(k-1)$-evine in $T$.

As noted above, $S^{k-1} \not\subseteq T$ implies that exactly two components of $T_i - x_i$ contain $(k-1)$-vertex paths beginning at the neighbor of $x_i$. Let $A_i$ and $B_i$ respectively be the subtrees of $T_i$ rooted at $x_i$ that are obtained by deleting the vertex sets of those two components. Both $A_i$ and $B_i$ contain all components of $T_i - x_i$ having no $(k-1)$-vertex path beginning at the neighbor of $x_i$. Together with $x_i$, these comprise $A_i \cap B_i$, and we call this rooted subtree $W_i$.

![Figure 2: $T_i$](image)

**Definition 4.3. $(k-1)$-evines containing $T_1$ and $T_2$.** By Corollary 2.9, we can determine the number of maximal $(k-1)$-evines and maximal $k$-evines in $T$ with each isomorphism type, and each maximal $k$-evine contains exactly one maximal $(k-1)$-evine with the same central edge. Therefore, just as we determined $T_1$ and $T_2$ in Lemma 4.1, we can also determine the maximal $(k-1)$-evines $S_1$ and $S_2$ whose central edges are the trunk edges in $T_1$ and $T_2$. From the list of maximal $k$-evines, we obtain a list of $(k-1)$-evines by deleting the vertices at distance $k$ from the central edge. The members of the list of maximal $(k-1)$-evines that are not generated in this process (paying attention to multiplicity) are $S_1$ and $S_2$. However, we do not yet know which of $\{S_1, S_2\}$ contains which of $\{T_1, T_2\}$. 

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For \( j \in \{1, 2\} \), let \( y_j z_j \) be the central edge of \( S_j \). Let \( C_j \) and \( D_j \) be the components of \( S_j - y_j z_j \), rooted at \( y_j \) and \( z_j \), respectively. Since \( S^{k-1} \not\subseteq T \), exactly two components of \( S_j - \{ y_j, z_j \} \) contain paths with \( k - 1 \) vertices starting from the neighbor of \( \{ y_j, z_j \} \). One is in \( C_j \) and one in \( D_j \); these are the major pieces of \( C_j \) and \( D_j \); see Figure 3.

The \( (k - 1) \)-vine \( S_j \) contains two maximal \( (k - 1) \)-vines, centered at \( y_j \) and \( z_j \). Let \( Y_j \) be the tree rooted at \( y_j \) obtained by deleting the major piece of \( C_j \) from the maximal \( (k - 1) \)-vine centered at \( y_j \). Let \( Z_j \) be the tree rooted at \( z_j \) obtained by deleting the major piece of \( D_j \) from the maximal \( (k - 1) \)-vine centered at \( z_j \). Thus the two maximal \( (k - 1) \)-vines contained in \( S_j \) are \( C_j \cup Y_j \) (centered at \( y_j \)) and \( D_j \cup Z_j \) (centered at \( z_j \)).

Offshoots from \( P \) at \( y_j \) having length less than \( k - 1 \) lie in both \( C_j \) and \( Y_j \), and the analogous statement holds for \( D_j \) and \( Z_j \). Finally, \( C_j \cup Y_j \) contains all of \( S_j \) except the vertices of \( D_j \) at distance \( k - 1 \) from \( z_j \), and \( D_j \cup Z_j \) contains all of \( S_j \) except the vertices of \( C_j \) at distance \( k - 1 \) from \( y_j \).

![Figure 3: Rooted trees in \( S_j \)](image)

We have found the \( (k - 1) \)-vines \( S_1 \) and \( S_2 \) that contain the \( (k - 1) \)-vines \( T_1 \) and \( T_2 \) that appear in \( T \) without being contained in \( k \)-vines, but we don’t yet know which contains which. Furthermore, if we knew that \( S_j \) is the \( (k - 1) \)-evine containing the copy of \( T_i \) that does not occur in a \( k \)-vine, we don’t know which of \( C_j \cup Y_j \) and \( D_j \cup Z_j \) is that occurrence of \( T_i \). If it is \( C_j \cup Y_j \), then \( y_j \) is \( x_i \); if it is \( D_j \cup Z_j \), then \( z_j \) is \( x_i \).

In the next several lemmas, we discuss isomorphisms of rooted trees, but we sometimes speak of two rooted trees being “the same”. In particular, whenever we speak of equality between two rooted trees among those we have defined above from the deck, we mean that these rooted trees found in \( T \) are isomorphic. For example, equality between two 4-tuples of rooted trees means that corresponding entries in the 4-tuple are isomorphic.

We next develop a tool used in the analysis. In this lemma, \( S_1 \) and \( S_2 \) denote any \( (k - 1) \)-vines containing rooted subtrees as defined above.

**Lemma 4.4.** Let \( S_1 \) and \( S_2 \) be \( (k - 1) \)-vines not containing \( S^{k-1} \), with rooted subtrees \( (C_1, Z_1, Y_1, D_1) \) and \( (C_2, Z_2, Y_2, D_2) \) as in Definition 4.3. If \( C_1 \cong Y_2, Z_1 \cong D_2, Y_1 \cong C_2, \) and \( D_1 \cong Z_2 \), then \( C_1 \cong Y_1 \) or \( D_1 \cong Z_1 \).
Proof. As in Definition 4.3, let the roots of \( C_1, D_1, C_2, D_2 \) be \( y_1, z_1, y_2, z_2 \), respectively. In each of these trees, the maximum distance from the root to a leaf is \( k - 1 \), and the leaves at distance \( k - 1 \) from the root are called \emph{peripheral vertices}. The given isomorphisms are isomorphisms as rooted trees and hence preserve the roots.

Since \( S^{k-1} \) does not appear, the common subtree \( W_1 \) in \( C_1 \) and \( Y_1 \) that is rooted at \( y_1 \) and contains no edge of a fixed longest path in \( S_1 \) has length less than \( k - 1 \). It is also the common subtree of \( C_2 \) and \( Y_2 \) rooted at \( y_2 \). The analogous observation holds for \( W_2 \) rooted at both \( z_1 \) and \( z_2 \). In Figure 4, copies of \( W_1 \) are drawn rounded, while copies of \( W_2 \) are drawn as triangles.

To facilitate discussion, we write \( A_1 \) for the isomorphic subtrees \( C_1 \) and \( Y_2 \), \( B_1 \) for \( C_2 \) and \( Y_1 \), \( A_2 \) for \( D_2 \) and \( Z_1 \), and \( B_2 \) for \( D_1 \) and \( Z_2 \), as shown in Figure 4. In one application where we consider the \((k - 1)\)-evines containing \( T_1 \) and \( T_2 \), these subtrees will indeed be copies of the subtrees of \( T_1 \) and \( T_2 \) having those names.

Iteratively, moving away from the roots, we will use the hypothesis that \( C_1 \not\cong Y_1 \) and \( D_1 \not\cong Z_1 \) (that is, \( A_1 \not\cong B_1 \) and \( A_2 \not\cong B_2 \)) to follow paths in the subtrees \( A_1, A_2, B_1, B_2 \) so that the offshoots from corresponding vertices in the paths in \( A_1 \) and \( B_1 \) are isomorphic, and similarly the offshoots from corresponding vertices in the paths in \( A_2 \) and \( B_2 \) are isomorphic. Since paths from the roots have finite length, this iteration leads to \( A_1 \cong B_1 \) and \( A_2 \cong B_2 \), contradicting the hypothesis.

At the roots of \( A_i \) and \( B_i \), we have the common offshoot \( W_i \) from a path of length \( k - 1 \). Consider the offshoot \( W_2 \) from \( z_2 \) in \( S_2 \). Not only is \( z_2 \) the root of \( Z_2 \), which is a copy of \( B_2 \), but also \( z_2 \) is the child outside \( W_1 \) of the root \( y_2 \) of \( Y_2 \), which is a copy of \( A_1 \). When we look at the copy of \( A_1 \) that appears as \( C_1 \) in \( S_1 \), it must also have a copy of \( W_2 \) as the offshoots from the child of \( y_1 \) not in \( W_1 \). We have obtained isomorphism between the offshoots at the children of \( y_1 \) in \( A_1 \) and \( B_1 \) along the path being grown from the root.

Furthermore, the copy of \( A_1 \) occurring as \( Y_2 \) has only one child of \( z_2 \) outside \( W_2 \). Hence when go to the copy of \( A_1 \) occurring as \( C_1 \) in \( S_1 \), also there is only one child outside \( W_2 \) for the vertex whose offshoots we have found to be \( W_2 \), thereby leaving a unique path to continue.
Similar arguments starting from $W_1$ in $S_2$ give us isomorphism between the offshoots from the child of the root outside $W_2$ in $A_2$ and $B_2$. With $S_2$ having copies of the same four rooted subtrees, we have the same isomorphisms there, though we could also obtain that by starting with the copies of $W_1$ and $W_2$ at the roots in $S_1$ and observing how they occur in the copies of the subtrees in $S_2$.

At the child of the root along the path, we have just found a copy of $W_2$ in $A_1$ in $S_1$, a copy of $W_1$ in $B_2$ in $S_1$, a copy of $W_2$ in $B_1$ in $S_2$, and a copy of $W_1$ in $A_2$ in $S_2$. Since the roots of these copies are one step from the centers of $S_1$ and $S_2$, the copies may contain peripheral vertices. The peripheral vertices are deleted when we view these offshoots as subgraphs of the “inner” subtrees, moving from $A_1$ to $A_2$ in $S_1$, from $B_2$ to $B_1$ in $S_1$, from $B_1$ to $B_2$ in $S_2$, and from $A_2$ to $A_1$ in $S_2$.

Since copies of $W_i$ are the same and at the same distance from the center, their truncations obtained by deleting peripheral vertices are the same, and in the inner copies ($A_2$ in $S_1$ with $B_2$ in $S_2$, and $B_1$ in $S_1$ with $A_1$ in $S_2$) we see the isomorphism at the next step moving away from the root. As we move along, the isomorphic offshoots we extract in $A_i$ and $B_i$ will be truncations of $W_1$ alternating with truncations of $W_2$.

Having obtained isomorphism at a new level using the inner subtrees, we view them and the unique edge continuing the path in the outer subtrees in the other $(k-1)$-evine. There we must truncate peripheral vertices to obtain the next level in the inner subtrees. As we go step-by-step, the same operations are occurring in $A_i$ and in $B_i$, so we obtain the isomorphism level by level.

In the remainder of this section, we use $S_1$ and $S_2$ to recognize the trunk edges in $T_1$ and $T_2$. Notation is as in Definitions 4.2 and 4.3. In $S_j$, the edge $y_jz_j$ is a trunk edge involving $x_1$ or $x_2$. If it is $y_j$ that is $x_i$, then $C_j$ is in fact $A_i$ or $B_i$, because $C_j \cup Y_j$ is then $T_i$.

When we write that two 4-tuples of rooted trees are equal, we mean that corresponding entries are isomorphic. Similarly, writing that a rooted tree is not in a set of rooted trees means that it is not isomorphic to any of the members, and the size of a set of rooted trees is the number of isomorphism classes in it.

**Lemma 4.5.** For a fixed choice of $i$ and $j$, if it is known that $S_j$ is the maximal $(k-1)$-evine whose central edge is the trunk edge of $T_i$, then the edge of $T_i$ serving as the trunk edge is recognizable.

**Proof.** By definition, $x_i$ is $y_j$ or $z_j$. If $x_i = y_j$, then $T_i = C_j \cup Y_j$ and the trunk edge of $T_i$ is in $B_i$. If $x_i = z_j$, then $T_i = D_j \cup Z_j$ and the trunk edge of $T_i$ is in $A_i$. Hence we want to determine which of $y_j$ and $z_j$ is $x_i$.

One of $\{C_j, D_j\}$ must be $A_i$ or $B_i$. By symmetry, we may label the subtrees so that $C_j \cong A_i$. If $D_j \not\cong B_i$, then $y_j = x_i$. Hence we may assume both $C_j \cong A_i$ and $D_j \cong B_i$. 

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Lemma 4.6. If some member of \( \{C_1, C_2, D_1, D_2\} \) does not belong to \( \{A_1, A_2, B_1, B_2\} \) (as a rooted tree), then the trunk edges of \( T_1 \) and \( T_2 \) are recognizable.

**Proof.** By symmetry, we may assume \( C_1 \notin \{A_1, A_2, B_1, B_2\} \). Thus \( y_1 \notin \{x_1, x_2\} \), so \( z_1 \in \{x_1, x_2\} \). That is, \( D_1 \cup Z_1 \in \{T_1, T_2\} \), so \( D_1 \in \{A_1, A_2, B_1, B_2\} \). The rooted trees \( A_1, A_2, B_1, B_2 \) need not be nonisomorphic, so \( D_1 \) may be isomorphic to more than one of them.

If \( D_1 \) is isomorphic to exactly one of them, which by symmetry we may assume is \( A_1 \), then \( z_1 = x_1 \) and we know that the trunk edge in \( T_1 \) lies in \( B_1 \), not \( A_1 \). Also, since \( D_1 \) is not isomorphic to \( A_2 \) or \( B_2 \), we have \( T_2 \subseteq S_2 \), and Lemma 4.5 completes the proof.

Hence we may assume that \( D_1 \) is isomorphic to more than one of \( \{A_1, A_2, B_1, B_2\} \). By symmetry, we again assume \( D_1 = A_1 \). If \( D_1 \) is not isomorphic to \( A_2 \) or \( B_2 \), then \( D_1 = A_1 \cong B_1 \) and it does not matter which candidate is the trunk edge of \( T_1 \). Since neither \( C_1 \) nor \( D_1 \) is in \( \{A_2, B_2\} \), we have \( T_1 \subseteq S_1 \) and \( T_2 \subseteq S_2 \) as actual subgraphs of \( T \), and Lemma 4.5 on \( T_2 \) and \( S_2 \) completes the proof.

In the remaining case, by symmetry, \( D_1 = A_1 \cong A_2 \). If also \( B_1 \cong B_2 \), then \( T_1 \cong T_2 \). Now it does not matter which \( T_i \) we view as lying in which \( S_j \) in \( T \), so we can assign them arbitrarily and use Lemma 4.5 to complete the proof.

Hence we may assume \( D_1 = A_1 \cong A_2 \) and \( B_1 \not\cong B_2 \). Again \( z_1 \in \{x_1, x_2\} \), so \( D_1 \cup Z_1 \) is \( T_1 \) or \( T_2 \). Since \( B_1 \not\cong B_2 \), now \( Z_1 \) is isomorphic to just one of \( B_1 \) or \( B_2 \); let it be \( B_j \). Now \( z_1 = x_j \), and the trunk edge of \( T_j \) is in \( B_j \). This leaves \( T_{3-j} \subseteq S_2 \), and Lemma 4.5 completes the proof. □

In the remaining cases, \( \{C_1, C_2, D_1, D_2\} \subseteq \{A_1, A_2, B_1, B_2\} \). In light of Lemma 4.5, our main task is to identify which of \( S_1 \) and \( S_2 \) contains which of \( T_1 \) and \( T_2 \). We break the analysis into three lemmas, depending on how many isomorphism classes of rooted trees comprise \( \{A_1, A_2, B_1, B_2\} \).
Lemma 4.7. If $|\{A_1, A_2, B_1, B_2\}| \leq 2$, then the trunk edges of $T_1$ and $T_2$ are recognizable.

Proof. If neither of $\{A_1, B_1\}$ is isomorphic to either of $\{A_2, B_2\}$, then having at most two isomorphism classes requires $A_1 \cong B_1$ and $A_2 \cong B_2$. In this case, $T_1$ and $T_2$ are symmetric and in each the choice of trunk edges from the two candidates does not matter. This case includes that of $A_1, A_2, B_1$, and $B_2$ being pairwise isomorphic.

In the remaining case, a member of $\{A_1, B_1\}$ is isomorphic to a member of $\{A_2, B_2\}$. By symmetry in the labeling, we may assume $A_1 \cong A_2$ and call this rooted tree $A$. If $B_1 \cong B_2$, then $T_1 \cong T_2$, and we can view $T_1$ and $T_2$ as contained in $S_1$ and $S_2$, respectively, applying Lemma 4.5 to find the trunk edges.

Hence we may assume $B_1 \not\cong B_2$. Since the four subtrees lie in two isomorphism classes, we may assume by symmetry in the indices that $B_1 \cong A_1 \cong A_2 \cong B_2 \not\cong A$. Now $T_1$ is symmetric and the choice of its trunk edge does not matter, so it remains to determine the trunk edge of $T_2$. If $B_2$ does not occur as any of the branches $\{C_1, C_2, D_1, D_2\}$, then $B_2$ cannot contain the end of a longest path and instead contains the trunk edge of $T_2$.

Let $B = B_2$. In the remaining case, $B \in \{C_1, C_2, D_1, D_2\}$. We have made no comment yet distinguishing branches of $S_1$ or $S_2$, so by symmetry we may assume $D_2 \cong B$. Now if $C_2 \not\cong A$, then neither branch of $S_2$ is $A_1$ or $B_1$, so $T_1 \not\subseteq S_2$ in $T$. This forces $T_2 \subseteq S_2$, and we apply Lemma 4.5. Hence we may assume $C_2 \cong A$ and $D_2 \cong B$. If $S_1 \cong S_2$, then we can apply Lemma 4.5 with $T_2 \subseteq S_2$ or $T_2 \subseteq S_1$. Hence we may assume $S_1 \not\cong S_2$.

Since $A_1 \cong B_1 \cong A \not\cong B \cong D_2$, we have $T_1 \subseteq S_2$ only if $Y_2 \cong A$. Therefore, if $Y_2 \not\cong A$, then we know $T_1 \not\subseteq S_2$, which forces $T_2 \subseteq S_2$, and we apply Lemma 4.5.

Hence we may assume $Y_2 \cong A$ and $T_1 \subseteq S_2$. We are confused about how to apply Lemma 4.5 only if $T_2$ occurs in both $S_1$ and $S_2$. For $T_2 \subseteq S_2$, we must have $Z_2 \cong A_2 \cong A$. That is, $C_2 \cong Z_2 \cong Y_2 \cong A$ and $D_2 \cong B$. On the other hand, confusion also requires having both $T_1$ and $T_2$ occur in $S_1$. With $S_1 \not\cong S_2$, by symmetry in the labels we may assume $C_1 \cong Z_1 \cong D_1 \cong A$ and $Y_1 \cong B$. See Figure 5.

![Figure 5: (k - 1)-evines in Lemma 4.4](image)

**Mina’s suggestion:**

Now we modify Figure 5 by relabeling to reverse $S_2$, viewing the subtrees $(A, A, A, B)$ as $(D_2, Y_2, Z_2, C_2)$ instead of $(C_2, Z_2, Y_2, D_2)$. With this relabeling, since $(C_1, Z_1, Y_1, D_1) =$
(A, A, B, A), we have \(C_1 \cong Y_2, Z_1 \cong D_2, Y_1 \cong C_2,\) and \(D_1 \cong Z_2.\) By Lemma 4.4, we conclude \(C_1 \cong Y_1\) or \(D_1 \cong Z_1.\) In other words, we have \(A \cong B,\) which is forbidden in this case.

(Here “or” is an issue. I weakened the statement of Lemma 4.4 a while ago because we were contradicting the assumption that both pairs are nonisomorphic. Here we already have \(D_1 \cong Z_1,\) so the current statement of Lemma 4.4 does not give us the claim. However, do we really need the assumption of \(C_1 \not\cong Y_1\) and \(D_1 \not\cong Z_1\) in Lemma 4.4 to keep the proof going? I think maybe the proof is clean enough now (having the one child to continue the path) that we just grind out the isomorphisms without assuming they don’t exist. Just in case, the old proof is still below.)

Since \(C_1 = Z_1 = A,\) the offshoots at \(y_1\) and \(z_1\) in \(S_1\) from a longest path are the same, both obtained by deleting the major piece of \(A.\) These are also the offshoots at the roots of \(Y_1\) and \(D_1\) that are not along a longest path. By comparing \(C_1\) and \(Z_1,\) which are both copies of \(A,\) we see that deleting the peripheral vertices of \(C_1\) (that is, \(A\)) and adding a copy of the root offshoots at a level above the root yields \(Z_1\) (that is, \(A\)). On the other hand, applying the same operation to \(D_1\) (that is, \(A\)) yields \(Y_1\) (that is, \(B\)). Hence we have failed to determine the trunk edges only if \(A = B,\) which does not hold in this final case. \(\square\)

**Lemma 4.8.** If \(|\{A_1, A_2, B_1, B_2\}| = 3,\) then the trunk edges of \(T_1\) and \(T_2\) are recognizable.

**Proof.** Again let \(W_1 = A_1 \cap B_1\) and \(W_2 = A_2 \cap B_2.\) By hypothesis, two of \(\{A_1, A_2, B_1, B_2\}\) are isomorphic. By Lemma 4.5, we can recognize the trunk edges in \(T_1\) and \(T_2\) unless each of \(T_1\) and \(T_2\) appears in both \(S_1\) and \(S_2\) as a subgraph and \(S_1 \not\cong S_2.\)

**Case 1:** The two isomorphic members of \(\{A_1, A_2, B_1, B_2\}\) come from the same \(T_i.\) By symmetry, we may assume \(A_1 \cong B_1 \cong A,\) so the choice of trunk edge for \(T_1\) does not matter. As noted above, we must have \(T_1\) in both \(S_1\) and \(S_2,\) so we have both \(A \in \{C_1, D_1\}\) and \(A \in \{C_2, D_2\}.\) However, since \(A\) is neither \(A_2\) nor \(B_2\) and \(T_2\) must also appear in both \(S_1\) and \(S_2,\) we may assume by symmetry in the labeling that \(C_1 = C_2 = A\) and that neither \(D_1\) nor \(D_2\) is \(A.\)

Since confusion requires \(S_1 \not\cong S_2,\) we may assume \(D_1 = A_2\) and \(D_2 = B_2,\) with \(T_2\) occurring as both \(Z_1 \cup D_1\) and \(Z_2 \cup D_2.\) In \(S_1,\) deleting the peripheral vertices of \(A\) (as \(C_1\)) and adding \(W_2\) above the root yields \(Z_1\) and hence \(B_2.\) In \(S_2,\) deleting the peripheral vertices of \(A\) (as \(C_2\)) and adding \(W_2\) above the root yields \(Z_2\) and hence \(A_2.\) Thus \(A_2 = B_2,\) which contradicts the hypothesis of this case.

**Case 2:** The two isomorphic members of \(\{A_1, A_2, B_1, B_2\}\) come one from \(T_1\) and one from \(T_2.\) By symmetry, we may assume \(A_1 \cong A_2 \cong A.\) Here all of \(\{A, B_1, B_2\}\) are different, but \(W_1 \cong W_2\) since each is obtained from \(A\) in the same way.
If \( A \notin \{C_1, C_2, D_1, D_2\} \), then the ends of \( T_1 \) and \( T_2 \) occupied by \( A \) do not occur at an end of \( S_1 \) or \( S_2 \), so the trunk edges in \( T_1 \) and \( T_2 \) both lie in \( A \). Hence we may assume \( C_1 \cong A \), by symmetry. If also \( D_1 \cong A \), then having \( T_1 \) and \( T_2 \) both appear in \( S_1 \) requires \( \{Y_1, Z_1\} = \{B_1, B_2\} \). Also, since \( C_1 \cong A \cong D_1 \), the rooted subtrees at \( y_1 \) and \( z_1 \) outside the major pieces both equal \( W_1 \). Hence both \( Y_1 \) and \( Z_1 \) are obtained by deleting the peripheral vertices of \( A \) and adding \( W_1 \) at the top. This yields \( Y_1 \cong Z_1 \) and \( B_1 \cong B_2 \), which contradicts the hypothesis of this case.

Hence we may assume \( D_1 \neq A \). By Lemma 4.6 and symmetry in the indices, we may now assume \( D_1 \cong B_2 \). Since \( T_1 \) and \( T_2 \) must both appear in \( S_1 \), we have \( Y_1 \cong B_1 \) and \( Z_1 \cong A \); that is, \( T_1 \cong C_1 \cup Y_1 \) and \( T_2 \cong Z_1 \cup D_1 \).

Now consider \( S_2 \). By reversing the labeling of \( S_2 \) if needed, we may assume \( T_1 = C_2 \cup Y_2 \) and \( T_2 = Z_2 \cup D_2 \). Hence \( C_2 \in \{A, B_1\} \) and \( D_2 \in \{A, B_2\} \). This leaves four cases.

(a) \( (C_2, D_2) = (A, B_2) \). Here \( S_2 \cong S_1 \), and we can apply Lemma 4.5 with either assignment of \( T_i \) to \( S_j \).

(b) \( (C_2, D_2) = (A, A) \). Here the argument in Case 1 about deleting the peripheral vertices from \( A \) and adding another copy of the root offshoots above the root yields \( Y_2 \cong Z_2 \) and hence \( B_1 \cong B_2 \), a contradiction.

(c) \( (C_2, D_2) = (B_1, B_2) \). In \( S_1 \), deleting the peripheral vertices of \( B_2 \) and adding \( W_1 \) above the root yields \( B_1 \). In \( S_2 \), deleting the peripheral vertices of \( B_2 \) and adding \( W_1 \) above the root yields \( A \). Hence \( B_1 \cong A \), a contradiction.

(d) \( (C_2, D_2) = (B_1, A) \). Now \( (C_1, Z_1, Y_1, D_1) = (A, A, B_1, B_2) \), and also \( (C_2, Z_2, Y_2, D_2) = (B_1, B_2, A, A) \). In particular, we have isomorphic pairs \( (C_1, Y_2), (Z_1, D_2), (Y_1, C_2) \), and \( (D_1, Z_2) \). By Lemma 4.4, \( A \in \{B_1, B_2\} \), again a contradiction. \( \square \)

Lemma 4.9. If \( |\{A_1, A_2, B_1, B_2\}| = 4 \), then the trunk edges of \( T_1 \) and \( T_2 \) are recognizable.

Proof. Here \( A_1, A_2, B_1, B_2 \) are distinct. Again by Lemma 4.5 we have the desired result unless \( S_1 \neq S_2 \) and each of \( T_1 \) and \( T_2 \) appears in both \( S_1 \) and \( S_2 \). By Lemma 4.6, we may assume \( \{C_1, C_2, D_1, D_2\} \subseteq \{A_1, A_2, B_1, B_2\} \).

Case 1: Some member of \( \{A_1, A_2, B_1, B_2\} \) is not in \( \{C_1, C_2, D_1, D_2\} \). By symmetry, we may assume \( A_1 \notin \{C_1, C_2, D_1, D_2\} \). In order to have \( T_1 \) appear in both \( S_1 \) and \( S_2 \), we must have \( B_1 \) as an end-subtree in both \( S_1 \) and \( S_2 \). By symmetry, we may assume \( D_1 \cong D_2 \cong B_1 \). Now, since \( T_2 \) appears in both \( S_1 \) and \( S_2 \) and \( S_1 \neq S_2 \), we may assume \( C_1 \cong A_2 \) and \( C_2 \cong B_2 \). To complete the copies of \( T_2 \) in \( S_1 \) and \( S_2 \), we must have \( Y_1 \cong B_2 \) and \( Y_2 \cong A_2 \).

At this point, the subtree \( W_2 \) that is \( A_2 \cap B_2 \) rooted at \( x_2 \) appears both in \( S_1 \) rooted at \( y_1 \) and in \( S_2 \) rooted at \( y_2 \). These are the portions of \( Y_1 \) and \( Y_2 \) that do not lie in \( D_1 \) and \( D_2 \), respectively. Since \( D_1 \cong D_2 \cong B_1 \), deleting the peripheral vertices of \( B_1 \) and placing \( W_2 \) atop the root of \( B_1 \) yields \( B_2 \) by looking at \( Y_1 \) in \( S_1 \) and \( A_2 \) by looking at \( Y_2 \) in \( S_2 \). Hence \( A_2 \cong B_2 \), contradicting the hypothesis.
Case 2: All of \{A_1, B_1, A_2, B_2\} appear in \{C_1, C_2, D_1, D_2\}, and hence there is a bijection via isomorphisms from one set of subtrees to the other. Recall that \(T_i = A_i \cup B_i\).

Since each of \(T_1\) and \(T_2\) appears in both \(S_1\) and \(S_2\), we may assume by symmetry that \(C_1 \cong A_1\), and hence \(Y_1 \cong B_1\) with \(W_1\) rooted at \(y_1\). Reversing the labeling of \(T_2\) if necessary, this leaves \(C_1 \cong A_1\), and hence \(Y_1 \cong B_1\) with \(W_1\) rooted at \(y_1\). Reversing the labeling of \(T_2\) if necessary, this leaves \(D_1 \cong B_2\) and \(Z_1 \cong A_2\), so \((C_1, Z_1, Y_1, D_1) = (A_1, B_2, A_1, A_2)\) (see Figure 4). By Lemma 4.4, we obtain \(A_1 \cong A_2\) or \(B_1 \cong B_2\), contradicting the hypothesis. □

Corollary 4.10. The trunk edges of \(T_1\) and \(T_2\) are recognizable.

Proof. With \(T_1, T_2, S_1, S_2\) and their subtrees \(A_1, B_1, A_2, B_2\) and \(C_1, D_1, C_2, D_2\) known from the deck, Lemmas 4.6 through 4.9 treat an exhaustive set of cases where arguments applying Lemmas 4.4 and 4.5 are used to recognize the trunk edges of \(T_1\) and \(T_2\), thereby determining which of \{\(A_1, B_1\)\} and which of \{\(A_2, B_2\)\} contain the ends of a longest path \(P\) in \(T\). □

5 Large Diameter and no Spi-center

In this section we continue and complete the reconstruction of trees having large diameter and no spi-center (that is, no copy of \(S^\ell+1\)). As in the previous section, we have \(r \geq n - 3\ell\) and \(n \geq 6\ell + 11\), which by Lemma 2.12 yields

\[
k \geq \left\lceil \frac{r - \ell - 4}{2} \right\rceil \geq \left\lceil \frac{n - 4\ell - 4}{2} \right\rceil \geq \left\lceil \frac{2\ell + 7}{2} \right\rceil = \ell + 4
\]

and thus allows us to recognize \(S^\ell+1 \not\subseteq T\).

In the previous section we determined the two maximal \((k - 1)\)-vines \(T_1\) and \(T_2\) whose centers are distance \(k - 1\) from the endpoints of any longest path in the \(n\)-vertex tree \(T\) whose \((n - \ell)\)-deck we are given. Since \(k - 1 \geq \ell + 1\) and \(S^\ell+1 \not\subseteq T\), in both \(T_1\) and \(T_2\) only two pieces have the full length \(k - 1\) (the “major” pieces). Furthermore, we have successfully oriented each of these \((k - 1)\)-vines, meaning that we know which is the “outer” major piece containing an endpoint of each longest path and which is the “inner” major piece extending toward the other \((k - 1)\)-vine.

Definition 5.1. In this section, for convenience we rename \(T_1\) and \(T_2\) as \(A\) and \(B\) (this will change), and we let their centers be \(a\) and \(b\). Along any full path \(P\), the distance between \(a\) and \(b\) is \(r + 1 - 2k\). Since \(k \geq (r - \ell - 4)/2\) by Lemma 2.12, the distance between \(a\) and \(b\) is at most \(\ell + 5\).
Inner peripheral vertices in $A$ and $B$ are vertices at distance $k - 1$ from $a$ and $b$ in the inner pieces. If there are multiple inner peripheral vertices in $A$, then let $a'$ be the vertex closest to $a$ in the inner piece at which paths to inner peripheral vertices in $A$ diverge. If there is no such vertex, then let $a'$ be the unique peripheral vertex in the inner piece, lying at distance $k - 1$ from $a$ along $P$. Define $b'$ similarly in $B$.

Lemma 5.2. All offshoots in $T$ from vertices between $a$ and $a'$ on $P$ (not including $a'$) are seen in full in $A$ (similarly between $b$ and $b'$ in $B$).

Proof. All vertices having distance at most $k - 1$ from $a$ are seen in $A$. An offshoot between $a$ and $a'$ containing a vertex with distance at least $k - 1$ from $a$ would contradict the definition of $a'$ (similarly for offshoots in $B$). □

The paths from $a$ to $a'$ and from $b$ to $b'$ both lie along $P$. Because we know $A$ and $B$ and which are the inner pieces, we know the distances from $a$ to $a'$ and from $b$ to $b'$. Knowing also that the distance from $a$ to $b$ is $r + 1 - 2k$, we know the order of these four vertices along $P$. We consider several cases, which we have just observed that we can recognize.

Lemma 5.3. If the vertices $a, b', a', b$ are distinct and occur along $P$ in that order, then $T$ is $\ell$-reconstructible.

Proof. Since we know $r$, we have a path of $r$ vertices in $T$ that we may assume is $P$. It suffices to know all the offshoots from $P$. The offshoots outside the path from $a$ to $b$ are given by the outer pieces of $A$ and $B$. Between $a$ and $b$, every offshoot at a vertex between $a$ and $a'$ is seen in full in $A$, and every offshoot at a vertex between $b'$ and $b$ is seen in full in $B$, by Lemma 5.2. With the vertices in this order, all offshoots are seen in full. □

Lemma 5.4. If $r \geq n - 2\ell$ and $n \geq 6\ell + 11$ and $a, b', a', b$ are distinct, then those vertices must occur in that order: $a, b', a', b$.

Proof. If the vertices occur in the order $a, a', b', b$, then the path from $a'$ to a peripheral vertex in $A$ outside $P$ has length at most $\ell$, since $S^{\ell+1} \not\subseteq T$. Similarly, the path from $b'$ to a peripheral vertex in $B$ outside $P$ has length at most $\ell$. Extending such paths from the peripheral vertices back through $a$ and $b$, respectively, to the ends of $P$ yields two paths that each have $2k - 1$ vertices, since each of the two paths then joins peripheral vertices in a $(k - 1)$ and pass through the center of the $(k - 1)$-vine. The two paths are disjoint, since the vertices on $P$ are in the order $a, a', b', b$, and each has at most $\ell$ vertices outside $P$. Hence $r \geq 4k - 2 - 2\ell$. Lemma 2.12 then yields $r \geq 2r - 2\ell - 8 - 2 - 2\ell$, which simplifies to $r \leq 4\ell + 10$. With $r \geq n - 2\ell \geq 4\ell + 11$, we have a contradiction. □
Lemma 5.5. If $a' = b'$, then $T$ is $\ell$-reconstructible.

Proof. By Lemma 5.2, we know everything in $T$ except the offshoots from $P$ at the vertex called $a'$ and $b'$. These offshoots have length less than $k - 1$, since $k - 1 \geq \ell + 1$ and $S^{\ell+1} \not\subseteq T$. Hence they appear in full in the $k$-vine centered at $a'$.

We know all the maximal $k$-vines in $T$; all the $k$-centers are along $P$, strictly between $a$ and $b$. We also know the central offshoots at all of those vertices other than $a'$. Hence we can exclude all the maximal $k$-vines whose offshoots from the central vertex appear as the offshoots from some vertex of $P$ between $a$ and $b$. The offshoots from the central vertex of the maximal $k$-vine that remains complete the reconstruction of $T$. Note that it does not matter if we exclude the "wrong" $k$-vine due to multiple $k$-vines having the same offshoots from the central vertex. \qed

Because $a'$ and $b'$ are defined even when $A$ or $B$ does not have multiple inner peripheral vertices, and in those cases the four vertices are ordered as in Lemma 5.3, for the remainder of this section we may assume $n - 3\ell \leq r \leq n - 2\ell - 1$, with the vertices $a, a', b, b'$ distinct and occurring in that order.

Lemma 5.6. If some longest path has vertices $x$ and $x'$ equidistant from the ends (and distance at least $k$ from the ends) such that the offshoots from the path at $x$ and $x'$ together have more than $\ell$ vertices, then $T$ is $\ell$-reconstructible.

Proof. First we observe that this case is recognizable. We consider subtrees containing an $r$-vertex path and offshoots from the path only at two vertices that are equidistant (with distance at least $k$) from the ends. If in some such subtree the offshoots have $\ell + 1$ vertices in total, then the subtree fits in a card (since $r \leq n - 2\ell - 1$), and we see it in a csc.

Since $S^{\ell+1} \not\subseteq T$, every longest path contains the path from $a$ to $b$, and thus $x$ and $x'$ lie along this path that contains all the $k$-centers in $T$.

Let $C_1$ be a largest subtree containing an $r$-vertex path such that the vertices along it with the same distance from the center as $x$ and $x'$ (these in fact are $x$ and $x'$ themselves) have degree 2. Since the offshoots at $x$ and $x'$ have at least $\ell + 1$ vertices, $C_1$ fits into a card, we see it, and it gives us in full all offshoots from the path other than those at $x$ and $x'$.

The exclusion argument with maximal $k$-vines used in the proof of Lemma 5.5 now tells us the offshoots at $x$ and the offshoots at $x'$ but not which is which. If $C_1$ is symmetric, then it does not matter which is which, so we may assume that $C_1$ is not symmetric along the path. Let $Q$ and $Q'$ denote the two sets of offshoots yet to be assigned to $x$ or $x'$.

Since the distance between $a$ and $b$ is at most $\ell + 5$, and in this section we know $k - 1 \geq \ell + 3$, both $A$ and $B$ contain the center of $P$. Since we also know the distance between $a$ and the center, we know the number $a_1$ of vertices in $A$ in offshoots from $P$ at vertices that are not on the other side of the center from $a$. Similarly define $b_1$ using $b$. Since we know all the
offshoots at vertices other than \( x \) or \( x' \), when \( |V(Q)| \neq |V(Q')| \) we can tell which of \( Q \) and \( Q' \) should be associated with the member of \( \{x, x'\} \) closer to \( a \) to reach a total of \( a_1 \) vertices on that side.

Hence we may assume \( |V(Q)| = |V(Q')| \). We may also assume \( a_1 \leq b_1 \). If \( a_1 = b_1 \), then let \( C_2 \) be a subtree having an \( r \)-vertex path plus the offshoots in \( B \) other than at \( \{x, x'\} \) (we see these in \( C_1 \)) plus offshoots at the member of \( \{x, x'\} \) in \( A \). Since \( C_2 \) omits at least half of the vertices outside \( P \), and \( n - r \geq 2l \), we have deleted enough so that \( C_2 \) fits in a card. We see \( C_2 \), and it tells us which of \( Q \) and \( Q' \) is attached to the member of \( \{x, x'\} \) in \( A \).

If \( a_1 < b_1 \), then instead we define \( C_2 \) again having an \( r \)-vertex path and offshoots at the member of \( \{x, x'\} \) in \( A \) but using only \( a_1 + 1 \) of the vertices in offshoots from \( P \) at vertices on the side of the center in \( B \). This distinguishes \( x \) and \( x' \) in \( C_2 \), which is again small enough to fit into a card and tell us which of \( Q \) and \( Q' \) is attached to \( x \).

The argument remains valid when \( x = x' \) as the unique center of a longest path when \( r \) is odd. Again \( C_1 \) gives us the offshoots at other vertices, and the exclusion argument for the maximal \( k \)-vines gives us the offshoots from the center, without needing to distinguish \( Q \) from \( Q' \). \( \square \)

**Theorem 5.7.** If \( r \geq n - 3l \) and \( n \geq 6l + 11 \), and \( S^{l+1} \not\subset T \), then \( T \) is \( \ell \)-reconstructible.

**Proof.** In light of Lemmas 5.3 and 5.5, we may assume \( n - 3l \leq r \leq n - 2l - 1 \) and that the vertices \( a, a', b', b \) are distinct and occur in that order along \( P \). Since we know \( A \) and \( B \), we know \( d(a, a') \) and \( d(b, b') \). By symmetry, we may assume \( d(a, a') \leq d(b, b') \).

For \( x \in V(P) \), let \( f(x) \) be the number of vertices in the offshoots from \( P \) at \( x \). We next prove \( f(a') + f(b') \geq \ell + 1 \). There are \( r - 2k \) vertices between \( a \) and \( b \) on \( P \). The offshoots at \( a' \) and \( b' \) contain peripheral vertices in \( A \) and \( B \). The paths from \( a \) and \( b \) to peripheral vertices have length \( k - 1 \). Hence the offshoots at \( a' \) and \( b' \) together contain at least \( 2k - 2 - (r - 2k) \) vertices, due to the order \( a, a', b', b \). By Lemma 2.12 and the case \( r \geq n - 3l \) and \( n \geq 6l + 11 \),

\[
4k - r - 2 \geq 2r - 2l - 8 - r - 2 \geq r - 2l - 10 \geq n - 5l - 10 \geq \ell + 1.
\]

This completes the proof that \( f(a') + f(b') \geq \ell + 1 \).

**Case 1:** \( d(a, a') = d(b, b') \). In this case \( a' \) and \( b' \) are equidistant from the center. Since we have proved \( f(a') + f(b') \geq \ell + 1 \), Lemma 5.6 implies that \( T \) is \( \ell \)-reconstructible.

**Case 2:** \( d(a, a') < d(b, b') \). Let \( a'' \) be the vertex at distance \( d(a, a') \) from \( b \) along the path from \( b \) to \( a \). Let \( b'' \) be the vertex at distance \( d(b, b') \) from \( a \) along the path from \( a \) to \( b \).

If \( f(a') + f(a'') \) or \( f(b') + f(b'') \) exceeds \( \ell \), then Lemma 5.6 completes the proof. Hence we may assume \( f(a') + f(a'') \leq \ell \) and \( f(b') + f(b'') \leq \ell \). On the other hand, we have proved \( f(a') + f(b') \geq \ell + 1 \).

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Since we know $A$ and $B$, we know the locations of all of $a, a', a'', b, b', b''$ along $P$. Since $f(a') + f(b') \geq \ell + 1$, omitting the offshoots at the four vertices $a', a'', b, b'$ leaves a subtree having an $r$-vertex path that fits in a card. Hence we see a largest csc $C_1$ with an $r$-vertex path in which the vertices at these specified distances along the path have degree 2; it shows us all offshoots at all the other vertices.

Since $f(a') + f(a'') \leq \ell$ and $r \leq n - 2\ell$, a largest subtree containing an $r$-vertex path and offshoots only at the vertices having the distances of $a'$ and $a''$ from the ends of the path fits in a card and shows us the offshoots at $a'$ and $a''$, but not which is at which vertex. Similarly, we obtain the offshoots at $b'$ and $b''$, but not which is at which of these two vertices.

If $d(b, b') > d(b, a)/2$, then $a, a', b', b'', b$ occur in that order along $P$. In this case, the offshoots at $b''$ and $a''$ appear in $B$, and we know them because we know all of $B$ and which of its major pieces is the inner one. This distinguishes which set of offshoots in each pair $(a', a'')$ and $(b', b'')$ is attached to which vertex, completing the reconstruction of $T$.

Hence we may assume $d(b, b') \leq d(b, a)/2$, and the six vertices occur in the order $a, a', b', b'', a''$, $b$ along $P$, including the possibility $b'' = b'$. Again we can use $B$ to see the offshoots at $a''$ and correctly assign offshoots to the pair $(a', a'')$, but we still must assign the offshoots at $b'$ and $b''$ to the correct vertex. If $b' = b''$, then there is only one set of offshoots and no need to distinguish them, so we may assume $d(b, b') \leq [d(b, a) - 1]/2 \leq (\ell + 4)/2$.

The offshoot(s) at $b'$ contain a path to an inner peripheral vertex of $B$. Hence $f(b') \geq k - 1 - d(b, b') \geq k - 3 - \ell/2$. Since $k \geq \ell + 4$ (using $n \geq 6\ell + 11$), we have $f(b') \geq \ell/2 + 1$. Since $f(b') + f(b'') \leq \ell$, we conclude $f(b') > f(b'')$, which allows us to assign the offshoots to $b'$ and $b''$ correctly, completing the reconstruction of $T$. \qed

# 6 Small Diameter

In this section we consider trees with smaller diameter ($r \leq n - 3\ell - 1$), first those having sparse cards and then those not having sparse cards. The key property when $r \leq n - 3\ell - 1$ is that any connected card containing an $r$-vertex path also has at least $2\ell + 1$ vertices outside that path. We note first a case that we have already handled.

**Corollary 6.1.** If $n \geq 6\ell + 5$ and $r \leq n - \ell$ and $T$ contains a sparse card that is a 3-legged spider, then $T$ is $\ell$-reconstructible.

**Proof.** A 3-legged spider has at most $(3r - 1)/2$ vertices. Such a tree as a card requires $(3r - 1)/2 \geq n - \ell \geq 5\ell + 5$, so $r \geq (10\ell + 11)/3$. In particular, $r \geq 3\ell + 6$. By Lemma 2.17, $T$ has exactly one spi-center. Now Lemma 3.1 applies and $T$ is $\ell$-reconstructible. \qed
Henceforth in this section we may assume that $T$ has no sparse card that is a 3-legged spider. We also assume $n \geq 6\ell + c$ and $r \leq n - 3\ell - 1$.

**Definition 6.2.** When discussing a particular sparse card $C$, the primary vertex is called $z$, and a fixed $r$-path $P$ is specified with vertices $v_1, \ldots, v_r$ in order, indexed so that $z = v_j$ with $j \leq (r + 1)/2$. An optimal sparse card is one that maximizes $j$ among all sparse cards; that is, the primary vertex is as close as possible to the center. The components of $T - z$ containing $v_{j-1}$ and $v_{j+1}$ are called $T_1$ and $T_2$, respectively. (Is $T_2$ well-defined here?) If $j = (r + 1)/2$, then $T_2$ is the side with more vertices, chosen arbitrarily if $T_1$ and $T_2$ also have the same size. The union of the offshoots from $P$ in $C$ is $W$, with $W^*$ being the union of the components of $T - z$ containing the offshoots in $W$. A long-legged card relative to $C$ is a connected card $C'$ having a leg longer than $C$ (that is, a path of length more than $r - j$ from a leaf to the nearest branch vertex). When $r \leq n - 3\ell - 1$ a long-legged card must have a branch vertex, since otherwise the card is a path, which requires $r \geq n - \ell$.

**Lemma 6.3.** The subtree $T_2$ defined from an optimal sparse card $C$ contains no full path. In particular, if $j < (r + 1)/2$, then every $r$-vertex path has exactly one endpoint in $T_2$.

**Proof.** Let $P$ be the full path in $C$, and suppose that $T_2$ contains a full path $P'$. Any two longest paths in $T$ share the center of $T$. Since $P'$ remains in $T_2$, it extends from the center along $P$ toward the primary vertex $z$ of $C$ but does not reach $z$, diverging earlier at a vertex $w$. In the other direction $P'$ cannot extend farther past the center than $P$ does, so we may assume that $P'$ shares that portion of $P$.

Using $P'$, we can now form a sparse card $C'$ with primary vertex $w$, since all of $T_1$ plus the offshoots from $P$ at $z$ lie in an offshoot from $P'$ at $w$. However, $w$ is closer to the center of $T$ than $z$ is, contradicting the optimality of $C$. \hfill \square

**Lemma 6.4.** Suppose $n \geq 6\ell + c$ and $r \leq n - 3\ell - 1$. If $T$ has an optimal sparse card $C$ of degree 3, and $T$ has a long-legged card relative to $C$, then $T$ is $\ell$-reconstructible.

**Proof.** We use notation $z, P, T_1, T_2, W, W^*$ as in Definition 6.2. Since $C$ has degree 3, $W$ is a single offshoot from $P$. Let $q$ be the maximum length of the legs in long-legged cards, so $q > r - j$. Let $C'$ be a long-legged card with a leg of length $q$, and let $w$ be the branch vertex at the end of the long leg $L$ in $C'$.

Let $W'$ be the subtree of $C'$ rooted at $w$ that is obtained by deleting the edges of $L$. We will show that $w$ lies in the offshoot $W^*$ from $P$ at $z$ in $T$. If $W^*$ omits $w'$ but contains some vertex $w'$ of $W'$ other than $w$, then the path from $w$ to $w'$ in $C'$ contains $z$, as does every path connecting $V(W^*)$ and $V(T) - V(W^*)$. Now the path of length $q$ leaving $w$ in $C'$ avoids $z$, which yields a path of length more than $q$ leaving $z$, which contradicts $q > r - j$. Hence if $w \notin V(W^*)$ then all of $W'$ is outside $W^*$. 31
Since $r < n - 3\ell$ and $C'$ is a card, $C'$ has at least $2\ell + 1$ vertices outside any path. In particular, $W'$ has at least $2\ell + 1$ vertices outside any longest path. Since $C$ is a card, $T_1 \cup T_2$ and the offshoots at $z$ other than $W$ together have at most $\ell$ vertices outside $P$. With $W'$ completely outside $W^*$, there are fewer vertices outside $P$ allowed to be in $W'$ than the number of such vertices that must be in $W'$. The contradiction yields $w \in V(W^*)$. With $w \in V(W^*)$, the path $L$ cannot visit both $T_1$ and $T_2$, so $C'$ must omit all of $T_1$ or all of $T_2$.

Since $C'$ is a card and omits all or $T_1$ or all of $T_2$, we have $j - 1 \leq \ell$. If there are two choices for $C'$ having legs of length $q$, then they must have the same vertex $w$ at the end of the leg, since otherwise by the maximality of $q$ each would have more than $2\ell$ vertices beyond the branch vertex at the end of the leg that would not appear in the other, preventing both from being cards. Hence there is only one choice for $w$.

Let $\hat{C}$ be any subtree of $T$ having a leg $L'$ of length at least $q$ and having at least $\ell - j + 1$ vertices in offshoots from its longest path. We claim that the branch vertex $u$ at the end of $L'$ in $\hat{C}$ lies in the subtree $W'$ defined above for any long-legged card $C'$. If $u \notin V(W')$ but some vertex in $\hat{C}$ that is outside $L'$ does lie in $W'$, then there is a path of length at least $q$ leaving $z$, which does not exist since $q > r - j$. Hence $w \notin V(W^*)$ requires that no vertex of $\hat{C}$ outside $L'$ is in $W'$. Having $u$ on $L$ in $C'$ between $w$ and the vertex at distance $j - 1$ from the end of $L$ would create a path with more than $r$ vertices. Having $u$ anywhere else relative to $L$ would not leave enough vertices outside $L$ to accommodate the vertices from $W'$, even with a longest path from $W'$ laid along $L$, since as a card $C'$ omits only $\ell$ vertices, of which at least $j - 1$ are in the path $P$ along $T_1$ or $T_2$, and $\hat{C}$ needs at least $\ell - j + 1$ vertices outside a longest path. (Either $u$ is on $P$ and we need $\ell - j + 1$ from the remaining $\ell - j$ outside $P$, or $u$ is outside $P$ and we need at least $\ell - j + 2$ there.) Hence $u \in V(W')$. This implies also that $\hat{C}$ avoids $T_1$ or $T_2$.

We have argued that the maximum length $q'$ of the offshoots at $w$ in a long-legged card satisfies $q' \leq j - 2 < \ell$. We also observed that such a card has at least $2\ell + 1$ vertices outside its longest path. Hence within some long-legged card where $W'$ has length $q'$, we can find a subtree $X$ with leg of length $q$ ending at branch vertex $u$ such that $X$ contains offshoots from $u$ outside the long leg that include a path of length $q'$ and a total of exactly $\ell + 1$ vertices. Such a subtree $X$ is a particular choice of $\hat{C}$ in the argument above, so any occurrence of $X$ in $T$ must have $u \in W'$. Furthermore, since $X$ has the maximum possible length beyond $u$, we must have $u = w$, and the offshoots in $X$ are contained in $W'$.

Now let $C_0$ be a largest subtree containing $X$ such that no additional vertices are added to the offshoots that came from $W'$ or to other offshoots from $w$. Since any such tree contains $X$, in $T$ its vertex $u$ is at $w$. Such a tree also omits at least $\ell$ vertices from $W'$, so it fits in a card. Therefore $C_0$ shows us the offshoots from $L$ in full, including $T_2$ and $T_1$ and the offshoots from the path joining $z$ and $w$ (but not at $w$).

Now, among the long-legged cards with leg of length $q$, choose $C'$ to minimize $d_{C'}(w)$, and within that minimize the size of a smallest offshoot $Y$ in $C$ from $L$ at $w$. If some offshoot
in \( T \) at \( w \) other than \( Y \) does not appear in full in \( C' \), then we can alter \( C' \) to add a vertex to that offshoot and delete a vertex from \( Y \). If \( Y \) disappears, then it decreases the degree of \( w \) or increases \( q \). Hence in \( C' \) we see in full all offshoots from \( L \) in \( T \) at \( w \), except possibly the one containing \( Y \).

It remains to determine the offshoots from \( L \) at \( w \). Let \( d \) be the degree of \( w \) in \( W' \), and let \( Q_1, \ldots, Q_d \) be the offshoots from \( w \) in \( W' \), in nonincreasing order of size. By the choice of \( C' \), \( Q_1, \ldots, Q_{d-1} \) are full offshoots in \( T \) at \( w \); we must find the smaller ones. Let \( C_3 \) be the family \( C_3 \) of subtrees having a leg of length \( q \) and \( d - 1 \) offshoots at the end of the leg, with \( d - 2 \) of them being \( Q_1, \ldots, Q_{d-2} \).

**Case 1:** \( d \geq 4 \). Since \( Q_1, \ldots, Q_{d-2} \) are the largest offshoots in \( W' \), they together have at least \( \ell + 1 \) vertices, and since the longest path from \( w \) in \( W' \) has length at most \( j - 2 \), these offshoots have at least \( \ell - j + 1 \) vertices outside a longest path. By the argument about \( \hat{C} \), in every such subtree the branch vertex at the end of the leg is in \( W' \), and it must equal \( w \) because moving farther into \( W' \) would not allow having an offshoot as large as \( Q_1 \).

A largest member of \( C_3 \) omits \( Q_d \) and hence fits in a card. It show us \( Q_{d-1} \) (which we already knew) as its smallest offshoot. Since we know \( T_2 \), we know the number of peripheral vertices in \( T_2 \), and hence we know the members of \( C_3 \) in which the smallest offshoot is obtained by deleting vertices of \( Q_{d-1} \). After excluding those members, a largest remaining member of \( C_3 \) shows us the next largest offshoot at \( w \). Continuing the exclusion argument allows us to find all the offshoots at \( w \).

**Case 2:** \( d = 3 \). In \( C' \) we see \( Q_1 \) and \( Q_2 \) in full. Together, \( Q_1 \) and \( Q_2 \) have at least \( \ell + 1 \) vertices, since \( Q_1 \) is no smaller than \( Q_2 \). Since the extension \( Q_3' \) of \( Q_3 \) in \( T \) is no bigger than \( Q_2 \), a member of \( C_3 \) whose second largest offshoot has the size of \( Q_3' \) still fits in a card. Since we know \( Q_2 \) and the number of peripheral vertices in \( T_2 \), we can exclude from \( C_3 \) all the members whose second offshoot arises by deleting vertices of \( Q_2 \). A largest remaining member shows us the full extension of \( Q_3 \). Continuing the exclusion argument allows us to find the remaining offshoots at \( w \).

**Case 3:** \( d = 2 \). We know \( Q_1 \), but we do not see in full the next-largest offshoot from \( w \) in \( T \) (unless it has the same size as \( Q_1 \) and we have several choices for \( C' \)). Let \( Q_2' \) be this offshoot in \( T \). A long-legged card \( C_4 \) with leg of length \( q \) having two offshoots \( Y_1 \) and \( Y_2 \) as equal in size as possible shows us \( Q_2' \) as \( Y_2 \) if the sizes of the two offshoots differ by more than 1, after which we can use exclusion to find any smaller offshoots. In the remaining case, we must work harder to find \( Q_2' \).

Since \( C_4 \) is a card, the remaining case is \( |V(Y_2)| = \lfloor (n - \ell - q - 1) \rfloor /2 \). With \( q \geq r - j + 1 \) and \( r \leq n - 3\ell - 1 \), we have \( n - q \geq n - r + j - 1 \geq 3\ell + j \). Hence

\[
|V(Y_2)| = \lfloor (n - \ell - q - 1)/2 \rfloor \geq \lfloor (2\ell + j - 1)/2 \rfloor > \ell.
\]

The offshoots from \( w \) in \( T \) that do not contain \( Y_1 \) or \( Y_2 \) are all completely omitted from \( C_4 \), and hence in total they have at most \( \ell \) vertices, smaller than \( Y_2 \).
Since $Y_1$ and $Y_2$ each have at least $\ell$ vertices, and $W^*$ has length $q'$, which is less than $\ell$, we can choose $C_4$ so that $Y_1$ and $Y_2$ each have the largest possible length, which will be the length of the offshoots from $w$ containing them in $T$. We then see in $C_4$ how many of these two offshoots have length $q'$. Since we know $Q_1$, we know whether it has length $q'$. By knowing how many of $Q_1$ and $Q'_2$ have length $q'$, we now know whether $Q'_2$ has length $q'$.

If $Q'_2$ has length $q'$, then let $C_5$ be the family of subtrees of $T$ having a leg of length $q$ at the end of which is a single offshoot with length $q'$ and at least $|Y_2|$ vertices. In every member of $C_5$, the vertex at distance $q$ from the leaf of the leg is $w$. If also $Q_1$ has length $q'$, then in some members of $C_5$ the offshoot arises by deleting vertices from $Q_1$. Since we know $Q_1$, we can exclude any such members, and a largest remaining member of $C_5$ shows us $Q'_2$.

If $Q'_2$ has length less than $q'$, then let $C'_6$ be the family of subtrees of $T$ having a leg of length $q$ at the end of which are two offshoots: one being a path of length $q'$ and one having $|V(Y_2)|$ vertices. Because $|V(Y_2)| > \ell$, the branch vertex must be $w$. Because we know $Q_1$, the number of paths of length $q + q'$, and the number of peripheral vertices in $T_2$, we can exclude the members of $C'_6$ in which the offshoot with $|V(Y_2)|$ vertices comes from $Q_1$. A largest remaining member of $C'_6$ shows us $Q'_2$.

Now that we know $Q_1$ and $Q'_2$, we obtain the remaining offshoots at $w$ to reconstruct $T$. If $|V(Q_2)| \geq \ell$, then let $C_6$ be the family of subtrees of $T$ having a leg of length $q$ at the end of which are two offshoots: one being $Q_1$. The presence of $Q_1$ fixes the branch vertex as $w$. Since we know $Q'_2$, and all the smaller offshoots at $w$ together have at most $\ell$ vertices, successively excluding the members of $C_6$ in which the second offshoot comes from a known offshoot at $w$ allows us to find all the offshoots at $w$.

If $|V(Q_2)| < \ell$, then there may be multiple offshoots at $w$ that are bigger then the $Q_2$ we see in $C'$. In this case let $C'_6$ be the family of subtrees of $T$ having a leg of length $q$ at the end of which are two offshoots: one showing exactly $\ell + 1$ vertices from $Q_1$ including a path of length $q_1$, where $q_1$ is the length of $Q_1$. This offshoot can only be from $Q_1$, and the branch vertex is fixed at $w$. Having $|V(Q_2)|$ or fewer vertices in the second offshoot still fits in a card. Since we know $Q'_2$, we can therefore employ an exclusion argument using $C'_6$ to find the remaining offshoots at $w$.

(Does the argument above work when $j = (r + 1)/2$?)

We maintain the definitions and notation from Definition 6.2.

**Lemma 6.5.** Suppose that $T$ has a sparse card $C$ whose primary vertex $z$ is the $j$th vertex on an $r$-path $P$ in $C$. Grown from an $r$-path $\langle u_1, \ldots, u_r \rangle$, let $C_1$ be a largest full subtree in $T$ such that $C_1$ has offshoots at $u_j$ from that path $P'$ that total exactly $\ell + 1$ vertices, and $u_2, \ldots, u_{j-1}$ have degree 2 in $C_1$. Under these conditions, the subtree $C_1$ fits in a card, satisfies $u_j = z$, and shows $T_2$ in full (if $j < (r + 1)/2$ or $|V(T_1)| < |V(T_2)|$).
Proof. Note first that, since $C$ has at least $2\ell + 1$ vertices in the union $W$ of its offshoots from $P$, such a subtree $C_1$ exists with $u_j = z$ in which the union $W'$ of the offshoots at $z$ from its $r$-path is contained in $W$. Also, the offshoots from $P$ in $T_2$ have altogether at most \( \ell \) vertices, since $C$ is a card.

We first prove $u_j = z$. If $u_j \notin V(P)$, then to avoid having a path longer than $P$, the path from $u_j$ to $V(P)$ must follow the portion \( \langle u_j, \ldots, u_r \rangle \) of $P'$ and reach $P$ in $T_2$ (we can choose this by symmetry if $j = (r + 1)/2$). Now $W'$ is contained in an offshoot from $T_2$, which cannot have $\ell + 1$ vertices. Hence $u_j \in V(P)$. If $u_j \notin \{z, v_{r+1-j}\}$, then we can choose paths from $P$ and $P'$ ending at $u_j$ such that their union is a path longer than $P$. If $u_j = v_{r+1-j}$, then both $W'$ and $u_{r+1-j}, \ldots, u_r$ are in $T_2$, which puts too many vertices into $T_2$ regardless of whether $v_{r+1-j}, \ldots, v_r$ are in $W'$. Hence $u_j = z$.

If $j < (r + 1)/2$, then any path of length $r - j$ extending from $z$ lies in $T_2$. Consider the portion of $C_1$ not in $T_2$. Since $T_1 \cup W^*$ has at least $j + 2\ell$ vertices, of which at most $j + \ell$ appear in $C_1$, we conclude that $C_1$ fits in a card, we see it as a csc, and it shows us all of $T_2$. \( \square \)

**Lemma 6.6.** Suppose $n \geq 6\ell + 7$ and $r \leq n - 3\ell - 1$. If $T$ has a sparse card $C$ of degree 3, and $T$ has no long-legged card relative to $C$, then $T$ is $\ell$-reconstructible.

Proof. With notation as in Definition 6.2, choose an optimal sparse card $C$. Among the candidates with $z = v_j$, choose $C$ to maximize the length $t$ of $W$. Note that $t \leq j - 1$, since $P$ is a longest path in $T$. By Corollary 6.1, we may assume that $C$ is not a spider. Hence $W$ is not a path, and $t$ is also the length of the full offshoot $W^*$, since otherwise vertices from a longer path in $W^*$ could be added while deleting leaves of $W$ not on that path. Since $r \leq n - 3\ell - 1$, there are at least $2\ell + 1$ vertices in $W$.

Let $C_1$ be as in Lemma 6.5, showing us all of $T_2$. Since $C$ has degree 3 and is a card, there is only one offshoot in $W^*$. Hence we can choose $C_1$ to have one offshoot from $P'$. Now either $W' \subseteq W^*$ or $W' \subseteq T_1$, with $t = j - 1$ required in the latter case. We know whether $t < j - 1$ holds, because we see $t$ in $C$.

**Case 1:** $t < j - 1$. As noted above, $W' \subseteq W^*$. Replace $C_1$ with a potentially larger tree $C'_1$ by dropping the requirement that $u_2, \ldots, u_{j-1}$ have degree 2. The offshoot $W'$ remains the same, along with the argument in Lemma 6.5 that $u_j = z$. Since $W'$ still omits at least $\ell$ vertices from $W^*$, $C'_1$ fits in a card. By its maximality, $C'_1$ shows us all of $T_2$, $T_1$, and any offshoots from $P$ at $z$ other than $W^*$.

We now know all of $T$ except $W^*$. In particular, we know $s$, where $s = |V(W^*)|$, and $t$, which is the length of both $W$ and $W^*$. By Lemma 6.4, we recognize that we are in the case where $T$ has no long-legged card relative to $C$, since otherwise we can reconstruct $T$. Hence $s - |V(W)| < j - 1$; otherwise, replacing $u_1, \ldots, u_{j-1}$ with vertices of $W^* - V(W)$ would produce a long-legged card.
Let $i = 1 + s - |V(W)| \leq j - 1$. Let $C_2$ be the family of cards consisting of a path $P''$ with vertices $u_i, \ldots, u_r$ having an offshoot $W''$ at $u_j$ that has $s$ vertices and length $t$. There is a card in $C_2$ in which $u_j = z$ and $W''$ is $W^*$. It suffices to recognize such a card in $C_2$.

We claim first that $u_j = z$ for any card in $C_2$. As in the similar argument for $C_1$, if $u_j \notin V(P)$, then the path from $u_j$ to $V(P)$ follows $\langle u_j, \ldots, u_r \rangle$ and reaches $P$ in $T_2$, which with $W''$ yields too big an offshoot from $T_2$. If $u_j \in V(P) - \{z\}$, then $W''$ is too big to be an offshoot from $P$, so $P$ enters $W''$. Since $t < j - 1$, to avoid having a path longer than $P$ the part of $P$ in $W''$ must come from $T_1$. Now $z$ has degree 2 in the card, which as $u_j, \ldots, u_r$ continues beyond $z$ implies that the card has at most $r + \ell$ vertices, a contradiction.

Suppose now that $j - i > t$. Since $u_j = z$ and $W^*$ has length $t$, the path $\langle u_i, \ldots, u_j \rangle$ cannot lie in $W^*$. Hence it must come from $T_1$, and any card in $C_2$ shows us $W^*$ to complete the reconstruction of $T$.

Hence we may assume $j - i \leq t$. Some cards in $C_2$ may take $\langle u_i, \ldots, u_{j-1} \rangle$ from $W^*$ and $W''$ from $T_1$. We aim to exclude these. When $j - i \leq t$, in $T$ there are copies of the spider $S_{j-i,j-1,r-j}$, and we know how many there are. We claim that all such spiders have branch vertex $z$. Since $t < j - 1$, the branch vertex cannot be in $T_1$ or $W^*$, since neither could provide a leg of length $j - 1$. If the spider has a branch vertex $z'$ in $T_2$ or in another offshoot from $z$, then it yields a full path in $T$ (contradicting Lemma 6.3) or a longer path than $P$.

Since $t < j - 1$, every copy of $S_{j-i,j-1,r-j}$ with branch vertex $z$ has leaves at distances $j - i, j - 1$, and $r - j$ from $z$ in $W^*$, $T_1$, and $T_2$, respectively. Since we know $T_1$ and $T_2$, we know the numbers of such leaves in $T_1$ and $T_2$ (including when $j = (r+1)/2$), so the count of spiders gives the number of vertices in $W^*$ at distance $j - i$ from $z$. This allows us to determine which cards in $C_2$ have the specified path of length $j - i$ from $z$ in $W^*$ and the offshoot from $P''$ as part of $T_1$ (or $T_2$ if $j = (r+1)/2$). After excluding such cards, the cards that remain show us $W^*$, completing the reconstruction.

**Case 2:** $t = j - 1 \geq \ell + 1$. Consider again the subtree $C_1$. Again $u_j = z$, the subtree fits in a card, and it shows us all of $T_2$. In $C_1$ we see in full all offshoots from $z$ in $T$ except $W^*$ and a smallest offshoot having length $j - 1$, which we may take to be $T_1$. The offshoot $W'$ and the path $\langle u_1, \ldots, u_{j-1} \rangle$ in $C_1$ lie in $W^*$ and $T_1$, but we don't know which is where.

Let $C_3$ be the family of full subtrees of $T$ whose vertices in positions $j$ and $r+1-J$ on the full path have degree 2; let $P'''$ denote the full path. Let $C_3$ be a largest member of $C_3$. Any member of $C_3$ has $z$ in position $j$ or $r+1-J$ along $P'''$, since $C$ prevents offshoots from $P$ having more than $\ell$ vertices, and here $j - 1 \geq \ell + 1$ forces such an offshoot if $z$ is elsewhere. Since $d_{C_3}(z) = 2$ and a path of length $r - j$ extends from $z$ in $C_3$, the subtree $C_3$ must be missing $T_1$ or $W^*$; since $j - 1 > \ell$, this means that $C_3$ (and any member of $C_3$) fits in a card.

Since we know $T_2$, we know the subtrees of $T_2$ that can arise when a vertex at distance $r + 1 - 2j$ from $z$ is restricted to degree 2. The part of $C_3$ outside such a subtree is a copy of $T_1$ or $W^*$, whichever is larger. If they have the same size, then we get both from
two candidates for $C_3$. Otherwise, we use an exclusion argument to find the other. Having obtained the larger one, say $X$, we know the number $n'$ of vertices in the smaller one, say $Y$. Knowing $X$, we know all the rooted subtrees of $X$ with $n'$ vertices. Excluding from $C_3$ the members that arise using such a subtree from $X$ in the middle and other end, a member of $C_3$ of the right size that remains shows us $Y$.

**Case 3:** $t = j - 1 \leq \ell$. Let $C'$ be a largest full subtree of $T$ having a leg with length more than $r - j$. Since by assumption $T$ has no long-legged card, $C'$ fits in a card. Since there are only $\ell$ vertices outside $C$, and $j - 1 \leq \ell$ while $|V(W^*)| \geq 2\ell + 1$ (so $W^*$ is bigger than $T_1$), the csc $C'$ shows us $W^*$ in full.

Recall $C_1$, in which we know $u_j = z$ and see all of $T_2$. As in Case 2, in $C_1$ we see all offshoots from $z$ other than a smallest of length $j - 1$, which we may take to be $T_1$. Again the offshoot $W'$ in $C_1$ may lie in $W^*$ or in $T_1$.

Having obtained $W^*$ from $C'$, we now obtain $T_1$ by an exclusion argument. Since $C$ omits exactly $\ell$ vertices and $j - 1 \leq \ell$, $T_1$ has at most $2\ell$ vertices, while $W^*$ has at least $2\ell + 1$. Knowing $T_2$ and $W^*$ (and extra offshoots at $z$), we know $|V(T_1)|$; call it $n'$. We also know the numbers of vertices in $W^*$ and $T_2$ that are farthest from $z$ and the number of $r$-vertex paths in $T$, so we can compute the number of vertices in $T_1$ that are farthest from $z$.

Now consider cscs with $|V(C')| - (|V(W^*)| - n')$ vertices having an $r$-vertex path and a leg with length more than $r - j$. Exclude all those in which the portion outside the leg comes from the known subtrees $W^*$ or $T_2$; such a csc that remains shows us $T_1$, completing the reconstruction of $T$. 

\[\square\]

**Lemma 6.7.** Suppose $n \geq 6\ell + r$ and $r \leq n - 3\ell - 1$. If $T$ has a sparse card, but no sparse card of degree 3, then $T$ is $\ell$-reconstructible.

**Proof.** Since all sparse cards are visible in the deck, we recognize such $T$. Among all sparse cards, choose $C$ to maximize the length $t$ of the longest offshoot from $P$ in $C$, and within that minimize the degree of $C$, and within that maximize the number of vertices in the largest offshoot of length $t$. **Check this choice; it guarantees that we see some offshoot of length $t$ in full, even if it is the smallest offshoot in $C$.**

By the choice of $C$, all offshoots from $P$ that appear in $C$ (all at $z$) are seen in full, except possibly for one smallest offshoot with length less than $t$ (or length $t$ if $C$ has no offshoots with length less than $t$). That is, a vertex missing from another offshoot could be added by deleting a leaf from a smallest offshoot. As usual, let $W$ be the union of the offshoots from $P$ that we see in $C$. Since $T$ has no sparse card of degree 3, there is more than one offshoot in $W$. Since $r \leq n - 3\ell - 1$, in $W$ there are at least $2\ell + 1$ vertices. The offshoots from $P$ in $T_1$ and $T_2$ and offshoots from $z$ not seen in $W$ total at most $\ell$ vertices, since $C$ is a card.

We claim that $T$ has no long-legged card relative to $C$. Such a card has at least $2\ell + 1$ vertices outside its longest path. Those vertices are too many to come from $T_1 \cup T_2$. If they
lie in offshoots from $P$ at $z$, then they come from only one such offshoot, since the length of the leg in the card exceeds $r - j$. Now using all of $P$ and some vertices from that offshoot yields a sparse card of degree 3, which in this case does not exist. (Are we using this?)

Let $C_1$ be a largest full subtree satisfying the requirements in Lemma 6.5. By Lemma 6.5, $u_j = z$, and we see all of $T_2$ in $C_1$. As in Lemma 6.6, we consider cases depending on whether $t < j - 1$. We know whether $t < j - 1$ holds, since we see $t$ and $j$ in $C$.

**Case 1: $t < j - 1$.** By Lemma 6.5, the branch vertex along the full path in $C_1$ is $z$. Since $t < j - 1$, this path is contained in $T_1 \cup T_2$. Hence the $\ell + 1$ vertices in $C_1$ in offshoots from the full path at $z$ are contained in offshoots from $P$ at $z$. As in Lemma 6.6, let $C_1$ grow to a larger subtree $C'_1$ by no longer requiring $u_2, \ldots, u_{j-1}$ to have degree 2. Again $u_j = z$, and the offshoots at $z$ remain the same. At least $\ell$ vertices in offshoots from $P$ at $z$ in $C$ are omitted by $C'_1$, so $C'_1$ fits in a card. By its maximality, $C'_1$ shows us all of $T_2$ and $T_1$.

We now know all of $T$ except some of the offshoots from $P$ at $z$. We know the total number $s$ of vertices in these offshoots, and we know their maximum length, $t$.

Let $C_2$ be the family of subtrees consisting of a full path $(u_1, \ldots, u_r)$ and one offshoot at $u_j$. Since we know $T_1$ and $T_2$, we know the number of peripheral vertices they have, and there are no other peripheral vertices (since $t < j - 1$). Since we know $T_2$, we also know all the members of $C_2$ in which the offshoot at $u_j$ comes from $T_2$. In the remaining members, $u_j = z$ and the full path comes from $T_1 \cup T_2$. All members of $C_2$ have fewer than $n - \ell$ vertices, since $T$ has no sparse card of degree 3. Hence largest remaining members of $C_2$ show us full offshoots from $P$ at $z$. An exclusion argument gradually omitting smaller members whose offshoots are contained in offshoots already found now allows us to obtain all the offshoots from $P$ at $z$ in $T$, which completes the reconstruction of $T$.

**Case 2: $t = j - 1$.** Again $C_1$ shows us $z$ and $T_2$. The path $(u_1, \ldots, u_{j-1})$ in $C_1$ may come from $T_1$ or from $W$.

First we use an exclusion argument to find all offshoots from $z$ in $T$ with length $j - 1$. Let $C_3$ be the family of full subtrees that have one offshoot of length $j - 1$ from the vertex $u_j$ in position $j$ on the full path. Since $T$ has no sparse card of degree 3, all members of $C_3$ fit in cards, and we see them. Since we know $T_2$, we can exclude the members in which $u_j$ and the offshoot come from $T_2$ (assuming $j < (r + 1)/2$). In the remaining members, $u_j = z$. A largest remaining member shows us a largest offshoot $W'$ from $z$ having length $j - 1$.

Knowing $T_2$ and $W'$, we know their numbers of peripheral vertices. We also know the number of $r$-vertex paths, since $r \leq n - \ell$. Hence we know the number of peripheral vertices in other offshoots from $z$. Therefore, for any $n'$, we know the number of members of $C_3$ in which the offshoot from the full path has $n'$ vertices and length $j - 1$ and comes from $W'$. A largest remaining member of $C_3$ shows us another offshoot of length $j - 1$. Continuing the exclusion argument shows us all offshoots of length $j - 1$ from $z$ in $T$, including $T_1$.

Finally, let $C_4$ be the family of full subtrees having one offshoot from the vertices in po-
position \( j \) on the full path, without regard to the length of the offshoot. Exclude the members in which the offshoot is contained in \( T_2 \) or any offshoot at \( z \) having length \( j - 1 \). A largest remaining member shows us a largest offshoot at \( z \) among those with length less than \( j - 1 \). Since we know all the peripheral vertices in offshoots at \( z \), continuing the exclusion argument shows us the remaining offshoots at \( z \). \( \square \)

**Lemma 6.8.** Suppose \( n \geq 6\ell + 2 \) and \( r \leq n - 3\ell - 1 \). If \( T \) has no sparse card, then \( T \) is \( \ell \)-reconstructible.

**Proof.** Since we have reconstructed \( T \) in all cases where \( T \) has a sparse card, we can recognize being in the case where \( T \) has no sparse card. We consider whether some connected card of \( T \) containing a full path has a leg of length at least \( r/2 \). Since a leg contains no branch vertex, such a leg in a card must lie along a longest path in the card.

**Case 1:** Some card containing an \( r \)-vertex path \( P \) has a leg of length at least \( r/2 \). Among such cards whose leg has maximum length, choose \( C_1 \) to maximize the maximum length of an offshoot from \( P \) in \( C_1 \) at the branch vertex \( z \) reached at the end of the leg. Within this choose \( C_1 \) to maximize the total number of vertices in the offshoots from \( P \) at \( z \). With \( P = \langle v_1, \ldots, v_r \rangle \), let \( z = v_j \), where \( j < (r + 1)/2 \) (the long leg implies \( j \neq (r + 1)/2 \)).

Let \( Q \) be the union of the offshoots in \( T \) from \( P \) at \( z \). Let \( T_1 \) and \( T_2 \) be the components of \( T - z \) containing \( v_{j-1} \) and \( v_{j+1} \), respectively. Since \( C_1 \) is a card, \( T_2 \) has at most \( \ell \) vertices in offshoots from \( P \). Because \( T \) has no sparse card, the portion of \( T_1 \) in \( C_1 \) cannot be just a path, and hence \( C_1 \) shows us \( Q \) in full.

**Case 1a:** \( Q \) has in total at least \( \ell + 1 \) vertices. Let \( C_2 \) be a largest subtree containing an \( r \)-vertex path \( P' \) with vertices \( u_1, \ldots, u_r \) such that \( \langle u_1, \ldots, u_j \rangle \) is a leg of \( C_2 \) and the offshoots from \( P' \) at \( u_j \) in \( C_2 \) form a specified rooted subgraph of \( Q \) with exactly \( \ell + 1 \) vertices. There is such a subtree with \( u_j = z \). Since \( C_1 \) is a card, there are at most \( \ell \) vertices in offshoots from any path of length \( r - j \) in \( T_2 \). Hence the \( \ell + 1 \) vertices from \( Q \) guarantee that \( u_j \) does not lie in \( T_2 \). Also \( u_j \) cannot lie in \( T_1 \cup Q \), since \( C_2 \) has two paths of length at least \( j - 1 \) from \( u_j \). Hence \( u_j \) must equal \( z \).

The path \( \langle u_{j+1}, \ldots, u_r \rangle \) in \( C_2 \) must lie in \( T_2 \). In \( C_2 \), there are exactly \( j + \ell \) vertices from \( T_1 \cup Q \). From \( C_1 \), we know that \( T_2 \) has at most \( r - j + \ell \) vertices. Hence \( C_2 \) has at most \( r + 2\ell \) vertices and fits in a card. We see it as a csc, and it shows us \( T_2 \) in full.

It remains to reconstruct \( T_1 \). Since we know \( T_2 \), we know its number of peripheral vertices. Knowing the number of \( r \)-vertex paths, we know the number of peripheral vertices in \( Q \cup T_1 \). Since we know \( Q \), we know its number of peripheral vertices, if any, so we also know the number of peripheral vertices in \( T_1 \).

Let \( C_3 \) be the family of subtrees of \( T \) having a leg of length at least \( r + 1 - j \). The blade of such a subtree is obtained by deleting \( r + 1 - j \) vertices from the leg. Knowing \( Q \) and
\[ T_2 \text{ and the numbers of peripheral vertices in each of } Q \text{ and } T_1, \text{ we know all members of } C_3 \text{ where the blades come from } Q \text{ or } T_2; \text{ exclude them. In a largest remaining such subtree, the blade is } T_1. \text{ That subtree fits in a card and we can see it, because it omits } Q, \text{ which has at least } \ell + 1 \text{ vertices.}

**Case 1b:** \( Q \) has in total at most \( \ell \) vertices. As remarked earlier, we see \( Q \) in full in \( C_1 \). Altogether there are at least \( 3\ell + 1 \) vertices outside \( P \). Since \( C_1 \) is a card, there are at most \( \ell \) vertices outside \( P \) in \( T_2 \), and \( Q \) has at most \( \ell \) vertices. Hence \( |V(T_1)| \geq j + \ell \).

Let \( C_3 \) be a largest subtree having an \( r \)-vertex path \( P' \) with vertices \( \langle u_1, \ldots, u_r \rangle \) such that \( u_j \) has degree 2 and the offshoots from \( \{u_1, \ldots, u_{j-1}\} \) have a total of \( \ell + 1 \) vertices. Since \( |V(T_1)| \geq j + \ell \), there exists such a subtree with \( u_j = z \). Since \( T_2 \) has at most \( \ell \) vertices outside \( P \), the offshoots from \( \{u_1, \ldots, u_{j-1}\} \) guarantee that \( u_j \) is not in \( T_2 \). Now its position along \( P' \) also prevents \( u_j \) from being in \( T_1 \) or \( Q \) (since otherwise there is too long a path). Hence \( u_j = z \). Now \( C_3 \) has at most \( r + 2\ell + 1 \) vertices, so it fits in a card, and we see a largest such subtree. It shows us \( T_2 \) in full.

Now let \( C_4 \) be the family of full subtrees having a leg of length \( r - j + 1 \). By the choice of \( C_1 \) as a card with longest leg, every such subtree has fewer than \( n - \ell \) vertices. Knowing \( T_2 \) and \( Q \), we can exclude from \( C_4 \) all members where the portion obtained by deleting the leg comes from \( T_2 \) or \( Q \). A largest remaining member of \( C_4 \) shows us \( T_1 \).

**Case 2:** No connected card containing an \( r \)-vertex path has a leg of length at least \( r/2 \). Let \( C' \) be a largest subtree having a leg of length at least \( r/2 \). Since \( T \) has no such card, \( C' \) has at most \( n - \ell - 1 \) vertices, and we see in full the subtree obtained by deleting \( \lceil r/2 \rceil \) vertices from the leg. If \( T \) is bicentral, meaning that \( r \) is even, then by an exclusion argument other cscs of this form allow us to find the other branch and reconstruct \( T \). Hence we may assume that \( T \) is unicentral (\( r \) is odd). Let \( z \) be the unique center of \( T \); it is the center of any \( r \)-vertex path.

**Case 2a:** No connected card containing an \( r \)-vertex path has degree 2 at \( z \). Let \( C_1 \) be a largest subtree containing an \( r \)-vertex path with \( d_{C_1}(z) = 2 \). By hypothesis, \( C_1 \) has at most \( n - \ell - 1 \) vertices, and the two offshoots from \( z \) appearing in \( C_1 \) are two largest offshoots of length \( (r-1)/2 \) from \( z \) in \( T \). By considering cscs that fix the largest offshoot of length \( (r-1)/2 \), we can then find the other offshoots of length \( (r-1)/2 \) from \( z \) by an exclusion argument. Finally, consider a largest subtree containing an \( r \)-vertex path on which the only branch vertex is the center. Since \( T \) has no sparse card, such a subtree has fewer than \( n - \ell \) vertices. We therefore see it in a card, and it shows us all the components of \( T - z \) except two smallest of length \( (r-1)/2 \), which we have already determined.

**Case 2b:** Some connected card containing an \( r \)-vertex path has degree 2 at \( z \). Let \( C \) be the family of such cards. An arm of a tree is an offshoot of maximum length from the set of central vertices. The length of an arm is \( \lceil (r-1)/2 \rceil \).

We first recognize whether \( T \) contains \( S_{\ell-1}^{2r} \). If \( S_{\ell-1}^{2r} \) lies in \( T \) but is too big to fit in a
card, then \( T \) has a sparse card, which is forbidden by hypothesis (or, Corollary 6.1 would apply). Thus if \( T \) contains \( S_{r-1}^2 \), then it fits in a card and we see it.

Suppose first that \( T \) does not contain \( S_{r-1}^2 \). This means that from \( z \) there are only two offshoots (arms) of length \((r - 1)/2\). A largest subtree having a leg of length at least \( r/2 \) fits in a card, by the hypothesis of Case 2, and hence such a subtree shows us a largest arm; let \( T_1 \) be that arm. Knowing \( T_1 \) and the number of \( r \)-vertex paths in \( T \), we find the number of peripheral vertices in the other arm, \( T_2 \). Hence as we consider smaller subtrees having a leg of length at least \( r/2 \), we know which how many of each arise by extending a long leg from \( T_1 \). After excluding those, we find \( T_2 \).

Now consider a largest subtree containing an \( r \)-vertex path on which the only branch vertex is \( z \). Since \( T \) has no sparse card and no third arm, we see this subtree, and it shows us all the other offshoots from \( z \).

Hence we may assume that \( T \) contains \( S_{r-1}^2 \), and we see it. Hence \( T \) has at least three arms (offshoots from \( z \) with length \((r - 1)/2\). Again consider a largest subtree containing an \( r \)-vertex path on which the only branch vertex is \( z \). Since \( T \) has no sparse card, we see this subtree and it shows us all other offshoots from \( z \) except two smallest arms, from which we see only the path with no branch vertex except \( z \).

Because \( C \) is nonempty, two largest arms together have at least \( n - \ell - 1 \) vertices, so they cannot fit in a card together with paths of length \((r - 1)/2\) from two other smallest arms. We conclude that \( T \) has exactly three arms, and we know a largest one.

Now consider a card in \( C \) whose two arms \( B_1 \) and \( B_2 \) differ least in size. If \(|V(B_1)| - |V(B_2)| > 1\), then \( B_2 \) is a second largest arm in \( T \), seen in full in the card. Now by considering cscs with two arms, one of which is \( B_1 \), we can exclude those where the other arm arises from \( B_2 \). A largest remaining such csc shows us the third (and smallest) arm in \( T \), completing the reconstruction.

Hence we may assume \(|V(B_1)| = \lceil (n - \ell - 1)/2 \rceil \) and \(|V(B_2)| = \lceil (n - \ell - 1)/2 \rceil \). Here \( B_1 \) and \( B_2 \) do not need to be arms of \( T \). Outside this card are exactly \( \ell \) vertices, so the third arm in \( T \) has at most \( \ell \) vertices. We consider another card replacing \( \ell \) vertices in this card by the missing vertices, as follows. Let \( B_1' \) be a rooted subtree of \( B_1 \) having \( \lceil (n - \ell - 1)/2 \rceil - \lfloor \ell/2 \rfloor \) vertices, and let \( B_2' \) be a rooted subtree of \( B_2 \) having \( \lceil (n - \ell - 1)/2 \rceil - \lfloor \ell/2 \rfloor \) vertices. Since \( n \geq 4\ell + 4 \), each of \( B_1' \) and \( B_2' \) has at least \( \ell + 1 \) vertices, and \( B_1 \) and \( B_2 \) were big enough to spare \( \lceil \ell/2 \rceil \) vertices. Hence when we examine cscs having a vertex \( z \) with one offshoot being \( B_1' \), another being \( B_2' \), and the remaining offshoots totaling \( \ell \) vertices, we can tell which offshoots are \( B_1' \) and \( B_2' \), and outside these we see the third arm.

We now know two arms, including a largest one. We know the number of peripheral vertices in each and the number of \( r \)-vertex paths. Hence we also know the number of peripheral vertices in the one unknown arm \( A \), and since we know all the other offshoots we know \(|V(A)|\). Also, the largest arm plus a leg extending it \((r - 1)/2\) vertices past \( z \) fits in a card. Hence among cscs with \(|V(A)| + 1 + (r - 1)/2 \) vertices having a leg of length at
least $r/2$, we can exclude all those that arise by growing legs from subtrees of the two known arms. What remains tells us the remaining arm, completing the reconstruction of $T$. \hfill \Box

References


