Cut-edges and factors in regular graphs

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Abstract

We study $2k$-factors in $(2r+1)$-regular graphs. From the characterization of graphs with 2-factors, it is known that every $(2r+1)$-regular graph with at most $2r$ cut-edges has a 2-factor; we characterize the connected graphs with $2r+1$ cut-edges that have no 2-factor. In addition, for $r \geq (3k-1)/2$, we prove that a $(2r+1)$-regular graph with at most $\frac{2r+1}{k} - 1$ cut-edges has a $2k$-factor.

1 Introduction

An *ℓ-factor* in a graph is an ℓ-regular spanning subgraph. In this paper we study the relationship between cut-edges and $2k$-factors in regular graphs of odd degree.

The relationship between edge-connectivity and 1-factors in regular graphs is well understood. Petersen [8] proved that every 3-regular graph with no cut-edge decomposes into a 1-factor and a 2-factor. Furthermore, the conclusion also holds for 3-regular graphs with cut-edges when the cut-edges all lie along a path. Schönberger [10] proved that in a 3-regular graph with no cut-edge, every edge lies in some 1-factor. Bäbler [1] extended Petersen's result to r-regular $(r-1)$-edge-connected graphs of even order. Berge [3] observed that also in this setting every edge lies in some 1-factor. Finally, a result of Plesník [9] implies most of these statements: If G is an r-regular $(r-1)$-edge-connected multigraph with even order, and G' is obtained from G by discarding at most $r-1$ edges, then G' has a 1-factor. The edge-connectivity condition is sharp: Katerinis [6] determined the minimum number of vertices in an r-regular $(r-2)$-edge-connected graph of even order having no 1-factor (see also [7] for a more general result).

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Petersen was in fact more interested in obtaining 2-factors. The result about 3-regular graphs whose cut-edges lie along a path implies that every 3-regular graph with at most two cut-edges has a 2-factor, Furthermore, there are graphs with three cut-edges (found originally by Sylvester) having no 2-factor. As a tool in a result about interval edge-coloring, Hanson, Loten, and Toft [5] generalized Petersen's result to regular graphs with larger odd degree.

Theorem 1.1 ([5]). *For $r \in \mathbb{N}$, every $(2r+1)$-regular graph with at most $2r$ cut-edges has a 2-factor.*

In this paper, we enhance Theorem 1.1 in two ways. We characterize the $(2r+1)$-regular graphs with $2r+1$ cut-edges that do not have 2-factors, and we generalize Theorem 1.1 to a sufficient condition for $2k$-factors. Namely, for $r \geq (3k-1)/2$, every $(2r+1)$-regular with fewer than $\left\lfloor \frac{2r+1}{k} \right\rfloor$ cut-edges has a $2k$-factor. (Sharpness?). Theorem 1.1 is the case $k = 1$.

Earlier, Bäbler [1] proved the weaker result that $(2r+1)$-regular graphs with no cut-edges have $2k$-factors when ***this requires a bound on k in terms of r, since 2-edge-connected $(2r+1)$-regular graphs need not have 1-factors***. Of course, since Petersen [8] proved that every regular graph of even degree has a 2-factor, when $k \leq r$ every $2r$-regular graph has a $2k$-factor. In a result with a similar flavor, Xiao and Liu [13] proved that a $(2kr+s)$-regular graph with at most $k(2r-3)+s$ cut-edges has a k-factor avoiding any given edge. For the relationship between edge-connectivity and k-factors, Bollobás, Saito, and Wormald [4] determined all the triples (r, h, k) such that every r-regular graph with edge-connectivity h (and even order when k is odd) has a k-factor.

For both of our results, we use the necessary and sufficient condition for the existence of ℓ-factors that was initially proved by Belck [2] and is a special case of the f-Factor Theorem of Tutte [11, 12]. When T is a set of vertices in a graph G, let $d_G(T) = \sum_{v \in T} d_G(v)$, where $d_G(v)$ is the degree of v in G. With $|T|$ for the size of a vertex set T, we also write $\|T\|$ for the number of edges induced by T and $\|A, B\|$ for the number of edges having endpoints in both A and B (when $A \cap B = \varnothing$). The characterization is the following.

Theorem 1.2. *A multigraph G has a ℓ-factor if and only if*

$$q(S,T) - d_{G-S}(T) \leq \ell(|S| - |T|) \tag{1}$$

for all disjoint subsets $S, T \subset V$, where the parity of a component Q of $G - S - T$ is the parity of $\|V(Q), T\| + \ell\,|V(Q)|$ and $q(S,T)$ is the number of odd components of $G - S - T$.

Since we consider only the situation where $\ell = 2k$, the criterion for components of $G - S - T$ to be odd simplifies to $\|V(Q), T\|$ being odd.

2 Cut-edges and 2-factors

In this section we generalize Theorem 1.1 to $2k$-factors in a way that for $k = 1$ enables us to characterize the $(2r+1)$-regular graphs with $2r+1$ cut-edges that do not have 2-factors. When there is no $2k$-factor, there is a violation of (1) with $l = 2k$. Such a violation leads to a contradiction if G has at most $2r/k$ cut-edges. When $k = 1$ and G has $2r+1$ cut-edges it forces equalities that describe the structure of the graph.

Definition 2.1. *Blistering* an edge uv in a $(2r+1)$-regular graph means replacing it with a connected graph that is $(2r+1)$-regular except for two vertices of degree $2r$ or one vertex of degree $2r-1$ plus two edges joining the vertex or vertices of deficient degree to u and v.

Theorem 2.2. *When $r \geq (3k-1)/2$, a $(2r+1)$-regular graph with at most $2r/k$ cut-edges has a $2k$-factor. Let G be a $(2r+1)$-regular graph with $2r+1$ cut-edges and no 2-factor. If $r > 1$, then G admits a vertex partition into sets R, S, T such that S and T are independent sets, $|T| = |S| + 1$, the cut-edges join T to distinct components of $G[R]$, and the remaining components of $G[R]$ are grown by blistering edges that join S and T. If $r = 1$, then there may also be components of $G[R]$ joined to T by three edges, and then $|T| - |S|$ is 1 plus the number of such components.*

Proof. Let G be a graph with no $2k$-factor; we study its structure. Given disjoint $S, T \subseteq V(G)$, let $R = V(G) - S - T$. With $\ell = 2k$ in (1), $q(S,T)$ is the number of components Q of $G[R]$ such that $\|V(Q), T\|$ is odd. Thus $q(S,T)$ has the same parity as $\|R, T\|$. In turn, $\|R, T\|$ has the same parity as $d_{G-S}(T)$, since the latter counts edges from R to T once and edges within T twice. Hence the left side of (2) is even, as is the right. Therefore, when G has no $2k$-factor, Theorem 1.2 yields disjoint vertex sets S and T in G such that

$$q(S,T) \geq 2k(|S| - |T|) + d_{G-S}(T) + 2. \tag{2}$$

Fix such sets S and T, and write q for $q(S,T)$.

Writing $d_{G-S}(T)$ as $2\|T\| + \|T, R\|$, we have

$$q \geq 2k(|S| - |T|) + 2\|T\| + \|T, R\| + 2 \tag{3}$$

For clarity, we will initially do the computations for the case $k = 1$. Subsequently we describe the extension to general k.

Summing the degrees in S and in T yields the following equations.

$$(2r+1)|S| = 2\|S\| + \|S, T\| + \|S, R\|$$
$$(2r+1)|T| = 2\|T\| + \|T, S\| + \|T, R\|$$

Substituting these into the result of multiplying (3) by $(2r+1)/2$ yields

$$\frac{2r+1}{2}q \geq 2\|S\| + \|S,R\| + (2r-1)\|T\| + \frac{2r-1}{2}\|T,R\| + (2r+1). \tag{4}$$

Now we break the components of $G[R]$ counted by q into several types.

counting variable:	q_1	q_2	q_3	q_4	q_5	q_6
#edges to T	1	1	1	3	3	≥ 5
#edges to S	0	1	≥ 2	0	≥ 2	≥ 0

We have $q = \sum_{i=1}^{6} q_i$ and $\|T,R\| \geq q_1+q_2+q_3+3q_4+3q_5+5q_6+t$, where t counts the edges joining T to even components of $G[R]$. Also $\|S,R\| \geq q_2+2q_3+2q_5+t'$, where t' counts the edges joining S to even component of $G[R]$.

Substituting for q, $\|T,R\|$, and $\|S,R\|$ in (4) yields

$$\frac{2r+1}{2}q \geq (2r-1)\|T\|+2\|S\|+q_2+2q_3+2q_5+t'+\frac{2r-1}{2}(q_1+q_2+q_3+3q_4+3q_5+5q_6+t)+(2r+1).$$

Now subtract $\frac{2r+1}{2}q$ and add q_1 to obtain

$$q_1 \geq (2r-1)\|T\| + 2\|S\| + 0\cdot q_2 + q_3 + (2r-2)q_4 + 2rq_5 + (4r-3)q_6 + t + t' + (2r+1). \tag{5}$$

The edge to each component of $G[R]$ counted by q_1 is a cut-edge. Since the terms on the right side of (5) include $2r+1$ and are all nonnegative when $r \geq 1$, a violation of the necessary and sufficient condition for 2-factors requires at least $2r+1$ cut-edges.

For the general case of $2k$-factors, the corresponding computations lead to the following generalization of (5).

$$\begin{aligned} kq_1 \geq\ & (2r+1-2k)\|T\| + 2k\|S\| + 0\cdot q_2 + kq_3 + (2r+1-3k)q_4 \\ & + (2r+1-k)q_5 + (4r+2-5k)q_6 + t + t' + (2r+1). \end{aligned}$$

Again the terms on the right are nonnegative and the edges counted by q_1 are cut-edges, so violating the necessary and sufficient condition implies that k times the number of cut-edges must be at least $2r+1$. In particular, if G has at most $2r/k$ cut-edges, then G has a $2k$-factor. The coefficient on q_4 is the reason the argument requires $r \geq (3k-1)/2$.

Returning to $k=1$, when G has exactly $2r+1$ cut-edges we can avoid a contradiction by having equality in (5). Hence T and S must be independent sets, $q_3 = q_5 = q_6 = t = t' = 0$, and $q_4 = 0$ unless $r = 1$. There are no even components of $G[R]$.

When $r > 1$, we have $q_4 = 0$ and hence $\|T,R\| = q_1 + q_2$. Any component Q of $G[R]$ counted by q is joined by one edge to each of S and T. Hence Q is a graph that is $(2r+1)$-regular except for two vertices of degree $2r$ or one vertex of degree $2r-1$. Such a component

Q can be suppressed by deleting its vertices and replacing the edges to S and T with a single edge from S to T. The smaller graph also satisfies these requirements on S and T to have no 2-factor, and adding such a component Q cannot introduce a 2-factor, because a cycle using the edge from Q to S must also use the edge from Q to T. Hence Q arises by blistering an edge joining S and T.

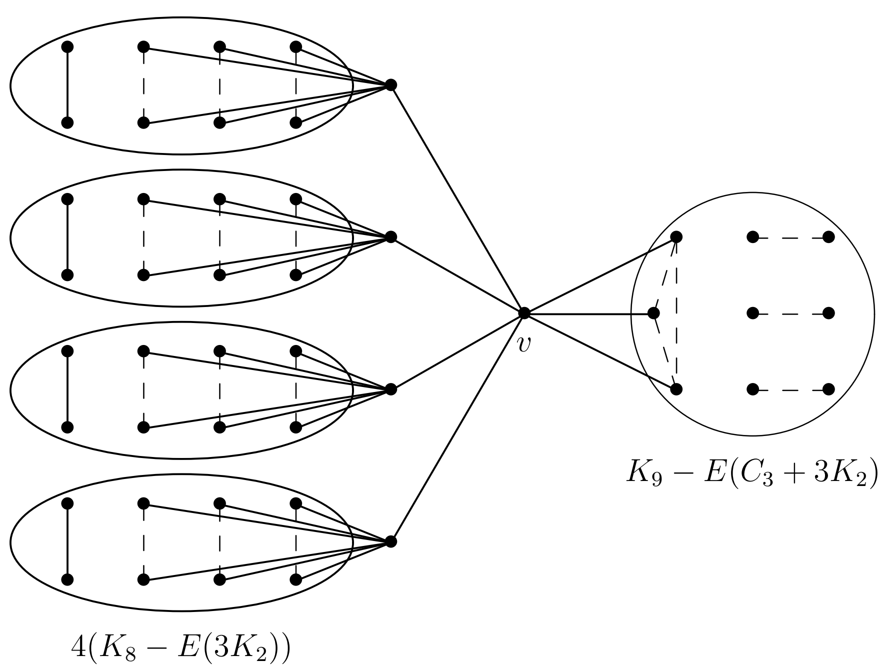
For $r > 1$, we have reduced (3) to $q_1 + q_2 \geq 2\,|S| - 2\,|T| + q_1 + q_2 + 2$. Furthermore, equality must hold, since this inequality was used in producing the later inequalities that must hold with equality. Hence $|T| = |S| + 1$.

When $r = 1$, we may have q_4 nonzero, which permits components of $G[R]$ with three edges to T and none to S. In this case, (3) reduces to $q_1 + q_2 + q_4 = 2\,|S| - 2\,|T| + q_1 + q_2 + 3q_4 + 2$, so now $|T| - |S| = 1 + q_4$. $\square$

3 Cut-edges and $2k$-factors

In Theorem 2.2, the statement for general k is that when $r \geq (3k-1)/2$, every $(2r+1)$-regular graph with at most $2r/k$ cut-edges has a $2k$-factor. Here we explore the question of whether this result is sharp.

For $k = 2$, we need $r \geq 5/2$, so the first case of interest is $r = 3$. In this case $2r/k = 3$, so we seek a 7-regular graph having four cut-edges, and no 4-factor. Such graphs exist.



Example 3.1. To form the graph G in the figure, begin with four disjoint copies K_8. Delete three disjoint edges from each. To each copy, add a new vertex adjacent to the endpoints of the three deleted edges, producing a 9-vertex graph in which the last vertex has degree 6 and the others have degree 7. Introduce a new vertex v adjacent to the vertex of degree 6 in each component; these four edges will remain as cut-edges in G.

The vertex v needs three more neighbors. From a copy of K_9, delete the edges of C_3+3K_2; this reduces the degree of three vertices to 6 and the degree of the others to 7. Make v adjacent to thos three vertices of degree 6 to complete G. Now G is 7-regular, and it has no 4-factor because v appears in no 4-regular subgraph (a 4-regular graph decomposes into cycles and hence cannot contain a cut-edge).

Extending this construction to larger r requires many more cut-edges, because we are prevent the 4-factor by ensuring that v has degree 3 in the graph obtained by deleting the cut-edges. Thus we construct a $(2r+1)$-regular graph have no 4-factor, but it has $2r-2$ cut-edges. We need a way to prevent 4-factors in near-$(2r+1)$-regular graphs with deficient vertices but no cut-edges. This may not be easy.

For $k=3$, we need $r \geq 4$, so we first seek a 9-regular graph with no 6-factor. With at most 8/3 cut-edges, a 6-factor must exist. Using the same construction method as in the example above, we can construct a 9-regular graph with four cut-edges that has no 6-factor (deleting the cut-edge leaves v with degree 5. Here it is unclear whether there is a construction with three cut-edges.

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