On the 3-Reconstructibility of Trees and Rooted Trees

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Abstract

A graph is $\ell$-reconstructible if it is determined by its multiset of subgraphs obtained by deleting $\ell$ vertices. Using centroids and rooted trees, we prove that trees with at least 22 vertices are 3-reconstructible.

1 Introduction

A graph is $\ell$-reconstructible if it is determined by its multiset of (unlabeled) subgraphs obtained by deleting $\ell$ vertices. The famous Reconstruction Conjecture of Kelly [4, 5] and Ulam [14] is that every graph with at least three vertices is 1-reconstructible. Kelly [5] made a stronger conjecture.

Conjecture 1.1 (Kelly [5]). For all $\ell \in \mathbb{N}$ there exists $M_\ell$ such that every graph with at least $M_\ell$ vertices is $\ell$-reconstructible (in particular, $M_1 = 3$).

We will prove that trees with at least 22 vertices are 3-reconstructible. It is conjectured that this holds with at least seven vertices, which would be sharp.

When discussing $\ell$-reconstructibility, the multiset of subgraphs of an $n$-vertex graph $G$ obtained by deleting $\ell$ vertices is called the $(n - \ell)$-deck or simply the deck of $G$, denoted...
We are given only the isomorphism class of each subgraph and do not know which vertices were deleted. The resulting \( \binom{n}{\ell} \) subgraphs are called the cards of the deck. For an \( n \)-vertex graph \( G \), being \( \ell \)-reconstructible means that no other graph has the same \( (n-\ell) \)-deck as \( G \), or \( D_{n-\ell}(G) = D_{n-\ell}(H) \) implies \( G \cong H \).

The elementary observation that motivates the extension from 1-reconstructibility to \( \ell \)-reconstructibility is the following.

**Observation 1.2.** For any graph \( G \), the \( k \)-deck \( D_k(G) \) determines the \( (k-1) \)-deck \( D_{k-1}(G) \).

**Proof.** Each card in \( D_{k-1}(G) \) appears in exactly \(|V(G)| - 1 \) cards in \( D_k(G) \). \( \square \)

Thus every graph that is \( \ell \)-reconstructible is also \((\ell-1)\)-reconstructible, and the goal is to determine for each graph \( G \) the maximum \( \ell \) such that \( G \) is \( \ell \)-reconstructible, which we may call its maximum reconstructibility.

For the family of all graphs, Nýdl [11] proved that \( M_\ell \) grows superlinearly in \( \ell \). For more restricted families, the threshold number of vertices to guarantee \( \ell \)-reconstructibility may be smaller. Spinoza and West [13] proved that all graphs having maximum degree at most 2 and at least \( 2\ell + 1 \) vertices are \( \ell \)-reconstructible. Furthermore, this is sharp, since the path \( P_{2\ell} \) and the disjoint union \( C_{\ell+1} + P_{\ell-1} \) of a cycle and a path have the same \( \ell \)-deck. In fact, ?? determined the maximum reconstructibility for all graphs with maximum degree at most 2.

For trees, nearly the same threshold is conjectured. Nýdl [10] conjectured that trees with at least \( 2\ell + 1 \) vertices are weakly \( \ell \)-reconstructible, meaning that no two such \( n \)-vertex trees have the same \( (n-\ell) \)-deck. Groenland et al. [3] found one counterexample by computer, consisting of two 13-vertex trees with the same 7-deck. In [7], The present authors proved that \( n \)-vertex acyclic graphs vertices form an \( \ell \)-recognizable family when \( n \geq 2\ell + 1 \) (and \((n, \ell) \neq (5, 2)\)), meaning that membership in the family is determined by the \( (n-\ell) \)-deck. The combination of \( \ell \)-recognizability and weak \( \ell \)-reconstructibility is \( \ell \)-reconstructibility, so the modification of Nýdl’s conjecture with the counterexample from [3] is now the following.

**Conjecture 1.3.** For \( n \geq 2\ell + 1 \) (except \((n, \ell) \in \{(5, 2), (13, 6)\}\)), every \( n \)-vertex tree is \( \ell \)-reconstructible, and this threshold is sharp.

For sharpness, Nýdl provided two trees with \( 2k \) vertices having the same \( k \)-deck. A short proof of this was provided by Kostochka and West [8] using the tools developed by Spinoza and West [13] for maximum degree 2. A spider is a tree having one vertex with degree at least 3. Let \( S_{a_1, \ldots, a_d} \) denote the spider consisting of one vertex of degree \( d \) that is the endpoint of paths with lengths \( a_1, \ldots, a_d \); the tree has \( 1 + \sum a_i \) vertices. Nýdl provided the spiders \( S_{k-1,k-1,1} \) and \( S_{k,k-2,1} \) with \( 2k \) vertices having the same \( k \)-deck.

According to Nýdl [12], Bondy and Hemminger [1] reported the existence of a preprint of Giles proving that sufficiently large trees are \( \ell \)-reconstructible, but this was apparently never
published. The result has now been proved in a recent paper by Groenland, Johnston, Scott, and Tan [3] that substantially generalizes and strengthen some of the earlier results. They proved that $n$-vertex trees are reconstructible from their $k$-decks when $k \geq \frac{8}{9}n + \frac{1}{9}\sqrt{8n + 5} + 1$.

For $k = n - 3$, this result implies that $n$-vertex trees are 3-reconstructible (reconstructible from the $(n - 3)$-deck) when $n \geq 194$. We extend this threshold for $\ell = 3$ to $n \geq 22$.

Results on $\ell$-reconstructibility are also known for degree lists, connectedness, random graphs, disconnected graphs, complete multipartite graphs, 3-regular graphs, and $r$-regular graphs that are not 2-connected. Kostochka and West [8] surveyed these results. In addition to the results on trees, the paper by Groenland et al. [3] also extends some of the other earlier results. They proved that the degree list of an $n$-vertex graph is reconstructible from the $k$-deck when $k \geq \sqrt{2n \log(2n)}$ and that connected $n$-vertex graphs are $\ell$-recognizable when $n \geq 10\ell$. We will use our earlier result [6] that degree lists of graphs with at least seven vertices are 3-reconstructible; in particular, we know the number of leaves of a tree from its $(n - 3)$-deck.

Our main tools are the $\ell$-reconstructibility of rooted trees from rooted connected subtrees (for $\ell \leq 3$, with some exceptions), and the idea of “centoid” in a tree. The term “centoid” has been used in other ways in the literature, but this definition seems most common.

**Definition 1.4.** A **centoid** of a tree is a vertex whose deletion minimizes the maximum number of vertices in a single component of the remaining forest. The cost of a vertex $v$ in a tree $T$ is the maximum number of vertices in a component of $T - v$. The cost of $T$, which we write as $c(T)$, is the minimum cost among the vertices of $T$. Thus a centoid is a vertex with minimum cost.

It is well known that a tree has a unique centoid or has two adjacent centoids, yielding unicentroidal or bicentroidal trees, respectively. We need this fact in a stronger form. Myrvold [9] heavily used centoids and a more detailed version of this lemma in proving that trees with at least five vertices are 1-reconstructible from only three cards; that is, every such tree has “reconstruction number” 3.

**Lemma 1.5.** Every $n$-vertex tree has either a unique centoid, with cost less than $n/2$, or two adjacent centoids, with cost exactly $n/2$.

**Proof.** If a vertex $v$ in a tree $T$ has cost greater than $n/2$, then its neighbor in the large component of $T - v$ has smaller cost. Hence a vertex with smallest cost has cost at most $n/2$. For such a vertex $v$ and any neighbor $u$, the forest $T - u$ has a component consisting of $v$ plus all the other components of $T - v$, thereby yielding cost at least $n/2$. Furthermore, if $v$ has cost less than $n/2$, then $u$ has cost more than $n/2$. Moving further away from $v$ increases the cost more.

Hence there are at most two centoids, and the cost is at most $n/2$. Furthermore, we have also shown that if the cost is less than $n/2$, then the centoid is unique. \qed
Definition 1.6. The pieces of a unicentroidal tree $T$ having centroid $z$ are the components of $T - z$; when we know $T$ and $z$, the neighbors of $z$ in the pieces are the roots of the pieces. In a bicentroidal tree, the two subtrees obtained by deleting the edge joining the centroids are the branches of the tree, and the roots of the branches are the centroids. The size of a piece or branch is the number of vertices.

Our overall approach is to identify centroids in certain connected cards, in which case the cards will give us subtrees of rooted trees that enable us to apply reconstruction of rooted trees to obtain pieces of the original tree. For this we need an analogue for rooted trees of the notion of pieces in unrooted trees.

Definition 1.7. A rooted tree is a tree with one vertex distinguished as a root; all other vertices are undistinguished, and there is no specification of order among neighbors of a vertex. The rooted pieces or $r$-pieces of a rooted tree are the rooted subtrees that are the components obtained by deleting the root, with the original neighbors of the root designated as the roots in the $r$-pieces. The size of an $r$-piece is its number of vertices.

A rooted connected card or $rc\ell$-card of a rooted tree $T$ with root $z$ is a rooted tree $T'$ with root $z$ obtained by deleting a leaf of $T$ other than $z$. More generally, the $rc\ell$-cards of a rooted tree with $n$ vertices and root $z$ are the rooted subtrees with $n - \ell$ vertices that have root $z$. The $rc\ell$-deck of a rooted tree is the multiset of its $rc\ell$-cards; the root vertex is known in each card, but otherwise the vertices are unlabeled.

A rooted tree is weakly $\ell$-reconstructible if it is determined by its $rc\ell$-deck; that is, no other rooted tree has the same $rc\ell$-deck.

We use “weakly $\ell$-reconstructible” because we are given that the full structure is a rooted tree, but this is the natural notion of $\ell$-reconstructibility for rooted trees.

2 Reconstructibility of rooted trees

In proving that trees are 3-reconstructible, we will use $\ell$-reconstructibility of rooted trees for $\ell \in \{1, 2, 3\}$. As we will discuss later, there will be some exceptions when $\ell \in \{2, 3\}$.

In the theory of reconstruction, proving reconstructibility for the graphs in a particular family $G$ often is done in two steps. First we show that every graph having the same deck as a graph in $G$ is also in $G$; this is showing that the family is recognizable. We can then restrict our attention to reconstructions in $G$ to show that only one graph (in $G$) has this deck; this is showing that the family is weakly reconstructible. Weakly $\ell$-reconstructible means doing this with the $(n - \ell)$-deck.

In the application of our result on rooted trees, we will know that we have the $rc\ell$-deck of a rooted tree. Thus we consider only rooted trees as reconstructions and ignore the problem
of showing that the deck comes from a rooted tree. In order to be precise, we therefore describe these results as proving weak reconstructibility.

**Theorem 2.1.** Rooted trees are weakly 1-reconstructible.

**Proof.** We assume that we are given the rc1-cards of a rooted $n$-vertex tree $T$. We use induction on $n$. When $n \leq 2$, there is only one $n$-vertex rooted tree. When $n = 3$, there are two $n$-vertex rooted trees, and they are distinguished by the number of rc1-cards. For the induction step, suppose $n > 3$, and let $z$ be the root of $T$.

Since $n \geq 4$ and $z$ is given in each card, $T$ has only one r-piece if and only if every rc1-card has only one r-piece. Hence we know whether $T$ has one r-piece or more than one.

If $T$ has only one r-piece, then let $T'$ be the rooted tree obtained from $T - z$ by designating the neighbor of $z$ in $T$ as the root $z'$. The rc1-cards of $T'$ are obtained from those of $T$ by deleting $z$ and designating its neighbor as the root $z'$. By the induction hypothesis, we can reconstruct $T'$ from these, and we reconstruct $T$ by adding $z$ to $T'$, adjacent to $z'$.

Hence we may assume that $T$ has more than one r-piece. Over all the rc1-cards of $T$, the r-pieces include the r-pieces of $T$ plus some rooted trees that are not r-pieces of $T$, obtained by deleting a leaf of an r-piece of $T$. In particular, all the largest r-pieces that arise are actual r-pieces of $T$, since they cannot arise from a larger r-piece by deleting a vertex.

If all r-pieces in all cards have only one vertex, then $T$ is a star rooted at the center. We can recognize this, so in the remaining case some r-piece of $T$ has more than one vertex.

Among all the largest r-pieces of the rc1-cards, let $M$ be one that occurs most often. Cards with fewer than the maximum number of r-pieces isomorphic to $M$ arise by deleting a leaf of an r-piece isomorphic to $M$. Let $v$ be a leaf of $M$, and let $L = M - v$, with the same root. Among the rc1-cards of $T$ with fewer than the maximum number of r-pieces isomorphic to $M$, find a card $T'$ with the maximum number of r-pieces isomorphic to $L$. Reconstruct $T$ by replacing an r-piece isomorphic to $L$ in $T'$ with an r-piece isomorphic to $M$. \(\square\)

Before we discuss weak 2-reconstructibility of rooted trees, we note some exceptions.

**Example 2.2.** Rooted trees with common rc2-decks. For $n \geq 3$, two $n$-vertex rooted trees have the same rc2-deck with one rc2-card. Let $\hat{P}_n$ denote the path $P_n$ as a rooted tree with an endpoint as the root. The only rc2-card of $\hat{P}_n$ is $\hat{P}_{n-2}$. Let $\hat{P}_n'$ denote the rooted tree consisting of $\hat{P}_{n-2}$ with two children added at its leaf. Again $\hat{P}_{n-2}$ is the only rc2-card.

For $n \geq 5$, two $n$-vertex rooted trees have the same rc2-deck with three rc2-cards. Let $\hat{Q}_n$ be the $n$-vertex rooted tree obtained from $P_{n-4}$ by adding two copies of $\hat{P}_2$ under the leaf. Let $\hat{Q}_n'$ be the $n$-vertex rooted tree obtained from $\hat{P}_{n-4}$ by adding $\hat{P}_1$ and $\hat{P}_3$ under the leaf. Both $\hat{Q}_n$ and $\hat{Q}_n'$ have rc2-deck consisting of two copies of $\hat{P}_{n-2}$ and one copy of $\hat{P}_{n-2}'$. These rooted trees are shown in Figure 1.
A rooted tree is trivial if it has only one vertex. The root-degree of a rooted tree is the degree of the root. We write a rooted tree with r-pieces $T_1, \ldots, T_d$ as $U(T_1, \ldots, T_d)$.

**Theorem 2.3.** Except for $\{\hat{P}_n, \hat{P}'_n, \hat{Q}_n, \hat{Q}'_n\}$, rooted trees are weakly 2-reconstructible. Knowing the number of leaves distinguishes between the exceptions.

**Proof.** We have the rc2-deck of an $n$-vertex rooted tree $T$, with root $z$ specified in each rc2-card. If $n = 2$, then $T = \hat{P}_2$. If $n = 3$, then $T \in \{\hat{P}_3, \hat{P}'_3\}$.

If $T$ has no vertices other than $z$ and its children, then $T$ has $\binom{n-1}{2}$ rc2-cards, all stars, and no other $n$-vertex rooted tree has this many rc2-cards. If $T$ has exactly one vertex other than $z$ and its children, then $T$ has $\binom{n-2}{2} + 1$ rc2-cards; again $T$ is determined. For $n = 4$, this leaves only $\hat{P}_4$ and $\hat{P}'_4$, which have the same rc2-deck but different numbers of leaves. This completes the proof for $n \leq 4$; we proceed inductively with $n \geq 5$.

Let $d^* = d_T(z)$. In the remaining cases, $T$ has at least two vertices that are not children of $z$. Now $d^*$ is the maximum of the root-degrees in the rc2-cards, and $d^* \leq n - 3$. The root-degree is the number of r-pieces of $T$, which we now know.

If $d^* = 1$, then $T$ has only one r-piece; let $z'$ be the child of $z$. Let $T'$ be the r-piece of $T$ (that is, $T' = T - z$ with $z'$ as root). The rc2-cards of $T'$ arise from those of $T$ by deleting $z$ and are rooted at $z'$. By the induction hypothesis, we can reconstruct $T'$ from its rc2-deck unless $T' \in \{\hat{P}_{n-1}, \hat{P}'_{n-1}, \hat{Q}_{n-1}, \hat{Q}'_{n-1}\}$, which holds if and only if $T$ is the corresponding member of $\{\hat{P}_n, \hat{P}'_n, \hat{Q}_n, \hat{Q}'_n\}$ (the cases $T \in \{\hat{Q}_n, \hat{Q}'_n\}$ do not arise when $d^* = 1$ until $n \geq 6$). We reconstruct $T$ by adding $z$ adjacent to $z'$ in $T'$; in the exceptional cases the number of leaves distinguishes between the two members of a pair with the same rc2-deck.

Hence we may assume $2 \leq d^* \leq n - 3$. When $n = 5$, we thus consider only instances with $d^* = 2$; they are $\hat{Q}_5$ and $\hat{Q}'_5$ (from Example 2.2) and $U(\hat{P}_1, \hat{P}_3)$. Since $U(\hat{P}_1, \hat{P}_3)$ has only two rc2-cards while $\hat{Q}_5$ and $\hat{Q}'_5$ have three, $U(\hat{P}_1, \hat{P}_3)$ is reconstructible; $\hat{Q}_5$ and $\hat{Q}'_5$ have the same rc2-deck but have different numbers of leaves. Hence we may assume $n \geq 6$.

Note that $T$ has at least two trivial r-pieces if and only if $T$ has an rc2-card with root-degree $d^* - 2$. In this case, we reconstruct $T$ by adding two trivial r-pieces to such a card.

Hence we may assume that all rc2-cards have root-degree at least $d^* - 1$, and $T$ has at most one trivial r-piece. Define a “slim card” to be an rc2-card with root degree exactly $d^* - 1$. There are at least $d^* - 1$ slim cards when $T$ has a trivial r-piece, with equality only when all other r-pieces are paths with at least three vertices. When $T$ has no trivial r-piece, the number of slim cards is the number of r-pieces isomorphic to $\hat{P}_2$, which is at most $d^*$. Hence we know whether $T$ has a trivial r-piece unless the deck has $d^* - 1$ or $d^*$ slim cards.

Suppose first that $T$ has exactly $d^*$ slim cards and has reconstructions both with and without a trivial r-piece. The reconstruction without is $U(\hat{P}_2, \ldots, \hat{P}_2)$, in which no rc2-card has an r-piece of size at least 3. This also requires $n = 2d^* + 1$, which yields $d^* \geq 3$ since $n > 5$. In a reconstruction having a trivial r-piece, $d^* - 1$ r-pieces are paths with size at
least 3, so some rc2-card does have an r-piece with size at least 3. Hence when \( T \) has \( d^* \) slim cards we can recognize whether \( T \) has a trivial r-piece.

Now suppose that \( T \) has exactly \( d^* - 1 \) slim cards. If \( T \) has a trivial r-piece, then all other r-pieces are paths with size at least 3. Here rc2-cards with two trivial r-pieces arise only by deleting two vertices from an r-piece of size 3, so there are at most \( d^* - 1 \) of them. If \( T \) has no trivial r-piece, then every r-piece except one is \( \hat{P}_2 \), and the other r-piece has size more than 2. In this case there are exactly \( \binom{d^*-1}{2} \) rc2-cards with two trivial r-pieces. Since \( \binom{d^*-1}{2} > d^* - 1 \) when \( d^* \geq 5 \), we may assume \( d^* \leq 4 \). If confusion remains, then the instances with one trivial r-piece or no trivial r-pieces are as below, where \( \hat{T}_m \) denotes any rooted tree with \( m \) vertices (with \( n \geq 9 \) when \( d^* = 3 \) and \( n \geq 5 \) when \( d^* = 2 \)). In each case, the two possibilities are distinguished by the number of rc2-cards.

<table>
<thead>
<tr>
<th>( d^* )</th>
<th>one trivial r-piece</th>
<th>#rc2-cards</th>
<th>no trivial r-piece</th>
<th>#rc2-cards</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>( U(\hat{P}_1, \hat{P}_3, \hat{P}_3, \hat{P}_3) )</td>
<td>9</td>
<td>( U(\hat{P}_2, \hat{P}_2, \hat{P}_2, \hat{T}_4) )</td>
<td>( \geq 10 )</td>
</tr>
<tr>
<td>3</td>
<td>( U(\hat{P}_1, \hat{P}<em>3, \hat{P}</em>{n-5}) )</td>
<td>5</td>
<td>( U(\hat{P}_2, \hat{P}<em>2, \hat{T}</em>{n-5}) )</td>
<td>( \geq 6 )</td>
</tr>
<tr>
<td>2</td>
<td>( U(\hat{P}<em>1, \hat{P}</em>{n-2}) )</td>
<td>2</td>
<td>( U(\hat{P}<em>2, \hat{T}</em>{n-3}) )</td>
<td>( \geq 3 )</td>
</tr>
</tbody>
</table>

Therefore, the rc2-deck determines whether \( T \) has a trivial r-piece. Now consider the case where the rc1-deck of \( T \) has a slim card. If either \( T \) has no trivial r-piece or \( T \) has both a trivial r-piece and a slim card with a trivial r-piece, reconstruct \( T \) from a slim card by adding \( \hat{P}_2 \) as an r-piece.

In the remaining case with a slim card, \( T \) has a trivial r-piece and the slim cards are the rc1-cards for the rooted tree \( T' \) obtained by deleting the trivial r-piece from \( T \). By Theorem 2.1, we can reconstruct \( T' \) from these and obtain \( T \) by adding a trivial r-piece.

Hence we may assume that \( T \) has no slim cards. Thus every r-piece of \( T \) has size at least 3, and every rc2-card has \( d^* \) pieces. Since \( d^* \geq 2 \) and every r-piece has size at least 3, every actual r-piece of \( T \) appears as an r-piece in some rc2-card. As in Theorem 2.1, the largest r-pieces that appear in rc2-cards are actual r-pieces of \( T \), and we see them. Let \( b \) be the maximum size of an r-piece of \( T \).

Suppose first that every r-piece of \( T \) has size \( b \). We recognize this case from rc2-cards whose r-pieces all have size \( b \) except for one with size \( b - 2 \), which must have lost two vertices because the others could not have lost any. Call such rc2-cards “pure” cards. Among all the pure cards, let \( C \) be one in which the multiplicity of some r-piece \( R \) in the list of r-pieces of size \( b \) is as small as possible (possibly 0). Reconstruct \( T \) by replacing the small r-piece in \( C \) with a copy of \( R \).

Hence we may assume that some r-piece of \( T \) has size less than \( b \). Now an rc2-card \( C' \) having a smallest r-piece over all the rc2-cards arises by deleting two vertices from a smallest r-piece of \( T \). In \( C' \) we see all r-pieces of \( T \) of size \( b \), with their multiplicities.

Let \( M \) be a largest r-piece of \( T \), and let \( d' \) be the number of r-pieces of \( T \) isomorphic to \( M \). Let \( T' \) be obtained from \( T \) by deleting the \( d' \) r-pieces of \( T \) isomorphic to \( M \). The
rc2-cards of \( T' \) are obtained from the rc2-cards of \( T \) having \( d' \) r-pieces isomorphic to \( M \) by deleting the copies of \( M \). If \( d' < d^* - 1 \), then \( T' \) has at least two r-pieces, each of size at least 3. Hence \( T' \) is not any of the exceptional rooted trees, and by the induction hypothesis we can reconstruct \( T' \) and replace the copies of \( M \) to obtain \( T \).

In the final case, \( T \) consists of \( d^* - 1 \) pieces isomorphic to \( M \) and one smaller piece \( R' \), and we know this. Let \( L \) be a rooted tree obtained from \( M \) by deleting one leaf; let \( a \) be the number of leaves whose deletion from \( M \) yields \( L \). In every rc2-card of \( T \) having \( d^* - 2 \) r-pieces isomorphic to \( M \) and one piece isomorphic to \( L \), the remaining r-piece is an rc1-card of \( R' \) (smaller than \( L \)). Each rc1-card of \( R' \) arises this way on exactly \( a(d^* - 1) \) cards. Hence we obtain the rc1-deck of \( R' \). By Theorem 2.1, we can reconstruct \( R' \) and thus \( T \). \( \square \)

**Definition 2.4.** A **broom** is a tree obtained from a star by growing a path from the center of the star. A **rooted broom** is a broom with root chosen as the endpoint of the path grown from the center of the star. In particular, a rooted path and a star rooted at its center are both degenerate examples of rooted brooms.

**Example 2.5.** In the statement of the next lemma, when \( \ell = 2 \) the second class includes \( \hat{P}_n \) and \( \hat{P}'_n \), which are brooms, plus one rooted tree \( \hat{P}''_n \) obtained by putting \( \hat{P}_1 \) and \( \hat{P}_2 \) below the leaf of \( \hat{P}_{n-3} \). Note that \( \hat{P}''_n \) has two leaves and two rc2-cards, while \( \hat{P}_n \) and \( \hat{P}'_n \) each have only one rc2-card. All rc2-cards for these three rooted trees are copies of \( \hat{P}_{n-2} \), but the three examples are distinguished by knowing the number of leaves and the number of rc2-cards.

**Lemma 2.6.** For \( \ell \geq 2 \) and \( n \geq \ell + 2 \), the rc\( \ell \)-cards of an \( n \)-vertex rooted tree \( T \) are pairwise isomorphic if and only if \( T \) is a rooted broom or \( T \) is formed by merging the leaf of \( \hat{P}_{n-\ell-1} \) with the root of a rooted tree with \( \ell + 2 \) vertices.

**Proof.** The rc\( \ell \)-cards of a rooted broom are a single rooted broom. In the second case described, every rc\( \ell \)-card is \( \hat{P}_{n-\ell} \).

Let \( T \) be a rooted tree with pairwise isomorphic rc\( \ell \)-cards. We may assume \( T \neq \hat{P}_n \). Let \( v \) be the branch vertex of \( T \) nearest to the root. If \( T \) is not in the class described, then at least \( \ell + 2 \) vertices lie below \( v \) in the tree. Also, the subtree rooted at \( v \) has at least two r-pieces, since \( v \) is a branch vertex.

It suffices to prove that a rooted tree \( \widetilde{T} \) with at least two r-pieces and at least \( \ell + 3 \) vertices has distinct rc\( \ell \)-cards if it is not a rooted star. If there are at most \( \ell \) vertices below the root outside the largest piece, then there is an rc\( \ell \) card with one r-piece and an rc\( \ell \)-card with more than one r-piece. If there are more than \( \ell \) vertices below the root outside a largest r-piece, then deleting \( \ell \) vertices from smallest r-pieces yields an rc\( \ell \)-card whose list of sizes of r-pieces differs from that of the rc\( \ell \)-card obtained by deleting vertices from largest r-pieces. \( \square \)
When we want to reconstruct rooted trees from the rooted subtrees obtained by deleting three vertices, more exceptions arise.

Example 2.7. Rooted trees with common rc3-decks. We describe specific rooted trees by attaching a rooted forest below the leaf of a rooted path. Several appear in Figure 1, including \( \hat{P}_n'' \) as defined in Example 2.5. With \( \hat{Q}_n \) and \( \hat{Q}'_n \) defined as in Example 2.2, let \( \hat{Q}_n'' \) be the rooted tree obtained by putting \( \hat{P}_1 \) and \( \hat{P}_3 \) below the leaf of \( \hat{P}_{n-4} \). Let \( \hat{B}_{n,t} \) be the \( n \)-vertex rooted broom with \( t \)-leaves (thus \( \hat{B}_{n,1} = \hat{P}_n \) and \( \hat{B}_{n,2} = \hat{P}_n' \)).

Every way of putting a rooted forest having \( \ell \) vertices below the leaf of \( \hat{P}_{n-\ell} \) yields an \( n \)-vertex rooted tree whose rc\( \ell \)-deck has a single card, \( \hat{P}_{n-\ell} \). It is more helpful to describe this as putting a rooted tree with \( \ell + 1 \) vertices under the leaf of \( \hat{P}_{n-\ell-1} \). When \( \ell = 2 \), the resulting trees are \( \hat{P}_n \) and \( \hat{P}_n' \). When \( \ell = 3 \), the tree whose root is put under the leaf of \( \hat{P}_{n-4} \) is one of \( \{ \hat{P}_4, \hat{P}_4', \hat{P}_n'', \hat{B}_{4,3} \} \).

More generally, every way of adding \( \ell + 1 \) vertices in \( d \) nonempty rooted trees under the leaf of \( \hat{P}_{n-\ell-1} \) yields an \( n \)-vertex rooted tree whose rc\( \ell \)-deck consists of \( d \) copies of \( \hat{P}_{n-\ell} \). For \((\ell,d) = (3,2)\) and \( n \geq 5 \), we obtain three trees: \( \hat{Q}_n \), \( \hat{Q}'_n \), and \( \hat{Q}_n'' \), where \( \hat{Q}_n'' \) is obtained from \( \hat{P}_{n-4} \) by adding \( \hat{P}_1 \) and \( \hat{P}_3 \) under the leaf.

For \( n \geq 6 \), several pairs of \( n \)-vertex rooted trees obtained from \( \hat{P}_{n-6} \) by adding one of two \( 6 \)-vertex rooted trees under the leaf have the same rc3-deck as listed below. Within each pair, the two rooted trees are distinguished by the number of leaves.

<table>
<thead>
<tr>
<th>under leaf of ( \hat{P}_{n-6} )</th>
<th>#rc3-cards</th>
<th>rc3-deck</th>
<th>#leaves</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U(\hat{P}_1, \hat{P}_4) ) or ( U(\hat{P}_1, \hat{P}_4') )</td>
<td>2</td>
<td>one ( \hat{P}<em>{n-3} ), one ( \hat{P}</em>{n-3}' )</td>
<td>2 or 3</td>
</tr>
<tr>
<td>( U(\hat{P}_2, \hat{P}_3) ) or ( U(\hat{P}_1, \hat{P}_4') )</td>
<td>3</td>
<td>two ( \hat{P}<em>{n-3} ), one ( \hat{P}</em>{n-3}' )</td>
<td>2 or 3</td>
</tr>
<tr>
<td>( U(\hat{P}_2, \hat{P}_4') ) or ( U(\hat{P}<em>1, \hat{B}</em>{4,3}) )</td>
<td>4</td>
<td>three ( \hat{P}<em>{n-3} ), one ( \hat{P}</em>{n-3}' )</td>
<td>3 or 4</td>
</tr>
</tbody>
</table>

Finally, let \( \hat{T}^+ \) denote the rooted tree obtained from a rooted tree \( \hat{T} \) by adding a trivial r-piece as an extra child of the root. If \( \hat{T} \) and \( \hat{T} \) are rooted trees that have the same rc\( \ell \)-deck
and also have the same \( \text{rc}(\ell-1) \)-deck, then \( \hat{T}^+ \) and \( \hat{T}^+ \) also have the same \( \text{rc}\ell \)-deck. Their deck consists of the common \( \text{rc}(\ell-1) \)-deck of \( \hat{T} \) and \( \hat{T} \) together with the cards of their common \( \text{rc}\ell \)-deck extended by adding a trivial \( r \)-piece. When \( \ell = 3 \), this occurs just when \( \{\hat{T}, \bar{T}\} \) is \( \{\hat{P}_{n-1}, \hat{P}'_{n-1}\} \) (for \( n \geq 4 \)) or \( \{\bar{Q}_{n-1}, \bar{Q}'_{n-1}\} \) (for \( n \geq 6 \)), since these are the only pairs of \( (n-1) \)-vertex trees having the same \( \text{rc}2 \)-deck.

When \( \ell = 3 \), all these examples have root-degree at most 2, except for root-degree 3 when \( n = 6 \) in the pair obtained by adding a trivial \( r \)-piece to \( \hat{Q}_5 \) or \( \hat{Q}'_5 \), and when \( n = 4 \) for \( \hat{B}_{4,3} \) in the set \( \{\hat{P}_4, \hat{P}'_4, \hat{P}''_4, \hat{B}_{4,3}\} \). Besides \( \hat{P}'_3 \) in the pair \( \{\hat{P}_3, \hat{P}'_3\} \) the examples with root-degree 2 are of those of the form \( \{\hat{T}^+, \bar{T}^+\} \) in the preceding paragraph where \( \hat{T} \) and \( \bar{T} \) are exceptions with root-degree 1 that also have the same \( \text{rc}2 \)-deck.

We mention two more pairs with seven vertices and root-degree 2: \( \{U(\hat{P}_2, \hat{P}'_2), U(\hat{P}_3, \hat{P}'_3)\} \) and \( \{U(\hat{P}_2, \hat{P}''_2), U(\hat{P}_3, \hat{P}'_3)\} \). The first pair have the same \( \text{rc}3 \)-deck consisting of two copies of \( \hat{P}_4 \) and two copies of \( \hat{P}'_4 \), and they are distinguished by the number of leaves. The second pair have the same \( \text{rc}3 \)-deck consisting of one copy of \( \hat{P}_4 \), one copy of \( \hat{P}'_4 \), and three copies of \( \hat{P}''_4 \), but both of these rooted trees have three leaves.

**Remark 2.8.** Rooted trees having the same \( \text{rc}2 \)-deck also have the same \( \text{rc}3 \)-deck. We showed in Theorem 2.3 that a rooted tree is determined by its \( \text{rc}2 \)-deck, except for \( \{\hat{P}_n, \hat{P}'_n, \hat{Q}_n, \hat{Q}'_n\} \).

If rooted trees with the same \( \text{rc}2 \)-deck are the same, then they also have the same \( \text{rc}3 \)-deck. Also, as noted in Example 2.7, \( \hat{P}_n \) and \( \hat{P}'_n \) have the same \( \text{rc}3 \)-deck, as do \( \hat{Q}_n \) and \( \hat{Q}'_n \).

**Theorem 2.9.** For \( n \geq 4 \), the \( n \)-vertex rooted trees not described in Example 2.7 are weakly 3-reconstructible. For the exceptions, it is sufficient to know also the number of leaves, except for the general pairs \( \{\hat{P}_n, \hat{P}'_n\} \) and \( \{\hat{Q}_n, \hat{Q}'_n\} \) and the 7-vertex pairs \( \{U(\hat{P}_2, \hat{P}'_2), U(\hat{P}_3, \hat{P}'_3)\} \) and \( \{U(\hat{P}_2, \hat{P}''_2), U(\hat{P}_3, \hat{P}'_3)\} \). Within each of these pairs, the two trees are distinguished by the number of copies of \( S_{2,1,1} \) as an unrooted subtree.

**Proof.** The behavior of the exceptions is verified by checking all the examples in Example 2.7. The examples having the same \( \text{rc}3 \)-deck are distinguished by their number of leaves, except for the two pairs listed in the theorem statement. There is one more copy of \( S_{2,1,1} \) in \( \hat{P}''_k \) than in \( \hat{P}'_k \) and there is one more copy of \( S_{2,1,1} \) in \( \hat{Q}''_k \) than in \( \hat{Q}'_k \).

Hence we may assume that we are given the \( \text{rc}3 \)-deck of an \( n \)-vertex rooted tree \( T \) not in the list of exclusions. Let \( z \) be the root of \( T \), specified in each \( \text{rc}3 \)-card. Let \( d^* = d_T(z) \). We use induction on \( n \).

**Step 1:** \( n \leq 6 \) or \( d^* \geq n - 3 \) or \( d^* = 1 \). For \( n \leq 4 \), all trees are exceptional.

Consider \( n = 5 \). The trees with \( d^* = 1 \) are in \( \{\hat{P}_5, \hat{P}'_5, \hat{P}''_5, \hat{B}_{5,3}\} \), all exceptional. Those with \( d^* = 2 \) are in \( \{\hat{Q}_5, \hat{Q}'_5, \hat{Q}''_5\} \), all exceptional. The only tree with \( d^* = 3 \) is \( U(\hat{P}_1, \hat{P}_1, \hat{P}_2) \), whose \( \text{rc}3 \)-deck is three copies of \( \hat{P}_2 \), and the only one with \( d^* = 4 \) is \( U(\hat{P}_1, \hat{P}_1, \hat{P}_1, \hat{P}_1) \), whose \( \text{rc}3 \)-deck is four copies of \( \hat{P}_2 \). These decks differ from those of the rooted trees in Example 2.7. Hence we may proceed inductively with \( n \geq 6 \).
Let \( s \) be the number of vertices of \( T \) other than \( z \) and its children. If \( s \leq 3 \), then \( T \) has a rooted star with \( n - 4 \) leaves as an rc3-card, while if \( s \geq 4 \) there is no such card. Furthermore, if \( s = 3 \), then there is exactly one such card, while if \( s < 3 \) there is more than one such card. Hence we can distinguish the cases \( s < 3 \), \( s = 3 \), and \( s > 3 \).

If \( s = 0 \), then \( T = \hat{B}_{n,n-1} \) and \( T \) has \( (n-1) \) rc3-cards (all stars). If \( s = 1 \), then \( T \) has \( n-3 \) trivial r-pieces and one 2-vertex r-piece, producing \( (n-2) + (n-3) \) rc3-cards. If \( s = 2 \), then the r-pieces of \( T \) are trivial except for two copies of \( \hat{P}_3 \), or one copy of \( \hat{P}_3^t \). In these three cases the numbers of rc3-cards are \( (n-3) + 2(n-4) \), \( (n-3) + (n-4) + 1 \), or \( (n-3)^2 \) + 1, respectively. These numbers and the count when \( s = 1 \) are distinct, except that \( (\frac{n}{3}) + 4 = (\frac{4}{3}) + 1 \) when \( n = 6 \), which occurs for the listed exceptions \( \hat{Q}_5^+ \) and \( \hat{Q}_5^{++} \). Hence the number of rc3-cards distinguishes all the nonexceptional rooted trees with \( s \leq 2 \), which corresponds to \( d^* \geq n - 3 \).

In the remaining cases, \( T \) has at least three vertices other than \( z \) and its children. Now \( d^* \) is the largest root-degree among the rc3-cards, and \( d^* \leq n - 4 \). We now know \( d^* \), which is the number of r-pieces of \( T \).

If \( d^* = 1 \), then \( T \) has only one r-piece; let \( z' \) be the child of \( z \). Let \( T' \) be the r-piece of \( T \), namely \( T - z \) with \( z' \) as root. The rc3-cards of \( T' \) are obtained from those of \( T \) by deleting \( z \) and naming \( z' \) as the root. By the induction hypothesis, we can reconstruct \( T' \) from its rc3-deck unless \( T' \) is one of the exceptional trees on \( n - 1 \) vertices, which holds if and only if \( T \) is one of the exceptional trees on \( n \) vertices. Outside the exceptional cases, we apply the induction hypothesis to \( T' \) and then reconstruct \( T \) by adding \( z \) above the root of \( T' \).

Hence we may assume \( 2 \leq d^* \leq n - 4 \). When \( n = 6 \), this leaves only instances with \( d^* = 2 \). There are six such rooted trees, and they occur in three pairs of two rooted trees having the same rc3-deck, as listed in Example 2.7: \{ \( U(\hat{P}_1, \hat{P}_4), U(\hat{P}_1, \hat{P}_4') \) (same as \( \{\hat{P}_5^+, \hat{P}_4^{++}\} \) ), \( U(\hat{P}_2, \hat{P}_3), U(\hat{P}_1, \hat{P}_4'') \) \}, and \( \{ U(\hat{P}_2, \hat{P}_3), U(\hat{P}_1, \hat{B}_{4,3}) \} \). Within each pair, the trees are distinguished by the number of leaves.

Hence in all other cases we have \( n \geq 7 \) and \( 2 \leq d^* \leq n - 4 \), and we know \( d^* \).

**Step 2:** At least three trivial r-pieces. Note that \( T \) has at least three trivial r-pieces if and only if \( T \) has an rc3-card with root-degree \( d^* - 3 \). In this case, we reconstruct \( T \) by adding three trivial r-pieces to such a card. Otherwise, all rc3-cards have root-degree at least \( d^* - 2 \), and \( T \) has at most two trivial r-pieces.

**Step 3:** Two trivial r-pieces. A card in the rc3-deck of \( T \) is \( j \)-slim if it has root-degree \( d^* - j \). A 2-slim card can arise by deleting two trivial r-pieces and a third vertex or by deleting one trivial r-piece and one r-piece of size 2; there are no 2-slim cards when \( T \) has no trivial r-pieces. There are at least \( d^* - 2 \) 2-slim cards when \( T \) has two trivial r-pieces, with equality only when all other pieces are paths of size at least 3. There are at most \( d^* - 1 \) 2-slim cards when \( T \) has exactly one trivial r-piece, with equality if and only if all other pieces are \( \hat{P}_2 \). Hence the number of trivial r-pieces is known unless the number of 2-slim
cards is \( d^* - 2 \) or \( d^* - 1 \).

Suppose first that \( T \) has exactly \( d^* - 1 \) 2-slim cards. If \( T \) has only one trivial r-piece, then \( T = \cup(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_3) \) and \( n = 2d^* \). Since \( n > 6 \), we have \( d^* \geq 4 \). No rc3-card has an r-piece of size at least 3. On the other hand, in a reconstruction with two trivial r-pieces and \( n = 2d^* \), some r-piece must have size at least 3, and with \( d^* \geq 4 \) we can see such a piece in some rc3-card. Hence in this case we either reconstruct \( T \) or know that every reconstruction has two trivial r-pieces.

Now suppose \( T \) has exactly \( d^* - 2 \) 2-slim cards. If \( T \) has only one trivial r-piece (Case 1), then \( T = \cup(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_3, \hat{T}') \) and \( n = 2d^* \). Since \( n > 6 \), we have \( d^* \geq 4 \). No rc3-card has an r-piece of size at least 3. On the other hand, in a reconstruction with two trivial r-pieces and \( n = 2d^* \), some r-piece must have size at least 3, and with \( d^* \geq 4 \) we can see such a piece in some rc3-card. Hence in this case we either reconstruct \( T \) or know that every reconstruction has two trivial r-pieces.

In Case 1, each 2-slim card has \( d^* - 3 \) pieces that are \( \hat{P}_2 \), plus one piece that is \( \hat{T}' \), and \( \hat{T}' \) must be a path since all pieces in Case 2 are paths. To obtain such a 2-slim card in Case 2, we must delete the two trivial pieces and can reduce at most one piece to \( \hat{P}_2 \). Since there must be \( d^* - 2 \) such 2-slim cards, in Case 2 the nontrivial pieces must all be \( \hat{P}_3 \). Since also the 2-slim cards can only leave one piece with size 3, in Case 2 we have \( T = \cup(\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_3) \), and in Case 1 we have \( T = \cup(\hat{P}_1, \hat{P}_2, \hat{P}_2, \hat{P}_3) \). All rc3-cards with three trivial r-pieces are \( \cup(\hat{P}_1, \hat{P}_1, \hat{P}_1, \hat{P}_2) \); in Case 1 there are three of these, but in Case 2 there are only two. Hence these cases are distinguished.

Therefore, no matter what \( d^* \) is we can recognize whether \( T \) has exactly two trivial r-pieces. If so, then the 2-slim rc3-cards are the rc1-cards of the tree \( T' \) obtained by deleting the trivial r-pieces of \( T \). By Theorem 2.1, we can reconstruct \( T' \), and we add two trivial r-pieces to obtain \( T \).

Step 4: One trivial r-piece. In all remaining cases, \( T \) has at most one trivial piece. If \( T \) has a 2-slim rc3-card, then \( T \) has a trivial piece and a piece of size 2; reconstruct \( T \) from a 2-slim card by replacing these pieces.

When the rc3-deck has no 2-slim card, suppose that there exist both a reconstruction \( T \) with a trivial piece and a reconstruction \( T^o \) with no trivial piece; we seek a contradiction.

Since there is no 2-slim card, r-pieces of \( T \) other than the trivial piece have size at least 3. The number of 1-slim cards with a trivial r-piece is then \( 2p \), where \( p \) is the number of r-pieces of \( T \) having size 3, since we can delete such a piece or delete two of its vertices (in only one way) and the trivial piece. In the rc3-deck of \( T^o \), the number of 1-slim cards with a trivial piece is \( q(q - 1) \), where \( q \) is the number of r-pieces of \( T^o \) having size 2; delete both vertices from one such piece and one from another. Hence \( 2p = q(q - 1) \).

Now consider 1-slim cards with no trivial piece. From \( T \), we must delete the trivial piece, so such a card has at most two r-pieces of size 2. From \( T^o \), we must delete one r-piece of size 2 or 3 and not create a trivial r-piece, so any such card will have at least \( q - 1 \) r-pieces of size 2. Thus \( q - 1 \leq 2 \).

These two computations reduce the possibilities to \((p, q) \in \{(1, 2), (3, 3)\} \). First consider
(p, q) = (3, 3). Each of T and \( T^o \) has six 1-slim cards with a trivial piece. In each such card of T, the remaining pieces all have size at least 3. However, each such card of \( T^o \) has a piece of size 2. Hence T and \( T^o \) cannot both be reconstructions in this case.

Now consider \((p, q) = (1, 2)\). If \( d^* = 2 \), then \( T \in \{ U(\tilde{P}_1, \tilde{P}_3), U(\tilde{P}_1, \tilde{P}_3') \} = \{ \tilde{Q}_5', \tilde{Q}_5' \} \) and \( T^o = U(\tilde{P}_2, \tilde{P}_3) = \tilde{Q}_5 \), listed as exceptions in Example 2.7 for \( n = 5 \). Hence we may assume \( d^* \geq 3 \). The two smallest r-pieces still may be \{ \tilde{P}_1 \tilde{P}_3 \}, \{ \tilde{P}_1 \tilde{P}_3' \}, or \{ \tilde{P}_2 \tilde{P}_2 \}, but now there is at least one more piece. The two 1-slim cards with a trivial r-piece arise by deleting three of the four vertices in the two smallest r-pieces, and they must be the same for any reconstruction, so in each case deleting the two smallest r-pieces yields the same rooted tree; call it Y. Let \( y_j \) be the number of rcj-cards of Y. In the three cases stated above, the number of rc3-cards of the resulting full tree is \( 2y_0 + 2y_1 + 2y_2 + y_3 \), or \( 2y_0 + 3y_1 + 3y_2 + y_3 \), or \( 2y_0 + 3y_1 + 2y_2 + y_3 \), respectively, where the coefficient of \( y_j \) is the number of ways of deleting \( 3 - j \) vertices from the two smallest r-pieces in forming cards. Since Y has at least four vertices, \( y_1 \) and \( y_2 \) are positive, so the numbers of rc3-cards differ in the three cases.

Hence the rc3-deck determines whether T has exactly one trivial r-piece; suppose that it does. We already reconstructed T in the case that T has a 2-slim card, so we may assume that all other r-pieces of T have size at least 3. Let S be the set of 1-slim cards having a trivial r-piece. As discussed earlier, \(|S| = 2p\), where p is the number of r-pieces of T with size 3. Let Y be the rooted tree obtained from T by deleting all the r-pieces of size 3. Any rc3-card in S shows Y and \( p - 1 \) of the 3-vertex pieces. Each such piece appears in \( 2p - 2 \) members of S. Hence when \( p \geq 2 \) we can determine from S the number of pieces of size 3 that are \( \tilde{P}_3 \) and the number that are \( \tilde{P}_3' \), thereby reconstructing all pieces of T. If \( p = 1 \), then since we know Y we know the rcj-deck of Y for each j. As earlier, since \( U(\tilde{P}_1, \tilde{P}_3) \) has two rc1-cards and two rc2-cards, while \( U(\tilde{P}_1, \tilde{P}_3') \) has three rc1-cards and three rc2-cards, we can tell from the size of the rc3-deck of T what the piece of size 3 is.

Hence we may assume that all r-pieces of T other than the one trivial piece have size at least 4. Let Y be the rooted tree obtained by deleting the trivial piece. Now the 1-slim cards of T are precisely the rc2-cards of Y. If Y is determined by its rc2-deck, then T is determined by adding a trivial piece to Y. If there is an alternative reconstruction \( \tilde{Y} \) from the rc2-deck of Y, then Y and \( \tilde{Y} \) also have the same rc3-deck, as observed in Remark 2.8. Now T and \( \tilde{T} \) have the same rc3-deck, where \( \tilde{T} \) arises from \( \tilde{Y} \) by adding a trivial piece. The resulting T and \( \tilde{T} \) are now \( Y^+ \) and \( \tilde{Y}^+ \), occurring as exceptions in Example 2.7. Otherwise Y is determined by its rc2-deck, which determines T.

**Step 5:** \( T \) has no trivial r-piece but has an r-piece of size 2. The only way to obtain a 1-slim rc3-card with a trivial r-piece when T has no trivial r-piece is to have two (or more) r-pieces of size 2 and delete both vertices from one of them and one from another. Hence in the remaining case if there is a 1-slim rc3-card with a trivial r-piece we reconstruct T from it by replacing the trivial r-piece with two copies of \( \tilde{P}_2 \).
We next need to be able to recognize when $T$ has a piece of size 2. Suppose that the rc3-deck also has a reconstruction $T^\circ$ with no r-piece of size at most 2. A 1-slim card of $T^\circ$ arises only by deleting an r-piece of size 3, and hence such a card has no r-piece of size 2. If $T$ has an r-piece of size 3, then $T$ has a 1-slim card with an r-piece of size 2. Hence in $T$ every r-piece other than one of size 2 has size at least 4. Now a 1-slim card of $T$ has at most one r-piece of size 3. This means that $T^\circ$ has at most two r-pieces of size 3 and therefore at most two 1-slim cards. Hence in $T$ there are at most two leaves outside the r-piece of size 2.

If there is only one 1-slim card, then $T = U(\hat{P}_2, \hat{P}_{n-3})$, and there are only three rc3-cards. However, there are four ways to delete at most three vertices from the r-piece of size 3 in $T^\circ$ (whether it is $\hat{P}_3$ or $\hat{P}_3'$), all leading to rc3-cards. Hence we may assume that there are two 1-slim cards, meaning that $T$ has exactly two leaves outside the smallest piece. Now $T$ has at most nine rc3-cards. If $d^* = 3$, then two r-pieces of size 3 and an additional piece of size at least 4 give $T^\circ$ more than nine rc3-cards.

Finally, if $d^* = 2$, then $T^\circ$ (and hence also $T$) has exactly seven vertices. Here $T \in \{U(\hat{P}_2, \hat{P}_4), U(\hat{P}_2, \hat{P}_4'), U(\hat{P}_3, \hat{P}_3')\}$ and $T^\circ \in \{U(\hat{P}_3, \hat{P}_3), U(\hat{P}_3, \hat{P}_3'), U(\hat{P}_3', \hat{P}_3')\}$. The possibilities for $T$ have four or five rc3-cards, respectively, while those for $T^\circ$ have four, five, or six, respectively. This yields the pairs $\{U(\hat{P}_2, \hat{P}_4), U(\hat{P}_3, \hat{P}_3')\}$ and $\{U(\hat{P}_2, \hat{P}_4'), U(\hat{P}_3, \hat{P}_3')\}$ which are exceptions having the same rc3-deck as described in Example 2.7. In each pair the trees are distinguished by the number of unrooted copies of $S_{2,1,1}$.

Hence we can recognize when $T$ has one r-piece of size 2 (and no trivial r-piece). Say that a 1-slim card having a piece of size 2 is defective. If $T$ has no defective card, then all pieces other than the one of size 2 have size at least 4, and the 1-slim rc3-cards of $T$ are the rc1-cards of the rooted tree $T'$ obtained by deleting the smallest r-piece from $T$. By Theorem 2.1, we can reconstruct $T'$ and add an r-piece of size 2 to obtain $T$.

If $T$ has a defective card, then $T$ has a piece of size 3. A defective card contains as r-pieces all the r-pieces of $T$ having size at least 4. It remains to determine the r-pieces of size 3; let there be $p$ that are copies of $\hat{P}_3$ and $q$ that are copies of $\hat{P}_3'$. Defective cards arise by deleting an r-piece of size 3 or by deleting the r-piece of size 2 and one leaf from an r-piece of size 3. Thus each copy of $\hat{P}_3$ is missing from two defective cards, and each copy of $\hat{P}_3'$ is missing from three defective cards. This yields $t = 2p + 3q$, where $t$ is the total number of defective cards. On the other hand $s = p + q + 1$, where $s$ is the number of pieces of size 3 in each defective cards. Since we see $s$ and $t$ from the rc3-deck, we compute $p$ and $q$.

**Step 6:** Every r-piece of $T$ has at least three vertices. Note the exceptions $U(\hat{P}_3, \hat{P}_3)$ and $U(\hat{P}_3, \hat{P}_3')$, which share rc3-decks with $U(\hat{P}_2, \hat{P}_4')$ and $U(\hat{P}_2, \hat{P}_4'')$, respectively, as discussed in Step 5. If the rc3-deck of $T$ is not from one of these exceptions, then in the remaining case we have determined that all r-pieces of $T$ have size at least 3.

Since $d^* \geq 2$ and every r-piece has size at least 3, every r-piece of $T$ appears as an r-piece of some rc3-card of $T$. Let $b$ be the maximum size of an r-piece in an rc3-card of $T$, and let
$c$ be the minimum such size, setting $c = 0$ if $T$ has a 1-slim card. The smallest r-piece(s) of $T$ have size $c + 3$. Every card having an r-piece of size $c$ (or 1-slim card if $c = 0$) has $d^* - 1$ r-pieces that are all the r-pieces of $T$ except one of size $c + 3$. Call these trim cards. (Note that $b \geq c + 3$, with equality possible.)

For every trim card, there is one r-piece of $T$ with size $c + 3$ that does not appear. If some trim card has an r-piece of size $c + 3$, then $T$ has more than one r-piece of size $c + 3$, and over all the trim cards we see all the pieces, in particular all the r-pieces of size $c + 3$. Choose a trim card $C$ in which an r-piece $R$ of size $c + 3$ appears fewer than the maximum number of times, and reconstruct $T$ from $C$ by replacing the smallest r-piece with a copy of $R$ (or adding $R$ as an r-piece if $c = 0$).

In the remaining case, $T$ has exactly one r-piece $R$ of size $c + 3$. Each trim card gives us the other r-pieces, with their multiplicities (hence we also know $b$). The set consisting of the smallest r-piece in each trim card is the rc3-deck of $R$. Now we can reconstruct $R$ by the induction hypothesis and hence reconstruct $T$ unless $R$ is in the set of exceptions.

Let $M$ be an r-piece of $T$ with $b$ vertices, and let $d'$ be the number of r-pieces of $T$ isomorphic to $M$. Obtain $T'$ from $T$ by deleting the r-pieces isomorphic to $M$. The rc3-cards of $T'$ are obtained from the rc3-cards of $T$ with $d'$ r-pieces isomorphic to $M$ by deleting the copies of $M$. If $d' < d^* - 1$, then $T'$ has at least two pieces, one of which is the exception $R$ and the others of which have at least four vertices. No exceptions in Example 2.7 fit this description, so by the induction hypothesis $T'$ is reconstructible from its rc3-deck, which we have obtained. After reconstructing $T'$, we add the copies of $M$ to obtain $T$.

Hence we may assume that $T$ consists of $d^* - 1$ r-pieces isomorphic to $M$ and one smaller piece $R$ with $s$ vertices (here $s = c + 3$). We know $M$, $s$, and $b$. First suppose that $s \leq b - 3$. Let $L$ be an rc2-card of $M$, and let $a$ be the number of copies of $L$ as an rc2-card of $M$. In every rc3-card of $T$ having $d^* - 2$ r-pieces isomorphic to $M$ and one piece isomorphic to $L$, the remaining r-piece is an rc1-card of $R$ (smaller than $L$). Each rc1-card of $R$ arises this way on exactly $a(d^* - 1)$ cards. Hence we obtain the rc1-deck of $R$. By Theorem 2.1, we can reconstruct $R$ and thus reconstruct $T$. If $s = b - 2$ and $d^* \geq 3$, then we can similarly use an rc1-card $L'$ of $M$ and obtain the rc1-deck of $R$ from cards with $d^* - 3$ r-pieces isomorphic to $M$ and two r-pieces isomorphic to $L$.

Next suppose $s = b - 2$ and $d^* = 2$. Let $L$ be an rc1-card of $M$, and let $a$ be the number of copies of $L$ as an rc1-card of $M$. In every rc3-card of $T$ having an r-piece isomorphic to $L$, the other r-piece is an rc2-card of $R$ (smaller than $L$). Each rc2-card of $R$ arises this way on exactly $a$ cards. Hence we obtain the rc2-deck of $R$. By Theorem 2.3, we can reconstruct $R$ and hence $T$ unless we have the common rc2-deck of $\tilde{P}_s$ and $\tilde{P}'_s$ or of $\tilde{Q}_s$ and $\tilde{Q}'_s$. To distinguish these, note that an rc3-card of $T$ having pieces of sizes $b - 3$ and $b - 2$ arises by deleting three vertices from $M$ (in which case the other piece is $R$) or by deleting two vertices from $M$ and one vertex from $R$. Hence the number of these cards is $ij + j'$, where $i$ is the number of leaves of $R$, $j$ is the number of rc2-cards of $M$, and $j'$ is the number of
rc3-cards of $M$. Since we know $M$, we know $j$ and $j'$ and can compute $i$. This distinguishes between $\hat{P}_s$ and $\hat{P}'_s$ and between $\hat{Q}_s$ and $\hat{Q}'_s$ for $R$.

In the final case, $s = b - 1$. Since we know $M$, we know the rc1-deck $L_1, \ldots, L_m$ of $M$, where $m$ is the number of leaves of $M$. Let $S$ be the multiset of rc3-cards of $T$ in which one piece has $b - 3$ vertices, one has $b - 1$ vertices, and the rest are isomorphic to $M$. Such cards arise by deleting three vertices from one copy of $M$ or by deleting one vertex from a copy of $M$ and two vertices from $R$. If for some member of $S$ the piece with $b - 1$ vertices is not in the rc1-deck of $M$, then that piece is $R$ and we complete the reconstruction.

Otherwise, $R$ is an rc1-card of $M$. Let $p$ be the number of rc3-cards of $M$, and let $q$ be the number of rc2-cards of $R$; at present $q$ is unknown. Let $k$ be the multiplicity of a given rc1-card $L$ of $M$. If $L \neq R$, then the number of members of $S$ in which the piece with $b - 1$ vertices is $L$ is $(d^* - 1)kq$. If $L = R$, then that number is $(d^* - 1)(kq + p)$. Since we know $|S|$ from the deck, and the multiplicities of the various rc1-cards of $M$ sum to $m$, we can compute $q = \frac{|S| - (d^* - 1)p}{(d^* - 1)m}$. Since we know the multiplicity of each rc1-card of $M$, we can now determine which one occurs too often as a piece of a member of $S$, and that is $R$. \hfill $\Box$

In applying Theorem 2.9, it will be helpful to know the number of leaves of a tree. In the unrooted setting, we proved in [6] that when $n \geq 7$, the $(n - 3)$-deck of an $n$-vertex graph determines its degree list. We have not yet found a short argument that directly determines the number of leaves of a tree from its $(n - 3)$-deck, though there is a short argument for obtaining the number of leaves from the $(n - 2)$-deck. The number of copies of $S_{2,1,1}$ is also determined by the $(n - 3)$-deck when $n \geq 8$, by Observation 1.2.

3 Trees with Cost at Most $(n - 4)/2$

For trees with small cost, we reduce 3-reconstructibility to weak 3-reconstructibility of rooted trees. We begin by showing that we can recognize the $(n - 3)$-decks of trees.

**Lemma 3.1.** If $T$ is an $n$-vertex tree with $n \geq 7$, then every graph having the same $(n - 3)$-deck as $T$ is a tree.

**Proof.** The $(n - 3)$-deck provides the 2-deck, so every reconstruction has $n - 1$ edges. All cards are acyclic, so reconstructions have no cycles of length at most $n - 3$. Therefore, a non-tree reconstruction $G$ must be $C_{n-1} + P_1$ or $C_{n-2} + P_2$ or the graph $C'$ consisting of $C_{n-2}$ plus a pendant edge and an isolated vertex.

The numbers of copies of $P_{n-3}$ and $K_{1,3}$ in these alternatives are $(n - 1, 0)$ or $(n - 2, 0)$ or $(n, 1)$, respectively; we know these values, since $n - 3 \geq 4$. A path has only four copies of $P_{n-3}$, but $4 < n - 2$. Hence we may assume that $T$ has a branch vertex. A non-tree reconstruction
allows only one copy of $K_{1,3}$, and then $G$ must be $C'$. However, the maximum number of copies of $P_{n-3}$ in a tree with only one copy of $K_{1,3}$ is 5, achieved by $S_{n-3,1,1}$, and $5 < n$. □

It sometimes is helpful to exclude “extreme” trees from consideration by recognizing the decks of such trees. Here we are deleting only two vertices.

**Lemma 3.2.** The connected cards in the $(n - 2)$-deck of an $n$-vertex tree $T$ are pairwise isomorphic if and only if $T$ is a star or a path or has at most five vertices.

**Proof.** The claim holds for stars and paths, and for $n = 5$ because all 3-vertex trees equal $P_3$. For $n \geq 6$, suppose that $T$ is not a star or a path but satisfies the condition.

A leaf-neighbor is a neighbor of a leaf. A tree is leaf-regular if all leaf-neighbors have the same degree. It is leaf-uniform if all leaf-neighbors are adjacent to the same number of leaves. If two leaf-neighbors $x$ and $y$ have different degrees or different numbers of leaves adjacent to them, then deleting one leaf adjacent to $x$ or one leaf adjacent to $y$ yields two nonisomorphic subtrees (in the first case the degree lists are different; in the second the leaves are grouped by their adjacent leaf-neighbors into sets of different sizes). Hence $T - w$ must be both leaf-regular and leaf-uniform for each leaf $w$.

Since $T$ is not a star, $T$ has at least two leaf-neighbors. Let $u$ and $u'$ be distinct leaf-neighbors, and let $v$ and $v'$, respectively, be leaves adjacent to $u$ and $u'$. Let $b$ and $b'$ be the numbers of leaves adjacent to $u$ and $u'$, respectively. If $b$ and $b'$ are at least 2, then because $T - v$ and $T - v'$ are leaf-uniform, and $u$ and $u'$ remain leaf-neighbors in both, we have $b' = b - 1$ and $b = b' - 1$, which is impossible.

Hence we may assume $b' = 1$. Suppose $b \geq 2$. Since the preceding paragraph applies to any two leaf-neighbors, the leaf-uniformity of $T - v'$ now implies that $T$ has no third leaf-neighbor, making $T$ a tree obtained from a star by replacing one edge with a path. Now deleting two vertices from the end of the path yields a different $(n - 2)$-card from the tree obtained by deleting two leaves adjacent to $u$, since $n \geq 6$.

Hence we may assume that every leaf-neighbor is adjacent to exactly one leaf. The leaf-neighbors at both ends of a longest path must now have degree 2. Since deleting the leaf at either end yields a leaf-regular tree having a leaf-neighbor with degree 2, all leaf-neighbors in $T$ have degree 2.

Given that $T$ is not a path, consider the distances from leaves to nearest branch vertices. Deleting the leaf and its neighbor from a shortest such path either reduces the number of leaves or reduces the minimum distance from a leaf to a branch vertex by 2. Deleting two leaves does neither and hence produces a different subtree with $n - 2$ vertices. □

In studying connected cards in the $(n - \ell)$-deck of an $n$-vertex tree, it is helpful to know which vertices of the original tree can be the centroid of the card. When $\ell$ is fixed,
Definition 3.3. A \textit{j-burl} in a tree $T$ is a vertex $v$ such that there are exactly $j$ vertices in $T - v$ outside the two largest components. The \textit{burl} of the tree $T$ is the set of these outside vertices when $v$ is the centroid of $T$. Recall that $c(T)$ denotes the cost of $T$, which is the number of vertices in a largest component of the forest obtained by deleting a centroid of $T$.

Lemma 3.4. Let $z$ be a centroid of an $n$-vertex tree $T$. Let $u$ be a centroid of a connected card $C$ in the $(n - \ell)$-deck of $T$. If $n > 2\ell$, then $z \in V(C)$. If $c(T) \leq (n - \ell + 1)/2$, then $u$ is $z$ or a neighbor of $z$. If $c(T) = (n - \ell + 2)/2$, then $u$ can have distance 2 from $z$ only if their common neighbor has degree 2 in $T$. If $c(T) = n/2$, then $u$ can be outside the neighborhoods of the centroids of $T$ only if the closer centroid is a $j$-burl with $j \leq (\ell - 4)/2$.

Proof. Omitting $z$ from a connected card requires the remaining vertices to lie in one piece of $T$, which requires $(n - \ell) \leq c(T) \leq n/2$.

Every component of $T - z$ has at most $c(T)$ vertices. Therefore, if $u$ is a vertex outside the closed neighborhood of $z$, in $T - u$ there is a component with at least $n - c(T) + 1$ vertices. In order to make $u$ a centroid of $C$, this component must be cut down to at most $(n - \ell)/2$ vertices. Hence $n - c(T) + 1 - \ell \leq (n - \ell)/2$, which simplifies to $c(T) \geq (n - \ell + 2)/2$.

When this inequality is violated, there is no such centroid. When it holds with equality, the large component of $T - u$ must have exactly $n - c(T) + 1$ vertices, so the common neighbor of $u$ and $z$ has degree 2.

Suppose that $T$ is bicentroidal and the centroid $u$ of $C$ has distance 2 from the closer centroid $z$ of $T$, and let $z$ be a $j$-burl in $T$. To become a piece of $C - u$, the component of $T - u$ containing $z$ must be trimmed to at most $(n - \ell)/2$ vertices. Hence $2 + j + n/2 - \ell \leq (n - \ell)/2$, which simplifies to $j \leq (\ell - 4)/2$. \hfill \Box

Corollary 3.5. Let $T$ be an $n$-vertex tree. If $c(T) \neq (n - 1)/2$, then every centroid in every connected $(n - 3)$-card $C$ of $T$ is or has a neighbor that is a centroid of $T$. If $c(T) = (n - 1)/2$ and $C$ has a centroid $u$ that is not a neighbor of the centroid $z$ of $T$, then $C$ is bicentroidal and the common neighbor of $u$ and $z$ has degree 2 in $T$ and is the other centroid of $C$.

Proof. Let $z$ be a centroid of $T$, and set $\ell = 3$ in Lemma 3.4. Now $c(T) \leq (n - 2)/2$ keeps the centroid of $C$ within distance 1 of $z$, while $c(T) = (n - 1)/2$ allows it to move one step farther when the common neighbor has degree 2 (the card is then bicentroidal).

When $T$ is bicentroidal, there is no $j$-burl when $j$ is negative, so the centroid of $C$ must be a neighbor of a centroid of $T$. \hfill \Box
Lemma 3.6. The $(n - \ell)$-deck $D$ of an $n$-vertex tree $T$ satisfies

$$c(D) = \begin{cases} c(T) & \text{if } c(T) \leq (n - \ell)/2, \\ \lfloor (n - \ell)/2 \rfloor & \text{if } c(T) > (n - \ell)/2. \end{cases}$$

Also, if $c(T) \leq (n - \ell)/2$, then the centroid of $T$ is a centroid in every connected card.

Proof. Let $z$ be a centroid of $T$, and let $X$ be a largest piece of $T$. By Lemma 3.4, $z$ appears in every connected card $C$ in $D$.

First suppose $c(T) \leq (n - \ell)/2$. Components of $C - z$ are contained in components of $T - z$ and hence have at most $c(T)$ vertices. Since $c(T) \leq (n - \ell)/2 = |V(C)|/2$, we conclude that $z$ is a centroid of $C$, and $c(C) \leq (n - \ell)/2$. Furthermore, if $C$ arises by deleting $\ell$ vertices outside $X$, then $X$ is still a piece, so $c(C) = c(T)$. Hence $c(D) = c(T)$.

Now suppose $c(T) > (n - \ell)/2$. Every connected card $C$ satisfies $c(C) \leq (n - \ell)/2$, so it suffices to construct a card $C$ with cost $\lfloor (n - \ell)/2 \rfloor$. Delete successive leaves of $X$ until exactly $\lfloor (n - \ell)/2 \rfloor$ vertices of $X$ remain. Complete the card by successively deleting other leaves outside $X$. The number of vertices remaining outside $X$ is $\lfloor (n - \ell)/2 \rfloor$, since the card has $n - \ell$ vertices. Thus $z$ is the unique centroid if $n - \ell$ is odd, while both $z$ and its neighbor $x$ in $X$ are centroids if $n - \ell$ is even. Hence $c(C) = \lfloor (n - \ell)/2 \rfloor$, and $c(D) = \lfloor (n - \ell)/2 \rfloor$. □

Theorem 3.7. For $n \geq 7$, trees with $n$ vertices and cost at most $(n - 5)/2$ are $3$-reconstructible.

Proof. Let $D$ be the $(n - 3)$-deck of such a tree $T$. By Lemma 3.6, we recognize that $T$ is in this family: every connected card has cost at most $(n - 5)/2$. Lemma 3.6 also implies that every connected card has the centroid $z$ of $T$ as its unique centroid. With $z$ distinguished in each connected card, the connected cards form the rc3-deck of $T$ as a rooted tree with root $z$. In addition, $c(T) \leq (n - 5)/2$ requires $d_T(z) \geq 3$. The rooted trees in Example 2.7 that have root-degree 3 and are not reconstructible from their rc3-decks have six vertices. Hence by Theorem 2.9 we can reconstruct $T$ from the deck. □

For general $\ell$, Lemma 3.6 will lead to $\ell$-reconstructibility of $n$-vertex trees having cost less than $\lfloor (n - \ell)/2 \rfloor$ if rooted trees are proved to be weakly $\ell$-reconstructible, since the centroid of $T$ is can be determined in every connected card and used as a root. This also needs a suitable threshold for $n$ and avoiding the exceptions to reconstruction of rooted trees.

Recognition of trees with cost at most $(n - 5)/2$ means that we can also recognize trees with cost at least $(n - 4)/2$. We will proceed in this way, successively recognizing and
reconstructing a subfamily of the remaining family of graphs. As the cost increases, this becomes more difficult. For cost \((n - 4)/2\), the proof of reconstruction is very similar to Theorem 3.7 once we prove that the family is recognizable. We will begin to need not only the number of vertices in the largest piece, but also the number in the next largest piece.

**Definition 3.8.** For a unicentroidal tree \(T\), let \(c'(T)\) denote the size of the second largest piece in \(T\), which may equal \(c(T)\); we call \(c'(T)\) the subcost of \(T\). An \((\frac{n-a}{2}, \frac{n-b}{2})\)-card is a connected card such that for a centroid \(u\) the two largest components of \(T - u\) have \((n-a)/2\) and \((n-b)/2\) vertices, with \(a \leq b\). For a deck \(D\), let \(c'(D)\) denote the maximum subcost among the connected cards with maximum cost.

**Henceforth always** when discussing a unicentroidal tree \(T\), we let \(z\) denote the centroid, with neighbor \(x\) in a largest piece \(X\) and neighbor \(x'\) in a second largest piece \(X'\). Note that the burl of \(T\) is \(T - R\), where \(R = V(X) \cup V(X') \cup \{z\}\).

By Lemma 3.6, we know the cost of \(T\) from the cost of its \((n - \ell)\)-deck when that cost is less than \([\lfloor (n - \ell)/2 \rfloor\). Now set \(\ell = 3\). When \(n\) is even, \(n - 3\) is odd, so Lemma 3.6 implies that \(c(D)\) can be \((n - 4)/2\) when \(c(T) \in \{(n - 4)/2, (n - 2)/2, n/2\}\). Our next task is to determine from the deck whether \(c(T)\) equals \((n - 4)/2\).

**Definition 3.9.** A card in the \((n - \ell)\)-deck of an \(n\)-vertex graph is **balanced** if it consists of two components having \([\lfloor (n - \ell)/2 \rfloor\) and \([\lfloor (n - \ell)/2 \rfloor\) vertices, respectively.

**Lemma 3.10.** If \(D\) is the \((n - 3)\)-deck of an \(n\)-vertex tree \(T\), where \(n\) is even and \(n \geq 20\), then \(c(T) = (n - 4)/2\) if and only if

\[(a)\] \(c(D) = (n - 4)/2\),
\[(b)\] \(D\) has no balanced cards, and
\[(c1)\] some connected card has cost \((n - 10)/2\), or
\[(c2)\] \(c'(D) = (n - 2j)/2\) for some \(j \in \{2, 3, 4\}\), and \(T\) has a \((\frac{n-2j}{2}, \frac{n-10}{2})\)-card. Also, when \(j = 4\) there is a \((\frac{n-6}{2}, \frac{n-8}{2})\)-card, and when \(j = 3\) there is a \((\frac{n-6}{2}, \frac{n-6}{2})\)-card and a \((\frac{n-8}{2}, \frac{n-8}{2})\)-card.

**Proof.** By Lemma 3.6, we may assume \(c(T) \in \{(n - 4)/2, (n - 2)/2, n/2\}\). In each case \(c(D) = (n - 4)/2\), by Lemma 3.6.

**Case 1:** \(c(T) = (n - 4)/2\). Any balanced card has components with \((n - 2)/2\) and \((n - 4)/2\) vertices, since \(n - 3\) is odd. Thus deleting any single vertex from \(T\) leaves a component with at least \((n - 2)/2\) vertices, contradicting \(c(T) = (n - 4)/2\). Hence (b) holds.

Define \(j\) by \(c'(T) = (n - 2j)/2\); note that \(j \geq 2\). Every piece of \(T\) has at most \((n - 4)/2\) vertices. By Lemma 3.6, the centroid of every connected card is \(z\), the centroid of \(T\). In any connected card obtained by deleting three vertices of \(X\), there remain \((n - 10)/2\) vertices of \(X\) and the entire second largest piece \(X'\). If \(j \geq 5\), then such a card has cost \((n - 10)/2\).
Hence we may assume \( j \in \{2, 3, 4\} \). The card described is a \( (\frac{n-2j}{2}, \frac{n-10}{2}) \)-card unless besides \( X \) and \( X' \) there is another piece in \( T \) with at least \((n-8)/2 \) vertices. This requires \( n \geq 1 + \frac{n-4}{2} + \frac{n-2j}{2} + \frac{n-8}{2} \), which simplifies to \( 2j \geq n - 10 \). Since \( 2j \leq 8 \), we obtain the \( (\frac{n-2j}{2}, \frac{n-10}{2}) \)-card unless \( n \leq 18 \). (When \( n = 18 \), for example, the spider \( S_{7,5,5} \) has cost \((n-4)/2 \), no connected card with cost 4 for (c1), and no \( (\frac{n-8}{2}, \frac{n-10}{2}) \)-card for (c2).)

For \( c'(D) = (n-2j)/2 \), we need a \( (\frac{n-4}{2}, \frac{n-2j}{2}) \)-card. We keep the two largest pieces to make such a card by deleting three vertices from the burl. These are available, since there are \( 1 + \frac{n-2j}{2} + \frac{n-4}{2} \) vertices outside the burl, leaving \( j + 1 \) vertices in the burl. Also \( c'(D) \leq c'(T) \) when each connected card has centroid \( z \), so \( c'(D) = c'(T) = (n-2j)/2 \).

Finally, deleting one vertex from \( X \) and two from the burl yields a \( (\frac{n-6}{2}, \frac{n-2j}{2}) \)-card. When \( j = 3 \), deleting two from \( X \) and one from \( X' \) yields a \( (\frac{n-8}{2}, \frac{n-8}{2}) \)-card.

**Case 2:** \( c(T) = (n-2)/2 \). Assume (a), (b), and (c). If \( c'(T) \geq (n-4)/2 \), then \( T \) has a balanced card, contradicting (b). Hence \( c'(T) \leq (n-6)/2 \).

Deleting \( \ell \) vertices reduces the cost by at most \( \ell \). Thus no card of \( T \) has cost at most \((n-10)/2 \), and (c1) fails. Hence (c2) holds, so \( c'(D) = (n-2j)/2 \) for some \( j \in \{2, 3, 4\} \), and with (a) there is a \( (\frac{n-4}{2}, \frac{n-2j}{2}) \)-card \( C \).

The centroid of \( C \) is in \( \{x, z, x'\} \), by Corollary 3.5. It cannot be \( x' \), since the component of \( T - x' \), containing \( z \), has at least \((n+6)/2 \) vertices, which cannot be cut to \((n-4)/2 \) by deleting three vertices. If the centroid is \( z \), then the second largest piece in \( T \) has at least \((n-2j)/2 \) vertices. Since the largest piece in \( T \) has \((n-2)/2 \) vertices, forming a \( (\frac{n-2j}{2}, \frac{n-10}{2}) \)-card with centroid \( z \) requires deleting at least four vertices.

Hence the centroid of \( C \) is \( x \). The component of \( T - x \), containing \( z \), has \((n+2)/2 \) vertices. Since \( C \) is a \( (\frac{n-4}{2}, \frac{n-2j}{2}) \)-card, \( C \) must arise by deleting three vertices from that component of \( T - x \). Thus \( X \) contains a piece \( Y \) of \( C \) with \((n-2j)/2 \) vertices.

Now consider the required \( (\frac{n-6}{2}, \frac{n-2j}{2}) \)-card \( C' \) if \( j \in \{3, 4\} \). The centroid of \( C' \) cannot be \( x \), since the two biggest components of \( T - x \) would together have to lose at least four vertices. Hence the centroid of \( C' \) is \( z \). Now, as earlier when we studied \( C \), there must be pieces as large as \((n-6)/2 \) and \((n-2j)/2 \) in \( T \), so again \( c'(T) \geq (n-2j)/2 \). Again we must delete at least four vertices from the two largest pieces of \( T \) to obtain a \( (\frac{n-2j}{2}, \frac{n-10}{2}) \)-card.

Hence \( j = 2 \), and \( C \) is a \( (\frac{n-4}{2}, \frac{n-4}{2}) \)-card. We have shown that the centroid of \( C \) is \( x \), with a piece \( Y \) contained in \( X \). Since \( Y \) uses all of \( X \) except the one vertex \( x \), we have \( d_T(x) = 2 \). Now deleting \( x \) and two vertices from \( X' \) yields a balanced card, contradicting (b).

**Case 3:** \( c(T) = n/2 \). Here \( T \) is bicentroidal, with centroids \( z \) and \( z' \). Let \( X \) be a second largest component of \( T - z \), with \( x \) its neighbor of \( z \); similarly define \( X' \) and \( x' \) from \( T - z' \). By (b), \( T \) has no balanced cards, which requires \( d_T(z), d_T(z') \geq 3 \) and \(|V(X)|, |V(X')| \leq (n-6)/2 \), so \( c'(T) \leq (n-6)/2 \). Since \( c(T) = n/2 \), every connected card has cost at least \((n-6)/2 \), so again (c1) cannot hold. Also, there is no \( (\frac{n-8}{2}, \frac{n-10}{2}) \)-card and no \( (\frac{n-8}{2}, \frac{n-8}{2}) \)-card, which are required when \( j \) in (c2) is 4 or 3, respectively. Hence \( j = 2 \)
and \( c'(\mathcal{D}) = (n - 4)/2 \).

Thus we have a \( \left(\frac{n-4}{2}, \frac{n-4}{2}\right) \)-card \( C \). Since \( c'(T) \leq (n - 6)/2 \), moving away from \( z \) or \( z' \) to a vertex \( w \) allows only the component of \( T - w \) containing \( z \) and \( z' \) to have more than \((n - 6)/2\) vertices. Hence \( C \) cannot exist. \( \square \)

**Theorem 3.11.** For even \( n \) with \( n \geq 20 \), trees with \( n \) vertices and cost \((n - 4)/2\) are 3-reconstructible.

*Proof.* Let \( \mathcal{D} \) be the \((n - 3)\)-deck of such a tree \( T \). By Lemmas 3.6 and 3.10, we recognize from \( \mathcal{D} \) that \( T \) is in this family. Lemma 3.6 also implies that every connected card has the centroid \( z \) of \( T \) as its unique centroid \((n - 3 \text{ is odd})\). Hence again the connected cards form the rc3-deck of \( T \) with root \( z \), and again \( d_T(z) \geq 3 \). Hence by the same proof as in Theorem 3.7, we can reconstruct \( T \) from the deck. \( \square \)

For trees with higher cost, there may be connected cards whose centroids are not the centroid of \( T \). Nevertheless, in many cases we will find a subset of the connected cards whose centroid can be identified as a particular vertex of \( T \). In that case we may be able to apply the same reconstruction argument.

**Lemma 3.12.** Let \( T \) be an \( n \)-vertex tree, and let \( \mathcal{D} \) be the \((n - 3)\)-deck of \( T \). If a vertex \( v \) in \( T \) and a subset \( \mathcal{D}' \) of \( \mathcal{D} \) can be identified such that \( \mathcal{D}' \) is the multiset of connected cards arising by deleting three vertices from one component \( H \) of \( T - v \), and in each card of \( \mathcal{D}' \) we know \( v \) and which neighbor of \( v \) is in \( H \), then \( T \) is 3-reconstructible.

*Proof.* For each card in \( \mathcal{D}' \), deleting the remaining vertices of \( H \) yields \( T - V(H) \). Over all cards in \( \mathcal{D}' \), the vertices that belong to \( H \) provide the rc3-deck of \( H \) rooted at its vertex neighboring \( v \).

Since we know the number of leaves of \( T \) and the number of leaves of \( T \) outside \( H \), we know the number of leaves of \( T \) in \( H \). Hence Theorem 2.9 allows us to reconstruct \( H \) unless \( H \in \{ \hat{P}_k', \hat{P}_{2k}', \hat{Q}_k', \hat{Q}_{2k}' \} \), where \( k = |V(H)| \). By Observation 1.2, we know the number of copies of \( S_{2,1,1} \) in \( T \), and from the cards in \( \mathcal{D}' \) we know the number of copies of \( S_{2,1,1} \) using at least one vertex outside \( H \). We therefore know the number of copies of \( S_{2,1,1} \) contained in \( H \). As noted in Theorem 2.9, we thus can reconstruct \( H \) and \( T \). \( \square \)

### 4 Trees with Cost \((n - 3)/2\)

For odd \( n \), Lemma 3.6 allowed us to recognize when the cost is at most \((n - 5)/2\), and in Theorem 3.7 we reconstructed such trees. The next two lemmas enable us to distinguish whether reconstructions from the deck have cost \((n - 3)/2\) or \((n - 1)/2\).
Lemma 4.1. If $D$ is the $(n-3)$-deck of an $n$-vertex tree $T$, where $n$ is odd and $n \geq 9$, then $c(T) = c'(T) = (n-3)/2$ if and only if $D$ has exactly one balanced card.

Proof. Necessity. If $c(T) = c'(T) = (n-3)/2$ and $T$ has centroid $z$, then $T-z$ has two components with $(n-3)/2$ vertices and two leftover vertices. Deleting those two vertices and $z$ yields a balanced card. When $n \geq 9$, this is the only balanced card deleting $z$. A balanced card keeping $z$ must delete a vertex $v$ in one of the large components of $T-z$, but then $T-v$ cannot have two components with $(n-3)/2$ vertices.

Sufficiency. A balanced card $C$ must lack a vertex from the path in $T$ connecting the two components of $C$. Thus $T$ has a vertex $z$ such that $T-z$ has two components with at least $(n-3)/2$ vertices, and only two since $n \geq 9$. If either such component has more vertices, then $T$ has more than one balanced card, since any leaf of such a component can be deleted. Hence $z$ has a neighbor in each large component of $T-z$, and the other two vertices are adjacent to $z$ or form $P_2$ adjacent to $z$. Now $c(T) = c'(T) = (n-3)/2$. \qed

Lemma 4.2. If $D$ is the $(n-3)$-deck of an $n$-vertex tree $T$ with cost at least $(n-3)/2$, where $n$ is odd and $n \geq 13$, then $c(T) = (n-3)/2$ and $c'(T) \leq (n-5)/2$ if and only if $D$ has no balanced cards and at least one of the following happens:

(a) $D$ has a card with cost at most $(n-9)/2$, or
(b) $D$ has a $(\frac{n-7}{2}, \frac{n-7}{2})$-card and a $(\frac{n-7}{2}, \frac{n-9}{2})$-card, or
(c) $D$ has a $(\frac{n-7}{2}, \frac{n-7}{2})$-card and among its bicentroidal cards has exactly one with a centroid of degree 2 or has one with both centroids of degree 2.

Proof. As usual, let $z$ be the centroid of $T$, let $X$ be a largest piece of $T$ (rooted at $x$), and let $X'$ be a next largest piece of $T$.

Case 1: $c(T) = (n-3)/2$. By Lemma 4.1, $c'(T) = (n-3)/2$ guarantees a balanced card, so we need only show that the conditions are necessary when $c'(T) \leq (n-5)/2$. Since the components of $T-z$ have at most $(n-3)/2$ vertices and the cards have $n-3$ vertices, $z$ is a centroid in every connected card.

Since $n-3$ is even, a balanced card has two components with $(n-3)/2$ vertices. When the path connecting them has more than one internal vertex, deleting any single vertex of $T$ leaves a component with at least $(n-1)/2$ vertices, contradicting $c(T) = (n-3)/2$. When the path has a single internal vertex, it is the centroid, contradicting $c'(T) \leq (n-5)/2$. Hence $T$ has no balanced cards.

A connected card $C$ obtained by deleting three vertices from $X$ has centroid $z$ and a piece with $(n-9)/2$ vertices. The card $C$ has another piece with $c'(T)$ vertices. If $c'(T) \leq (n-9)/2$, then $c(C) = (n-9)/2$, and (a) holds. If $c'(T) = (n-7)/2$, then $C$ is a $(\frac{n-7}{2}, \frac{n-9}{2})$-card, and deleting two vertices from $X$ and one from the burl yields a $(\frac{n-7}{2}, \frac{n-7}{2})$-card, confirming (b).

If neither (a) nor (b) holds, then $c'(T) = (n-5)/2$, and $z$ is a 3-burl. Deleting two vertices from $X$ and one from $X'$ yields a $(\frac{n-7}{2}, \frac{n-7}{2})$-card. Since $X'$ has only $(n-5)/2$
vertices, all bicentroidal cards have centroids $x$ and $z$ and arise by deleting three vertices outside $X$. When $n \geq 13$, we have $|V(X')| \geq 4$. Hence when $d_T(x) > 2$, the only way to create a bicentroidal card with a centroid of degree 2 is to delete the three vertices of the burl. If $d_T(x) = 2$, then there may be many bicentroidal cards with one centroid having degree 2, but we can also make one with both centroids having degree 2 by deleting the three vertices of the burl.

**Case 2:** $c(T) = (n-1)/2$. We show that (a), (b), and (c) all fail when $T$ has no balanced cards. If $c'(T) = (n-1)/2$, then a balanced card arises by deleting $z$ and leaves of $X$ and $X'$. If $c'(T) = (n-3)/2$, then a balanced card arises by deleting $z$, the leaf neighbor of $z$, and a leaf of $X$. Hence $c'(T) \leq (n-5)/2$. If $d_T(x) = 2$, then deleting $x$ and two vertices of $X'$ yields a balanced card, so we must have $d_T(x) \geq 3$.

Deleting $\ell$ vertices reduces the cost by at most $\ell$. Hence no card of $T$ has cost at most $(n-9)/2$, and (a) fails.

Deleting three vertices outside $X$ yields cards with $x$ a centroid. Since $d_T(x) > 2$, in such cards $x$ is the unique centroid and the cost is $(n-5)/2$. Deleting one vertex of $X$ and two outside $X$ yields a bicentroidal card with centroids $x$ and $z$. All other cards have centroid $z$.

Both (b) and (c) require a $(n-7, n-7)$-card, which must have centroid $z$. Thus $T - z$ has two components with at least $(n-7)/2$ vertices, so $c'(T) \geq (n-7)/2$. With $c'(T) \geq (n-7)/2$, we must delete at least four vertices to obtain a $(n-7, n-7)$-card. Hence (b) fails.

Finally, suppose that (c) holds. When $c'(T) = (n-5)/2$, a $(n-7, n-7)$-card with centroid $z$ requires deleting four vertices from $X \cup X'$, which is not allowed. Hence we may assume $c'(T) = (n-7)/2$. With $T$ being a $(n-1, n-7)$-tree, $z$ is a 3-burl in $T$. Using $d_T(x) \geq 3$, we have observed that all bicentroidal cards have centroids $x$ and $z$. Since $z$ is a 3-burl in $T$ and $d_T(x) \geq 3$, no such card has 2-vertices as both centroids.

Since bicentroidal cards arise only by deleting one vertex of $X$ and two outside $X$, no bicentroidal card has $z$ as a centroid of degree 2. Hence there must be exactly one bicentroidal tree with $x$ being a centroid of degree 2. Such cards arise only when $d_T(x) = 3$ and the deletions are the leaf neighbor of $x$ and two vertices outside $X$. However, there is more than one way to delete two vertices outside $X$, so (c) cannot hold. □

**Theorem 4.3.** For $n \geq 13$ with $n$ odd, $n$-vertex trees with cost $(n-3)/2$ are reconstructible.

**Proof.** Lemma 4.1 allows us to recognize $(n-3, n-3)$-trees. Lemma 4.2 allows us to recognize when $c(T) = (n-3)/2$ and $c'(T) \leq (n-5)/2$. Consider at tree $T$ in these cases.

**Case 1:** $c'(T) \leq (n-5)/2$. All bicentroidal cards have centroids $x$ and $z$ and are obtained by deleting three vertices outside $X$. Hence $X$ appears as one branch in all such cards, and the other branch provides the rc3-deck of $T - V(X)$, rooted at $z$; let $Y' = T - V(X)$. If there is only one branch that appears in all bicentroidal cards, then it is $X$, and Lemma 3.12 applies to reconstruct $Y'$ and $T$.

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If all the bicentroidal cards are the same, then the rc3-cards of $Y'$ are all the same. Hence those cards must be brooms or paths, by Lemma 2.6. Since $d_Y(z) \geq 2$, we conclude that $Y$ is a star with centroid $z$, and actually $c'(T) = 1$. Now we can reconstruct $T$ from any bicentroidal card by adding three leaves to the branch that is a star.

**Case 2:** $c'(T) = (n - 3)/2$. Here $T$ is a $\left(\frac{n-3}{2}, \frac{n-3}{2}\right)$-tree and $z$ is a 2-burl. Since $(n - 7)/2 \geq 3$ when $n \geq 13$, any $\left(\frac{n-5}{2}, \frac{n-7}{2}\right)$-card shows whether the two vertices of the burl are adjacent to each other or only to $z$. A connectee card of $T$ is bicentroidal if and only if it arises by deleting three vertices outside $X$ (having centroids $z$ and $x$) or three vertices outside $X'$ (having centroids $z$ and $x'$).

Let $h$ be the number of vertices in $\{x, x'\}$ with degree 2 in $T$. We have $h = 2$ if and only if every bicentroidal card has a centroid of degree 2. In the bicentroidal cards having a centroid of degree 3, that centroid is then $z$. The branch not containing $z$ is $X$ or $X'$ with its root located. Over all such cards, we obtain the rooted $X$ and $X'$ (identical or not).

We claim $h = 0$ if and only if no bicentroidal card has degree 2 at both centroids. Reducing $z$ to degree 2 requires deleting both vertices of the burl, but the third deleted vertex cannot be a neighbor of the other resulting centroid. In bicentroidal cards with one centroid of degree 2, that centroid now is $z$. These cards form the rc1-deck of $T$ (minus the burl) as a tree rooted at $z$. By Theorem 2.1, we can reconstruct this subtree and hence $T$.

In the remaining case, some but not all bicentroidal cards have two centroids of degree 2, which implies $h = 1$. By symmetry, let $d_T(x) = 2$. In the bicentroidal cards with two centroids of degree 2, one branch is $X$, and after deleting $z$ the other is an rc1-card of $X'$ with root $x'$. If only one branch is common to all these cards, then it is $X$, and by Theorem 2.1 we can reconstruct $X'$. Otherwise, the bicentroidal cards are all the same. When both branches appear in all these cards and are distinct unrooted trees, then we can see which is $X$ in the unique balanced card. If they are the same unrooted tree, with different roots, then the one that arises from $X'$ must be the rooted broom, by Lemma 2.6. □

5 Trees with Cost $(n - 2)/2$ and Subcost at most $(n - 4)/2$

In light of Lemma 3.10 and Theorem 3.11, when $n$ is even we henceforth restrict to $n \geq 20$ so that in reconstruction arguments we may assume that every reconstruction from $\mathcal{D}$ has cost at least $(n - 2)/2$. By Lemma 3.10, we can recognize from the $(n - 3)$-deck of $T$ that $c(T) \in \{(n - 2)/2, n/2\}$. To distinguish these two cases, we consider the subcost.

**Definition 5.1.** When $T$ is an $n$-vertex tree with $c(T) = (n - a)/2$ and $c'(T) = (n - b)/2$, we say that $T$ is an $\left(\frac{n-a}{2}, \frac{n-b}{2}\right)$-tree. A $\left(\frac{n-a}{2}, \frac{n-b}{2}\right)$-vertex in a tree $T$ is a vertex $v$ such that $T - v$ has largest component with $(n - a)/2$ vertices and next largest with $(n - b)/2$ (possibly $a = b$). In particular, an $\left(\frac{n-a}{2}, \frac{n-b}{2}\right)$-vertex in a $\left(\frac{n-a}{2}, \frac{n-b}{2}\right)$-tree is a centroid.
Lemma 5.2. For $n$ even, an $n$-vertex tree $T$ is a $(n-2, n-8)$-tree with $b \geq 8$ if and only if $c(T) \geq (n-2)/2$ and the $(n-3)$-deck of $T$ has a card with cost $(n-8)/2$.

Proof. If $c(T) = (n-2)/2$ and $c'(T) \leq (n-8)/2$, then deleting three vertices from the largest piece of $T$ yields the desired card. If $c(T) = n/2$, or if $c(T) = (n-2)/2$ and $c'(T) \geq (n-6)/2$, then $c(C) \geq (n-6)/2$ for each connected card $C$. \hfill \Box

Theorem 5.3. For $n \geq 20$ with $n$ even, $n$-vertex trees with cost $(n-2)/2$ and subcost at most $(n-8)/2$ are 3-reconstructible.

Proof. Lemmas 3.6 and 3.10 enable us to recognize the case $c(T) \geq (n-2)/2$, and from Lemma 5.2 we recognize that $c(T) = (n-2)/2$ and $c'(T) \leq (n-8)/2$.

For $1 \leq j \leq 3$, a connected card $C$ obtained by deleting $j$ vertices of $X$ and $3-j$ vertices outside $X$ has centroid $z$ and cost $(n-2-2j)/2$, since the components of $C-z$ contained in $X$ and $X'$ have $(n-2-2j)/2$ and at most $(n-8)/2$ vertices, respectively. Connected cards deleting three vertices outside $X$ have centroid $x$ and cost $(n-4)/2$, because the piece rooted at $z$ has $(n-4)/2$ vertices. Thus the set $D'$ of connected cards with cost $(n-6)/2$ consists precisely of those obtained by deleting two vertices from $X$ and one outside $X$.

If $T$ has a $(n-8, n-8)$-card $C$, then $X'$ has $(n-8)/2$ vertices and $C$ arises by deleting three vertices from $X$. In this case the full burl at $z$ has four vertices, is present in $C$, and is distinguishable from what remains of $X$ and $X'$ since $n \geq 20$. Cards obtained by deleting three vertices from the burl are $(n-2, n-8)$-cards and show $X$ and $X'$ completely. Hence we obtain $X$ and $X'$ and the burl at $z$ to reconstruct $T$.

If $T$ has no $(n-8, n-8)$-card, then $c'(T) \leq (n-10)/2$. Now cards with cost $(n-8)/2$ are obtained by deleting three vertices of $X$, and what remains of $X$ is the unique largest piece. By Lemma 3.12, $T$ is 3-reconstructible. \hfill \Box

By Lemmas 3.6, 3.10, and 5.2, the $(n-3)$-deck of an $n$-vertex tree $T$ allows us to recognize when $c(T) \geq (n-2)/2$ and $c'(T) \geq (n-6)/2$, for even $n$.

Lemma 5.4. For $n \geq 14$ with $n$ even, an $n$-vertex tree $T$ satisfying $c(T) \geq (n-2)/2$ and $c'(T) \geq (n-6)/2$ is a $(n-2, n-8)$-tree if and only if

(a) the $(n-3)$-deck $D$ has a $(n-6, n-8)$-card and a $(n-6, n-8)$-card, and

(b) $T$ has a balanced card or the number of disconnected cards with components having $n/2$ and $(n-6)/2$ vertices is not 1.

Proof. To allow the second-largest piece to have $(n-8)/2$ vertices, the condition of having a $(n-8, n-8)$-card in the $(n-3)$-deck may require $(n-8)/2 \geq 3$, which simplifies to $n \geq 14$.

Case 1. $T$ is a $(n-2, n-8)$-tree. Deleting two vertices of $X$ and one from $X'$ or from the burl yields the two cards required by (a). If $d_T(x) = 2$, then deleting $x$ and two vertices of $X'$ yields a balanced card. Hence if $T$ has no balanced card, then $d_T(x) \geq 3$. 

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There are three vertices in the burl, so a card having components with \( n/2 \) and \( (n - 6)/2 \) vertices cannot be obtained by deleting \( z \). Also the component of \( T - x' \) containing \( z \) has \( (n + 6)/2 \) vertices, so \( T - x' \) cannot contain such a card. Hence with \( z \) remaining and \( d_T(x) \geq 3 \), obtaining such a card requires \( d_T(x) = 3 \) and a leaf neighbor of \( x \), and we must delete \( x \), its leaf neighbor, and a leaf of the component of \( T - x \) containing \( z \). Hence when \( d_T(x) = 3 \) and \( x \) has a leaf neighbor there is more than one such card, while if \( d_T(x) \neq 3 \) or \( x \) has no leaf neighbor then there is no such card.

**Case 2.** \( c(T) = (n - 2)/2 \) and \( c'(T) \geq (n - 4)/2 \). First suppose \( c'(T) = (n - 4)/2 \). The centroid \( z \) is a 2-burl in \( T \). Let \( C \) be a \( (\frac{n-6}{2}, \frac{n-8}{2}) \)-card of \( T \). The centroid \( u \) of \( C \) is a 3-burl in \( T \); hence \( u \) must be a \( k \)-burl in \( T \) with \( k \geq 3 \). Thus \( u \neq z \). Now the component of \( T - u \) containing \( z \) has at least \( (n + 2)/2 \) vertices and cannot reach \( (n - 6)/2 \) by deleting three.

When \( c'(T) = (n - 2)/2 \), the analogous argument uses the required \( (\frac{n-6}{2}, \frac{n-6}{2}) \)-card \( C \). Here \( z \) is a 1-burl, \( u \) is a 2-burl in \( C \), and again the component of \( T - u \) containing \( z \) has at least \( (n + 2)/2 \) vertices.

**Case 3.** \( c(T) = n/2 \). Let \( z \) and \( z' \) be the centroids of \( T \). Let \( Y \) and \( Y' \) be the components of \( T - zz' \), with \( z \in V(Y) \) and \( z' \in V(Y') \). If \( T \) has a \( (\frac{n-6}{2}, \frac{n-6}{2}) \)-card \( C \), then the centroid of \( C \) must be \( z \) or \( z' \) (a centroid \( w \) anywhere else would leave too big a component of \( T - w \) to cut down to \( (n - 6)/2 \) vertices). By symmetry, we may let \( z \) be the centroid. Cutting the big piece of \( T - z \) down to \( (n - 6)/2 \) vertices requires \( C \) to be obtained by deleting three vertices of \( Y' \). Now the second-largest component of \( T - z \) must be contained in \( Y \) and have \( (n - 6)/2 \) vertices, so \( z \) is a 2-burl.

In this setting, the centroid of a \( (\frac{n-6}{2}, \frac{n-8}{2}) \)-card cannot be in \( Y \); the piece containing \( z' \) would be too big. Hence the centroid of such a card \( C' \) must be \( z' \), and deleting three vertices from \( Y \) yields a piece with \( (n - 6)/2 \) vertices. No more vertices can be deleted, so \( Y' - z' \) must have a component with \( (n - 8)/2 \) vertices, and \( z' \) is a 3-burl.

By the argument above, every subtree with at least \( (n - 4)/2 \) vertices contains \( z \) or \( z' \), so \( T \) has no balanced cards. In order to obtain a card with components having \( n/2 \) and \( (n - 6)/2 \) vertices, it is necessary to delete one centroid and delete no vertices from the resulting component with \( n/2 \) vertices. Since also a component with \( (n - 6)/2 \) vertices is needed, the only way to do this is to delete \( z \) and the two vertices of the burl at \( z \). Hence there is only one such card, and (b) fails.

\[ \square \]

**Theorem 5.5.** For \( n \geq 20 \) with \( n \) even, \( (\frac{n-2}{2}, \frac{n-6}{2}) \)-trees are 3-reconstructible.

*Proof.* By Lemma 5.4 and earlier cases, we recognize that \( T \) is a \( (\frac{n-2}{2}, \frac{n-6}{2}) \)-tree.

Label \( z, X, x, X', x' \) as usual. Since there are \( (n + 2)/2 \) vertices outside \( X \), cards obtained by deleting three vertices outside \( X \) have centroid \( x \) and cost \( (n - 4)/2 \). Connected cards obtained by deleting one vertex of \( X \) and two outside \( X \) also have cost \( (n - 4)/2 \), with centroid \( z \), but they have subcost at most \( (n - 6)/2 \). Connected cards obtained by deleting
at least two vertices of $X$ have cost at most $(n - 6)/2$. Thus $T$ has $(n - 4, n - 4)$-cards if and only if $d_T(x) = 2$, and $x$ is the centroid of such cards.

Let $\mathcal{D}'$ be the set of $(n - 4, n - 4)$-cards. If $\mathcal{D}' \neq \emptyset$, then $d_T(x) = 2$. In cards in $\mathcal{D}'$, one piece is always $X - x$. The other piece yields the rc3-deck of $T - V(X)$, rooted at $z$. If only one piece appears in all cards of $\mathcal{D}'$, then it is $X$. By Lemma 3.12, we can reconstruct $T$.

If each $(n - 4, n - 4)$-card has the same two pieces, then by Lemma 2.6 $T - V(X)$ is a rooted broom or is formed by merging the leaf of a rooted path with the root of a rooted tree with five vertices. When $(n - 6)/2 \geq 2$, this is impossible with $z$ being a 3-burl in $T$.

Hence we may assume $d_T(x) \geq 3$ and $\mathcal{D}' = \emptyset$. Now consider cards with cost $(n - 4)/2$. They arise with centroid $x$ by deleting three vertices outside $X$ or with centroid $z$ by deleting one vertex of $X$ and two outside $X$; in both cases the largest piece is unique. The root of the largest piece has degree 2 in the card if in the first case we deleted the three vertices of the burl or in the second case $x$ is a 1-burl and we deleted it leaf neighbor.

Therefore, we recognize that $x$ is not a 1-burl by there being exactly one card $C$ with cost $(n - 4)/2$ such that the neighbor of the centroid in the largest piece has degree 2. When this happens, the unique largest piece is rooted at $z$, and deleting $z$ from it gives us $X'$. Deleting the largest piece from $C$ gives us $X$, rooted at $x$. To find the configuration of the burl we examine any $(n - 6, n - 8)$-card. Such cards arise only by deleting three vertices of $X \cup X'$; we distinguish the burl within it because $n \geq 16$.

In the remaining case, when there is more than one such card $C$, we know that $x$ is a 1-burl in $T$. Consider the $(n - 4, n - 4)$-cards in which the centroid is a 3-burl whose neighbor in the largest piece has degree 2. Such cards arise only by deleting the leaf neighbor of $X$ and two vertices of $X'$. Since $n \geq 18$, we see the burl. Reconstruct $X$ from the largest piece by giving the root a leaf neighbor, so we know the number of leaves of $X'$. The second largest pieces in these cards form the rc2-deck of $X'$; reconstruct $X'$ using Theorem 2.3.

**Lemma 5.6.** For $n \geq 16$ with $n$ even, an $n$-vertex tree $T$ satisfying $c(T) \geq (n - 2)/2$ and $c'(T) \geq (n - 4)/2$ is a $(n - 2, n - 4)$-tree if and only if

(a) the $(n - 3)$-deck $D$ has a $(n - 6, n - 6)$-card and a $(n - 4, n - 4)$-card, and

(b1) $T$ has exactly one balanced card, or (b2) and at least one of \{(c1), (c2)\} hold:

(b2) $D$ has at least $2+q$ balanced cards, where all large components of balanced cards have at least $q$ leaves, and their small components (except maybe one) are isomorphic.

(c1) No connected card has a vertex $y$ whose deletion leaves largest components with $n/2$ and $(n - 12)/2$ vertices and has a 2-neighbor in the component with $n/2$ vertices, or

(c2) $T$ has a $(n - 4, n - 8)$-card in which the piece of order $(n - 8)/2$ has a 2-burl or 3-burl at distance at most 3 from the centroid of the card.

**Proof.** The exceptional trees listed are all bicentroidal; they include $S_{2,(n-6)/2,n/2}$ in the degenerate case where $j = 2$.

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Case 1: \( T \) is a \((\frac{n-2}{2}, \frac{n-4}{2})\)-tree. With \( |V(X)| = (n-2)/2 \) and \( |V(X')| = (n-4)/2 \), the centroid \( z \) is a 2-burl. Deleting two vertices from \( X \) and one from \( X' \) yields a \((\frac{n-6}{2}, \frac{n-6}{2})\)-card; deleting the burl and one leaf from \( X \) yields a \((\frac{n-4}{2}, \frac{n-4}{2})\)-card. Hence (a) holds.

Deleting \( z \) and the burl yields a balanced card, and this is the only balanced card omitting \( z \). A non-leaf vertex whose deletion produces a component with at least \((n+4)/2\) vertices cannot be deleted in forming a balanced card. Hence a balanced card containing \( n/2 \) components of \( (n-4)/2 \) values and requires \( n - x \) vertices to be connected, meaning \( d_T(x) = 2 \). Hence if \( d_T(x) \geq 3 \), then \( T \) has exactly one balanced card, in which case (b1) holds.

Henceforth in this case we may assume that (b1) fails and \( d_T(x) = 2 \). The components of order \((n-4)/2\) in balanced cards are \( X' \) when \( z \) and the burl are deleted, or \( X - x \) when the deleted vertices are \( x' \) and \( y\). Since \( T - x \) has components with \((n+2)/2\) and at most \((n-4)/2\) vertices, forming a balanced card also requires deleting two vertices from the large component of \( T - x \) and requires \( X - x \) to be connected, meaning \( d_T(x) = 2 \). Hence if \( d_T(x) \geq 3 \), then \( T \) has exactly one balanced card, in which case (b1) holds.

The number of balanced cards is 1 plus the number of subtrees of \( T - V(X) \) obtained by deleting two vertices. Let \( q' \) be the number of leaves of \( T - V(X) \) (note that \( T - V(X) \) is the large component in one balanced card, so \( q' \) is at least the value \( q \) defined in the statement of (b2)). If \( q \geq 4 \), then we have more than \( q \) ways to delete two leaves from \( T - V(X) \). If \( q = 2 \), then \( T - V(X) \) is a path and we have three ways to delete two vertices. If \( q = 3 \), then \( T - V(X) = S_{1,1,(n-4)/2} \), and we have three ways to delete two leaves plus one way to delete two vertices from the long leg, since \((n-4)/2 \geq 2 \). Hence when (b1) fails, the number of balanced cards is at least \( 2 + q \), and (b2) holds.

If \( T \) has no connected card having a vertex \( y \) whose deletion leaves largest components with \( n/2 \) and \( (n-12)/2 \) vertices, then (c1) holds. Hence we may assume that \( y \) is such a vertex in such a card \( C \). Note that \( y \) is a 2-burl in \( C \). Also, \( y \) cannot be \( z \), since \( T - z \) has no component with at least \( n/2 \) vertices.

The largest component of \( T - y \) contains \( z \), the two vertices in the burl of \( T \), the large components of \( T - z \) not containing \( y \), and the \( s \) internal vertices of the \( z \)-path in \( T \). This amounts to \((n+4)/2 + s \) vertices if \( y \in X' \), and \((n+2)/2 + s \) vertices if \( y \in X \). To reduce to \( n/2 \) in forming \( C \), exactly \( s + 2 \) of these vertices must be deleted if \( y \in X' \), or \( s + 1 \) if \( y \in X \). Therefore \( s \leq 2 \), and \( y \) has distance at most \( 3 \) from \( z \).

The \( s + 1 \) or \( s + 2 \) vertices that were deleted from \( T \) to form the largest component of \( C - y \) were not in the burl of \( y \), which in \( C \) has only two vertices. Hence the burl of \( y \) can have more than three vertices only if \( s = 0 \) and \( y \in X \). That is, \( y \) is the neighbor of \( z \) in \( X \). In this case we can form the desired \((\frac{n-4}{2}, \frac{n-8}{2})\)-card by deleting three vertices of \( X \), including one or two from the burl at \( y \). In all other cases the burl in \( T \) at \( y \) has two or three vertices, and we obtain the desired \((\frac{n-4}{2}, \frac{n-8}{2})\)-card by deleting the appropriate number of vertices from \( X \) and \( X' \) without disturbing the burl at \( y \).

Case 2: \( T \) is a \((\frac{n-2}{2}, \frac{n-2}{2})\)-tree. Given (a), let \( C \) be a \((\frac{n-6}{2}, \frac{n-6}{2})\)-card with centroid \( u \).
Since the centroid $z$ of $T$ is a 1-burl while $u$ is a 2-burl in $C$, vertex $u$ cannot be $z$. For any $v \in V(T)$ with $v \neq z$, the largest component of $T - v$ has at least $(n + 2)/2$ vertices. Deleting three vertices cannot reduce that component below $(n - 4)/2$ vertices. Hence $T$ cannot have such a card $C$, and (a) fails.

**Case 3:** $T$ is bicentroidal. Let $z$ and $z'$ be the centroids of $T$, with $Y$ the branch containing $z$ and $Y'$ the branch containing $z'$.

If $v \in V(T) \setminus \{z, z'\}$, then $T - v$ has a component with at least $(n + 2)/2$ vertices, which cannot be reduced to $(n - 6)/2$ by deleting three. Hence the centroid of a $(\frac{n-6}{2}, \frac{n-6}{2})$-card $C$ must be $z$ or $z'$; by symmetry, we may assume it is $z$. Since the component $Y'$ of $T - z$ containing $z'$ has $n/2$ vertices, $C$ must arise by deleting three vertices of $Y'$. Hence $T - z$ has exactly $(n - 6)/2$ vertices in its second-largest component $X$, so $z$ is a 2-burl in $T$. Now a $(\frac{n-4}{2}, \frac{n-4}{2})$-card $C'$ must have centroid $z'$ and arise by deleting one vertex from $Y'$ and two from $Y$. To form the piece of $C'$ with $(n - 4)/2$ vertices that is contained in $Y'$, the second-largest component $X'$ of $T - z'$ must have $(n - 4)/2$ or $(n - 2)/2$ vertices, in which case $z'$ is a 1-burl or a 2-vertex in $T$, respectively.

Because $z$ is a 2-burl in $T$ and $|V(X)| = (n - 6)/2$, there is no balanced card omitting $z$. One way to create a balanced card is to delete $z'$, a leaf of $Y$, and either the leaf neighbor of $z'$ (if $X'$ has $(n - 4)/2$ vertices) or a leaf of $X'$ (if $X'$ has $(n - 2)/2$ vertices). Since $Y$ is a nontrivial tree, there are at least two such cards, and hence (b1) cannot hold.

If $|V(X')| = (n - 4)/2$, then deleting $z'$ and its leaf neighbor and a leaf of $Y$ is the only way to form a balanced card. The number of balanced cards is then the number of leaves of $Y$. The component of order $(n - 2)/2$ in any balanced card is obtained from $Y$ by deleting one leaf. If the minimum number of leaves in such a component is $q$, then $Y$ has at most $q + 1$ leaves. Hence $T$ has at most $q + 1$ balanced cards, so (b2) cannot hold in this case.

In the remaining case, $|V(X')| = (n - 2)/2$ and $d_T(z') = 2$. Still $z$ is a 2-burl. There are now two ways to form balanced cards. Type 1 is to delete $z'$ and two vertices from $Y'$; the component of order $(n - 4)/2$ is then contained in $Y$. Type 2 is to delete $z'$ and one leaf each from the trees $Y$ and $X'$; the component of order $(n - 4)/2$ is then contained in $X'$.

The components of order $(n - 4)/2$ in the balanced cards are the subtrees of $Y$ obtained by deleting two vertices and the subtrees of $X'$ obtained by deleting one leaf. These must be isomorphic, except possibly for one. If all subtrees obtained by deleting two vertices from $Y$ are isomorphic, then $Y$ is a path or a star, by Lemma 3.2. The star is forbidden, since $z$ is a 2-burl in $T$ and $Y$ has $n/2$ vertices, with $n \geq 10$. However, $Y$ may be a path. Now every subtree obtained from $X'$ by deleting a leaf must also be a path, except possibly for one.

If $X'$ is a path (with $(n - 2)/2$ vertices), then attaching $z'$ to it makes $T$ constructed from $P_{n/2} + P_{n/2-1}$ by adding $z'$ with one neighbor in each component, where the neighbor $z$ in the component with $n/2$ vertices has distance 2 from a leaf; we return to this case later.

If $X'$ is not a path, then deleting any leaf except one yields a path, so $X' = S_{1,1,n/2-4}$. Now there are two balanced cards in which the component having $(n - 4)/2$ vertices is not
a path, obtained by deleting the end of the long leg of $X'$ and one of the two leaves of $Y$.

The other case is that all subtrees of $X'$ obtained by deleting a leaf are isomorphic; let $Z$ be that subtree. Now all leaves of $T$ in $X'$ have the same distance from $z'$, and all vertices at a given distance from $z'$ have the same degree. Furthermore, since $Y$ has at least three subtrees obtained by deleting two vertices, at least two such subtrees of $Y$ are also isomorphic to $Z$. In at least one of those copies of $Z$ in $Y$, we delete one vertex from the burl of $T$, leaving $z$ with degree 2 and a leaf neighbor in $Z$. In another copy, we delete two vertices of $X$, leaving $z$ with degree 3 or a leaf at distance 2. (If there is only one subtree deleting two vertices of $X$, then $Y$ is a broom with three leaves or a path; the broom fails.) The resulting trees cannot be isomorphic and have the properties of $Z$ unless $Z$ is a path. Now $Y$ and $Y'$ are both paths, with $z$ having distance 2 from one end. This yields $T = S_{2,(n-6)/2,n/2}$, a special case of the construction above where $z'$ is made adjacent to an endpoint of $P_{n/2-1}$.

In these two cases we are left with the spider $S_{2,(n-6)/2,n/2}$ or a tree with two (nonadjacent) vertices of degree 3 and the rest of degree at most 2, in which the legs at one branch vertex (which we have called $z$) have length 2 and $(n-6)/2$. Now consider conditions (c1) and (c2). Deleting $z$ from the connected card obtained by deleting three vertices from the leg of length $(n-6)/2$ at $z$ leaves components with $(n-12)/2$ and $n/2$ vertices, and the neighbor $z'$ of $z$ in the component with $n/2$ vertices has degree 2. Hence (c1) fails.

Hence $T$ must satisfy (c2). The centroid of a $(\frac{n-4}{2},\frac{n-8}{2})$-card must be a 2-burl in the card. Hence the only choices for the centroid of such a card are $z$ and the neighbor $x'$ of $z'$ in $Y'$, which has distance 2 from $z$. To have $x'$ as the centroid, note that the component of $T - x'$ containing $z$ has $(n+2)/2$ vertices. We must delete three vertices there to get down to a piece with $(n-4)/2$ vertices. However, the piece with $(n-8)/2$ vertices is then a path without the required 2-burl or 3-burl.

To have $z$ as the centroid, we must delete one vertex from the leg of length $(n-6)/2$ at $z$, since we cannot find a piece with $(n-4)/2$ vertices there. However, again the piece with $(n-8)/2$ vertices is then a path and cannot contain the required 2-burl or 3-burl.

**Theorem 5.7.** For $n \geq 20$ with $n$ even, $(\frac{n-2}{2},\frac{n-4}{2})$-trees are reconstructible.

**Proof.** Among the remaining trees, Lemma 5.6 allows us to recognize that $T$ is a $(\frac{n-2}{2},\frac{n-4}{2})$-tree from its $(n-3)$-deck $D$ (since $n \geq 20$). Let $z, X, x, X', x'$ be as usual.

In a $(\frac{n-2}{2},\frac{n-4}{2})$-tree, the centroid $z$ is a 2-burl. A $(\frac{n-6}{2},\frac{n-6}{2})$-card has centroid $z$, is obtained by deleting two vertices from $X$ and one from $X'$, and determines whether the vertices of the burl are adjacent. Hence it suffices to reconstruct $X$ and $X'$ with their roots $x$ and $x'$ located.

As discussed in Lemma 5.6, here $d_T(x) \geq 3$ if and only if $T$ has exactly one balanced card, obtained by deleting $z$ and the burl. Consider this case. For $v \in V(T) - \{x, z\}$, always $T - v$ has a component with at least $(n+4)/2$ vertices. Hence the centroid of a connected
card can only be \( z \) or \( x \). Furthermore, since \( T - x \) has a component with \((n + 2)/2\) vertices, the centroid is \( x \) only when three vertices outside \( X \) are deleted, which means that when the centroid is \( x \) the centroid has degree at least 3 in the card.

Cards in which the centroid has degree 2 therefore have centroid \( z \); they are obtained by deleting the burl and one leaf of \( X \). All such cards are \( (\frac{n-4}{2}, \frac{n-6}{2}) \)-cards in which one piece is \( X' \) and the other piece is an rc1-card of \( X \). If the two pieces in these cards are not always the same, then we know \( X' \) and reconstruct \( X \) by Theorem 2.1. If the two pieces are always the same, then \( X \) has the property that all vertices at a given distance from the root \( x \) have the same number of children, so we can locate \( x \) within \( X \) as the larger component of the balanced card. We then also have \( X' \).

Hence we may assume \( d_T(x) = 2 \). In every \( (\frac{n-4}{2}, \frac{n-6}{2}) \)-card, the centroid is a 1-burl, so the centroid must be \( z \) and not \( x \). Such cards are obtained by deleting one vertex from the burl and either one leaf each from \( X \) and \( X' \) or two vertices from \( X \). In the \( (\frac{n-4}{2}, \frac{n-6}{2}) \)-cards where the centroid has a neighbor with degree at least 3, that neighbor is \( x' \). When \( x' \) is in the piece with \((n - 4)/2 \) vertices, that piece is \( X' \). For the cards with \( x' \) in the piece with \((n - 6)/2 \) vertices, the pieces with \((n - 4)/2 \) vertices form the rc1-deck of \( X \) rooted at \( x \), and we reconstruct \( X \) by Theorem 2.1. Thus we have both \( X' \) and \( X \) to reconstruct \( T \).

Finally, suppose that in every \( (\frac{n-4}{2}, \frac{n-6}{2}) \)-card both neighbors of the centroid have degree 2. When \((n - 4)/2 \geq 3 \), this requires \( d_T(x') = 2 \). Let \( y \) be the neighbor of \( x \) in \( X \). In the \( (\frac{n-4}{2}, \frac{n-4}{2}) \)-cards where the centroid has a neighbor with degree at least 3, the centroid is \( x \); these cards arise by deleting three vertices outside \( X \). If in some such card the neighbors of the centroid both have degree at least 3, then \( d_T(y) \geq 3 \). Now the \( (\frac{n-4}{2}, \frac{n-4}{2}) \)-cards in which the centroid has exactly one neighbor of degree 2 arise by deleting the two vertices of the burl and one leaf of \( X' \). The piece whose root has degree at least 3 in these cards is \( X - x \), and the other piece gives the rc1-deck of \( X' \) with \( z \) prepended at the root. By Theorem 2.1, we reconstruct \( X' \).

Otherwise, \( d_T(y) = 2 \), and the connected cards in which the centroid has one neighbor of degree 2 and one of degree 3 are obtained by deleting one vertex of the burl and two vertices from \( X' \). The piece whose root has degree 2 is always \( X - x \), rooted at \( y \). The other piece is obtained from an rc2-card of \( X' \) by prepending \( z \) and a vertex of the burl at the root. This gives us the rc2-deck of \( X' \) (two copies of it when the vertices of the burl are not adjacent, which we can determine from the \( (\frac{n-6}{2}, \frac{n-6}{2}) \)-cards). Since we know the total number of leaves in \( T \) and the number of leaves of \( T \) in \( X \) and in the burl, by Theorem 2.3 we can reconstruct \( X' \) with its root.
6 High-Cost Trees with Special Structure

In this section we prove 3-reconstructibility of two special classes of trees, each of which is a family whose \( n \)-vertex members have cost at least \( (n - 2)/2 \). With these results in hand, further recognition arguments can assume that the deck does not come from a tree in these families.

**Definition 6.1.** A \( j \)-vertex or \( j^+ \)-vertex is a vertex with degree \( j \) or at least \( j \), respectively. Similarly, a \( j \)-neighbor or \( j^+ \)-neighbor is a neighbor that is a \( j \)-vertex or \( j^+ \)-vertex, respectively. A full vertex is a \( 3^+ \)-vertex that is not a 1-burl. A caterpillar is a tree having a single path incident with all edges; equivalently, it is a tree not containing \( S_{2,2,2} \) as a subgraph.

**Theorem 6.2.** For \( n \geq 10 \), every \( n \)-vertex caterpillar having maximum degree at most 3 is 3-reconstructible.

**Proof.** Let \( \mathcal{D} \) be the \((n - 3)\)-deck of such an \( n \)-vertex tree. When \( n \geq 2\ell + 1 \), acyclic graphs are \( \ell \)-recognizable ([7]), and the number of edges is known from the 2-deck, so all reconstructions are trees. Also we know the degree list, since \( n - \ell \geq 5 \) and these trees have no \( 4^+ \)-vertex. Finally, since \( n - \ell \geq 7 \) we see that a tree with the given deck has no copy of \( S_{2,2,2} \), and hence it is a caterpillar. Thus this family of \( n \)-vertex graphs is \( \ell \)-recognizable.

Let \( T \) be a reconstruction of \( \mathcal{D} \). We know the number \( s \) of 3-vertices in \( T \) and hence the number of vertices in a longest path in \( T \); it is just \( n - s \). Let the end-distance of a 3-vertex be its minimum distance from an endpoint of a longest path.

**Case 1:** \( s \leq 1 \). If \( s = 0 \), then \( T \) is a path, which we know from the maximum degree. If \( s = 1 \), then the end-distance of the 3-vertex is the least \( j \) such that \( T \) has only one copy of \( S_{1,1,j+1} \). By Observation 1.2, we can count the copies of \( S_{1,1,j+1} \) in \( T \) if \( j + 4 \leq n - \ell \), and \( j \leq (n - 2)/2 \) suffices. We locate the 3-vertex on the \((n - 1)\)-vertex path to reconstruct \( T \).

**Case 2:** \( s = 2 \). We seek the distance \( d \) between the two 3-vertices and the end-distance of one of them. Let \( H_i \) be the \((i + 5)\)-vertex subtree consisting of a path with \( i + 1 \) vertices plus two leaves appended at each end. The value \( d \) is the only \( i \) such that \( H_i \subseteq T \). By Observation 1.2, we find \( H_d \) in \( T \) if \( d + 5 \leq n - 3 \).

In this case, let \( r \) be smallest among the two end-distances. We know the number \( t_j \) of copies of \( S_{1,1,j} \) in \( T \) if \( j + 3 \leq n - \ell \). If \( j \leq r \), then \( t_j = 4 \), except that \( t_j = 6 \) if \( j = d + 1 \leq r \). If \( j = r + 1 \), then \( t_j = 3 \) unless \( d = n - 3 - 2r \) (here both end-distances equal \( r \), and \( t_j = 2 \)) or \( d = 4 \) (here \( t_j = 5 \)). Thus, if we find some \( j \) with \( t_j = 5 \), then \( r = j - 1 \). Otherwise, \( r \) is the least \( j - 1 \) such that \( t_j = 3 \), or if \( t_j \) is never odd for \( j \leq n - 6 \), then \( r = (n - d - 3)/2 \).

If \( d \geq n - 7 \geq 3 \) and we do not find \( H_d \), then the two end-distances sum to at most 4 and are measured from opposite ends of the path. They are the same and equal \( r \) if \( t_{r+1} = t_r - 2 = 2 \), and otherwise they are the two values of \( r \) such that \( t_{r+1} = t_r - 1 \).
Case 3: $s = 3$. The approach is similar to Case 2. The distances $d$ and $d'$ from the middle 3-vertex to the other two are the least $i$ and $j$ such that we find $H_i$ and $H_j$ as subgraphs (possibly $i = j$), unless one of those distances is at least $n - 7$. In that case, since $s = 3$, the tree $T$ arises from a path with $n - 3$ vertices by adding leaf-neighbors to vertices with end-distances 1, 1, and 2.

Otherwise, we know $d$ and $d'$. We need to know the end-distance for a 3-vertex having a given distance ($d$ or $d'$) from the middle 3-vertex. Let $H_{i,j}'$ consist of a path $P$ with $i + 2 + j$ vertices plus a leaf neighbor of the vertices at distances 1 and $j$ from the opposite endpoints of $P$. If $i + j \leq n - 7$, then we can count the copies of $H_{i,j}'$ in $T$. Consider $H_{d,j}'$ and $H_{d,r}'$ for various $j$. Since the middle 3-vertex has distance at most $(n - 4)/2$ from some endpoint of a longest path in $T$, for $d$ or $d'$ we find at least two (at least four if $d = d'$) copies of $H_{d,j}'$ for all $j$ from 1 up to some value $r$, with only one copy of $H_{d,r+1}'$ existing (two or three copies if $d = d'$ and $r$ is or is not $(n - 4 - 2d)/2$). This value $r$ is the end-distance for the 3-vertex at distance $d$ from the middle 3-vertex, and this determines $T$.

Case 4: $s = 4$. All connected cards having a path $P$ with $n - 4$ vertices arise by deleting leaf neighbors of three 3-vertices, leaving one whose distance from an endpoint of $P$ is its end-distance. Let $r$ be the least such end-distance.

In $T$ there are six subgraphs that are copies of $H_i$ for various $i$, corresponding to the six pairs of 3-vertices. Since we can delete the two 3-vertices not in the pair, we see all these distances, except the largest in the case $r = r' = 1$, where the distance is $n - 5$. Hence we know all six pairwise distances.

From these values, we determine the order of the distances between 3-vertices along the path. Let the positions be $a, b, c, d$ in order, up to translation. The largest among the distances gives us $d - a$. The next largest is $c - a$ or $d - b$; up to reversal, we may let it be $c - a$. Now we know $d - c$; it is $(d - a) - (c - a)$. This leaves $\{b - a, c - b, d - b\}$. Among these three values, $b - a$ and $c - b$ are the two with sum $c - a$, and then which of the two is $c - b$ is the one who sums with $d - c$ is $d - b$.

We now know $a, b, c, d$ in order, except for translation and reversal. The value $r$ is the end-distance for the 3-vertex as position $a$ or $d$. To distinguish between these, we use the counts for the subgraphs $H_{i,j}'$ of Case 3. We associate position $a$ with the least end-distance if $T$ has fewer copies of $H_{b-a,r+1}'$ than $H_{b-a,r}'$, or position $d$ if $T$ has fewer copies of $H_{d-c,r+1}'$ than $H_{d-c,r}'$. However, more care is needed in the special case $c - b = r$. In that case, we can examine subgraphs with three 3-vertices to distinguish the two possibilities, and they are isomorphic if $d - c = b - a$.

Case 5: $s \geq 5$. Let $D'$ be the set of connected cards having a path with $n - s$ vertices; such cards arise by deleting leaf neighbors of three distinct 3-vertices. Let $v$ and $w$ be the 3-vertices closest to the two ends of a longest path in $T$. The cards in $D'$ having 3-vertices farthest apart have $v$ and $w$ as 3-vertices, telling us their end-distances. If they have distinct
end-distances, then the various cards in \( D' \) having a 3-vertex closest to an endpoint of the path \( P \) with \( n - s \) vertices give us the positions of all the other 3-vertices, reconstructing \( T \).

If \( v \) and \( w \) have the same end-distance \( r \), then consider the cards in \( D' \) where all 3-vertices have end-distance more than \( r \). These arise by deleting the leaf neighbors of \( v \), \( w \), and one other 3-vertex; still at least two 3-vertices remain. Among these, the cards having 3-vertices farthest apart fix the end-distances of the second 3-vertex from each end.

Let \( q \) be the minimum end-distance among these two 3-vertices. If they both have end-distance \( q \), then look at one card in \( D' \) having one 3-vertex with end-distance \( q \) and no 3-vertex with end-distance \( r \). This shows us the remaining 3-vertices, and we know exactly where to add the three missing leaves. If only one of these two has end-distance \( q \), then consider the cards in \( D' \) where the least end-distance of 3-vertices is \( q \). Such cards are missing the leaf neighbors of \( v \), \( w \), and one other 3-vertex not having end-distance \( q \). Since \( s \geq 5 \), over all such cards we obtain the positions of the other 3-vertices, since the 3-vertex with end-distance \( q \) distinguishes the two ends of \( P \). \( \square \)

Note that when the centroid of an \( n \)-vertex tree is a 2-vertex or a 1-burl, the cost of the tree is at least \( (n - 2)/2 \).

**Theorem 6.3.** Let \( D \) be the \((n - 3)\)-deck of an \( n \)-vertex tree \( T \) having a connected card with a full vertex. If \( T \) is unicentroidal with no unicentroidal card having a full vertex as centroid, or if \( T \) is bicentroidal with both centroids having degree 2, then \( T \) is 3-reconstructible, without knowing in advance the cost or subcost.

**Proof.** Since cards are subgraphs of \( T \), a card with a full vertex requires a full vertex in \( T \).

If \( T \) is unicentroidal with a full vertex \( z \) as centroid, then deleting two vertices from a largest component of \( T - z \) and one vertex from a next largest component leaves a connected card in which \( z \) is the unique centroid (and a full vertex). Hence \( z \) must be a 1-burl or a 2-vertex. Thus \( T \) has cost at least \((n - 2)/2\); it is a \((\frac{n-2}{2}, \frac{n-2}{2})\)-tree, a \((\frac{n-3}{2}, \frac{n-3}{2})\)-tree, or a \((\frac{n-1}{2}, \frac{n-1}{2})\)-tree. Furthermore, if a neighbor of \( z \) is a full vertex, then it becomes the centroid of a unicentroidal card when three vertices are deleted from the other nontrivial piece. Hence every non-leaf neighbor of the centroid in \( T \) is also a 1-burl or a 2-vertex.

When \( T \) is bicentroidal and both centroids are 2-vertices, again if a neighbor of a centroid is a full vertex there is a card in which it is a full vertex and the unique center. Hence in this case also every non-leaf neighbor of a centroid in \( T \) is a 1-burl or a 2-vertex.

Let an **optimal card** be a connected card minimizing \( r \), where in a unicentroidal tree \( r \) is the distance from the centroid to a closest full vertex, and in a bicentroidal card \( r \) is the average distance from the two centroids to a closest full vertex. Since we have forbidden full vertices as centroids of cards, \( r \geq 1/2 \).

For a full vertex \( v \) in \( T \), the minimum average distance from \( v \) to a centroid in any connected card containing \( v \) as a full vertex occurs only in a card obtained by deleting three
vertices outside the piece containing \( v \). In optimal cards, there may be two choices for a vertex \( v \) having average distance \( r \) from the centroid(s), possibly one in each nontrivial piece.

If no piece (or branch) occurs in all optimal cards, then each nontrivial piece of \( T \) has a full vertex at the same distance from the centroid in \( T \). If these are \( v \) and \( v' \), then some optimal cards have \( v \) as the closest full vertex (distance \( r \)) from their centroid, while other cards have \( v' \) as this vertex. Each optimal card contains one of the two resulting pieces, so over all optimal cards we obtain both. In each card we see the centroid(s) and whether they have leaf neighbors, and in all cases we can assemble \( T \).

If all the optimal cards have the same piece containing the full vertex closest to the centroid, then either that piece arises from the same vertex in \( T \) or it arises from two different full vertices in \( T \). In the latter case, the non-constant pieces of the optimal cards provide two copies of the rc3-deck of the constant piece. In the former case, they give the rc3-deck of the piece of \( T \) not containing the constant piece, and we reconstruct by Lemma 3.12.

In particular, when the neighbor \( x \) of a centroid \( z \) is the centroid in the optimal cards, the rc3-deck of the non-constant piece tells us whether \( z \) is a 1-burl or a 2-vertex, which distinguishes between \( \left( \frac{n-2}{2}, \frac{n-3}{2} \right) \)-trees and bicentroidal trees in this family. The same thing happens to distinguish between \( \left( \frac{n-1}{2}, \frac{n-3}{2} \right) \)-trees and \( \left( \frac{n-1}{2}, \frac{n-1}{2} \right) \)-trees in this family. □

In this result we do not include the case of bicentroidal trees having a centroid \( z \) that is a 1-burl, because \( z \) becomes the unique centroid in a connected card both when three vertices are deleted from the other branch (leaving a constant piece with \( (n-4)/2 \) vertices) and when two vertices are deleted from the other branch and one from the branch containing \( z \) (leaving a \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-tree with neither piece constant).

7 \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-Trees

At this point we know that the remaining trees (when \( n \) is even) have cost at least \( (n-2)/2 \), and those with cost \( (n-2)/2 \) are \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-trees. Distinguishing these from bicentroidal trees using the \( (n-3) \)-deck is somewhat difficult. We need more structure in the cards.

**Definition 7.1.** A special card in the \( (n-3) \)-deck of an \( n \)-vertex tree is a connected card with cost \( (n-4)/2 \) in which the centroid has degree at least 3 (so the largest piece is unique) and the neighbor of the centroid in the largest piece is a 1-burl having a neighbor of degree 2. A superspecial card is a special card such that the centroid is a 1-burl (so it is a \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card) and its neighbor in the second largest piece has degree 2.

Recall that every reconstruction from the \( (n-3) \)-deck of a tree has the same number of leaves, as remarked before Lemma 3.1. For unicentroidal trees, label \( z, x, X, x', X' \) as usual. We first recognize two subclasses of \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-trees.
Lemma 7.2. Let $D$ be the $(n-3)$-deck of some tree that is a $(\frac{n-2}{2}, \frac{n-2}{2})$-tree or is bicentroidal. Let $t$ be the number of leaves in every tree whose $(n-3)$-deck is $D$.

(A) All reconstructions are $(\frac{n-2}{2}, \frac{n-2}{2})$-trees where the centroid has no 2-neighbor if and only if $D$ has exactly $t+2$ balanced cards.

(B) All reconstructions are $(\frac{n-2}{2}, \frac{n-2}{2})$-trees where the centroid has exactly one 2-neighbor if and only if conditions (1), (2), (3) below all hold:

1. $D$ has at least $t+2$ balanced cards,
2. $D$ has a special card, and either
   (3a) $D$ has a special card that is not superspecial, or
   (3b) the rooted pieces of order $(n-4)/2$ in superspecial cards are not all isomorphic.

Proof. Recall that when $n$ is even, balanced cards have components with $(n-2)/2$ and $(n-4)/2$ vertices. Also, the centroid of a $(\frac{n-2}{2}, \frac{n-2}{2})$-tree is a 1-burl.

Case 1: $T$ is a $(\frac{n-2}{2}, \frac{n-2}{2})$-tree such that the centroid $z$ has no 2-neighbor. Since $z$ has no 2-neighbors, deleting one of its non-leaf neighbors yields three components. Even if one of them has a single vertex and can be deleted, the large component has $(n+2)/2$ vertices and cannot be trimmed small enough. Hence every balanced card must delete $z$ and its leaf neighbor and keep the other two neighbors of $z$. The remaining vertex deleted must be another leaf of $T$, and there are $t-1$ choices for it.

Case 2: $T$ is a $(\frac{n-2}{2}, \frac{n-2}{2})$-tree such that the centroid $z$ has exactly one 2-neighbor. We may assume $d_T(x) = 2$ and $d_T(x') \geq 3$. In addition to the $t-1$ balanced cards described in Case 1, $T$ has balanced cards obtained by deleting $x$ and two vertices from the $n/2$-vertex component of $T - x$ containing $z$. Since this tree has another leaf outside the leaf neighbor of $z$, we obtain at least three additional balanced cards, satisfying (B1).

To obtain a special card, delete three vertices from $X$. The centroid will now be $x'$, having degree at least 3 as required. The neighbor of $x'$ in the piece with $(n-4)/2$ vertices is $z$, which is a 1-burl and has a 2-neighbor $x$. Hence any card obtained by deleting three vertices of $X$ is a special card, satisfying (B2). If in such a card $x'$ is not a 1-burl or $x'$ has a non-leaf $3^+$-neighbor other than $z'$ in $X'$, then this special card is not superspecial. Hence if (B3a) fails, then $x'$ is a 1-burl in $T$ and has a 2-neighbor in $X'$.

The pieces with $(n-4)/2$ vertices in these special cards are the rc3-cards of $X$ with $z$ prepended as the root. If they are all isomorphic, then by Lemma 2.6 the rooted tree is a star or has root-degree 1 (since $n - 3 - 1 \geq 2$). This contradicts $z$ being a 1-burl. Hence if (B3a) fails, then (B3b) holds.

Case 3: $T$ is a $(\frac{n-2}{2}, \frac{n-2}{2})$-tree such that the centroid $z$ has two 2-neighbors. By the same argument as in the beginning of Case 2, $T$ has at least $t+5$ balanced cards. Hence $T$ cannot satisfy condition (A). If $T$ satisfies condition (B), then let $C$ be a special card with centroid $u$. Recall that $C$ is a connected card with cost $(n-4)/2$ in which $d_C(u) \geq 3$ and the neighbor of $u$ in the largest part is a 1-burl. Hence $u$ cannot be $x$ or $x'$, since they have
degree 2 in \( T \), and \( u \neq z \) since \( z \) has no \( 3^+ \)-neighbor. For any other vertex \( y \), in \( T - y \) there is a component with at least \( (n + 4)/2 \) vertices, which cannot be cut down to \( (n - 4)/2 \) vertices by deleting three vertices. Hence \( T \) has no special card, and (B2) fails.

**Case 4:** \( T \) is bicentroidal, with centroids \( z \) and \( z' \) and branches \( Y \) containing \( z \) and \( Y' \) containing \( z' \). When \( z \) and \( z' \) both have degree at least 3, no balanced card contains both of them. When either is a full vertex, it must lie in every balanced card. When \( z \) or \( z' \) is a 1-burl, we obtain balanced cards by deleting it, its leaf neighbor, and one leaf from the larger remaining component. When both centroids are 1-burls, this generates exactly \( t \) balanced cards, and there are no others. When one centroid is a 1-burl and the other is a full vertex, the number of balanced cards is at most \( t - 2 \). Hence a bicentroidal card with no centroid of degree 2 cannot satisfy the conditions on balanced trees specified in (A) or (B).

If either centroid has degree 2, then we obtain balanced cards by deleting it and one leaf from each resulting component, or two vertices from the larger component. In all cases, this yields at least \( t + 2 \) balanced cards. Thus the condition on balanced cards in (A) cannot be satisfied, but those in (B) are satisfied.

Hence we may assume \( d_T(z') = 2 \) and that (B) holds. First suppose also \( d_T(z) = 2 \). Now let \( C \) be a special card. By Corollary 3.5, the centroid \( u \) of \( C \) is a neighbor of \( z \) or \( z' \). Also \( d_C(u) \geq 3 \), so \( u \notin \{z, z'\} \). Also \( u \) cannot be the non-centroidal neighbor of \( z \) or \( z' \), because the piece of \( C \) with \( (n - 4)/2 \) vertices would contain the neighbor of \( u \) in \( \{z, z'\} \), which has degree only 2 in \( T \) and hence cannot be a 1-burl in \( C \). Thus \( T \) cannot have a special card.

We may therefore assume \( d_T(z) \geq 3 \). A special card \( C \) cannot have centroid \( z' \) since \( d_T(z') = 2 \). It cannot be a non-centroidal neighbor of \( z \) because deleting such a vertex leaves a component with \( (n + 4)/2 \) vertices, too large to be reduced to \( (n - 4)/2 \) by deleting three vertices. It cannot be the non-centroidal neighbor of \( z' \), because the neighbor of that vertex in the big piece would be \( z' \), which is not a 1-burl. Hence by Corollary 3.5 the centroid of \( C \) must be \( z \). Now since \( d_T(z') = 2 \), and in \( C \) the neighbor of \( z \) in the large piece must be a 1-burl, that neighbor must be \( x \). Since the large piece in \( C \) must have \( (n - 4)/2 \) vertices, and the component of \( T - z' \) containing \( z \) has \( n/2 \) vertices, \( z \) must be a 1-burl in \( T \), and every special card is obtained from \( T \) by deleting three vertices from \( Y' \).

With \( z \) being a 1-burl and \( z' \) having degree 2, every special card is now superspecial. Hence (B3a) fails. Now we require the pieces of order \( (n - 4)/2 \) in these cards not to be isomorphic, but they are all obtained from \( Y \) by deleting \( z \) and its leaf neighbor. Hence (B3b) fails, and (B) cannot be satisfied by a bicentroidal card.

When \( n \) is even, we still must distinguish \( (\frac{n-2}{2}, \frac{n-2}{2}) \)-trees whose centroid has two \( 2^+ \)-neighbors from bicentroidal trees. We do this by breaking into further cases depending on the distances from the centroid to branch vertices or leaves.
Definition 7.3. For \( a \leq b \), a tree has an \( a, b \)-spear if the centroid is a 1-burl and the distances from the centroid to the nearest vertex with degree other than 2 in the two nontrivial pieces are \( a \) and \( b \) (the shorter distance may be in either piece). The \( (n-3) \)-deck of an \( n \)-vertex tree has \( a, b \)-type when there is a \( \left( \left\lfloor \frac{n-5}{2} \right\rfloor, \left\lceil \frac{n-5}{2} \right\rceil \right) \)-card and \( (a, b) \) is the lexicographically least pair \((i, j)\) such that some such card has an \( i, j \)-spear. When \( n \) is even we are considering \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-cards, while when \( n \) is odd these are \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-cards.

The next lemma captures an argument we need several times.

Lemma 7.4. Let \( D \) be the \( (n-3) \)-deck of a bicentroidal tree \( T \). If \( n \geq 10 \) and the properties below hold, then \( T \) has one centroid of degree 2, and the other is a 2-vertex or a 1-burl.

(a) \( D \) has an \( a, b \)-type with \( a \geq 2 \), and

(b) \( D \) has no connected card whose centroid is a full vertex.

Proof. Let \( z \) and \( z' \) be the centroids of \( T \), with branches \( Y \) containing \( z \) and \( Y' \) containing \( z' \). By Corollary 3.5, in every connected card the centroid is a neighbor of \( z \) or \( z' \). Deleting two vertices from one of \( \{Y, Y'\} \) and one from the other yields a connected card with centroid \( z \) or \( z' \). If \( z \) or \( z' \) is a full vertex, then in this way we obtain a connected card whose centroid is a full vertex, as long as \( n/2 - 1 \geq 4 \), violating (b). Hence each of \( z \) and \( z' \) is a 1-burl or a 2-vertex. If they are both 1-burls, then deleting vertices as described yields a \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card whose centroid has a neighbor of degree 3, violating (a).

Lemma 7.5. Let \( D \) be the \( (n-3) \)-deck of a tree that is bicentroidal or is a \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-tree whose centroid has two 2-neighbors. For \( n \geq 14 \), every reconstruction from \( D \) is a \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-tree having a 2,2-spear if and only if conditions (a), (b), (c) below hold.

(a) \( D \) has 2,2-type,

(b) \( D \) has no connected card whose centroid is a full vertex, and

(c1) some \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card has a path \( \langle \text{centroid}, 3^+\text{-vertex}, 3^+\text{-vertex} \rangle \), or

(c2) for every \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card, there is a 3\(^{+}\)-vertex within distance 2 of the centroid, or there is no 3\(^{+}\)-vertex at distance exactly 3 from the centroid.

Proof. Let \( T \) be a tree whose \( (n-3) \)-deck is \( D \).

Case 1: \( T \) is a \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-tree having a 2,2-spear. The centroid \( z \) is a 1-burl, with nontrivial pieces \( X \) and \( X' \) rooted at \( x \) and \( x' \) of degree 2, respectively. The respective noncentroidal neighbors \( y \) and \( y' \) of \( x \) and \( x' \) are 3\(^{+}\)-vertices, since \( T \) has a 2,2-spear. Every connected card has centroid in \( \{x, z, x'\} \). Since \( T \) has \( (n+2)/2 \) vertices outside \( X \), the cards with centroid \( x \) are those missing three vertices outside \( X \), and they are \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-cards. Similarly for \( x' \). Hence every \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card has centroid \( z \), and \( z \) has no 3\(^{+}\)-neighbor. Since we can delete two vertices from one piece and one from the other while leaving \( y \) and
Let $y'$ as $3^+$-vertices (when $(n-2)/2 \geq 5$), (a) holds. Also, (b) holds since $x, z, x'$ are the only possible centroids and cannot become full vertices.

A $(\frac{n-4}{2}, \frac{n-4}{2})$-card with centroid $x$ is obtained by deleting three vertices outside $X$. If $y$ has a $3^+$-neighbor, then (c1) holds. Similarly, (c1) holds if $y'$ has a $3^+$-neighbor. If neither $y$ nor $y'$ has a $3^+$-neighbor, then the $(\frac{n-4}{2}, \frac{n-4}{2})$-cards with centroid $x$ or $x'$ still have $y$ or $y'$ as a $3^+$ neighbor of the centroid. There are also $(\frac{n-4}{2}, \frac{n-4}{2})$-cards with centroid $z$; these arise by deleting the leaf neighbor of $z$ and one leaf each from $X$ and $X'$. Whether or not $y$ and $y'$ are reduced to 2-vertices in such a card, the vertices at distance 3 from $z$ are neighbors of $y$ or $y'$ and have degree at most 2. Hence (c2) holds when (c1) fails.

**Case 2:** $T$ is a $(\frac{n-2}{2}, \frac{n-2}{2})$-tree not having a 2, 2-spear. Again label the path $\langle y, x, z, x', y' \rangle$ through the centroid $z$, with $y$ and $y'$ being the nonleaf neighbors of $x$ and $x'$ other than $z$. By hypothesis $d_T(x) = d_T(x') = 2$. Since $T$ does not have a 2, 2-spear, by symmetry we may assume $d_T(y) = 2$. Since all connected cards have centroid in $\{x, z, x'\}$, all $(\frac{n-4}{2}, \frac{n-6}{2})$-cards have centroid $z$. With $d_T(y) = 2$, this violates (a).

**Case 3:** $T$ is a bicentroidal tree. Let $Y$ and $Y'$ be the branches, rooted at the centroids $z$ and $z'$, respectively. By Lemma 7.4, we may assume $d_T(z') = 2$ and that $z$ is not full. Let $x$ be the non-leaf neighbor of $z$ in $Y$, and let $x'$ be the neighbor of $z'$ in $Y'$.

If $d_T(z) = 2$, then deleting three vertices from one branch yields a card with centroid $x$ or $x'$; by (b), neither $x$ nor $x'$ is full. Since $D$ has 2, 2-type, we must have a $(\frac{n-4}{2}, \frac{n-6}{2})$-card. The centroid of such a card is a 1-burl and cannot be $z$ or $z'$. Hence $x$ or $x'$ is a 1-burl, but now $d_T(z) = d_T(z') = 2$ violates (a); that is, no $(\frac{n-4}{2}, \frac{n-6}{2})$-card can have a 2, 2-spear.

Hence we may assume that $z$ is a 1-burl in $T$ and $d_T(z') = 2$. Since $d_T(z') = 2$, the centroid in any $(\frac{n-4}{2}, \frac{n-6}{2})$-card having a 2, 2-spear must be $z$ or $x'$, which in either case yields $d(x') \geq 3$. Deleting three vertices from $Y$ yields cards with centroid $x'$ and cost $(n-4)/2$. If $x'$ is a full vertex, this violates (b). Hence $x'$ is a 1-burl, and to avoid violating (a) the non-leaf neighbor $y'$ of $x'$ in $Y'$ must have degree 2. Also, deleting three vertices from $Y'$ makes $z$ the centroid of a $(\frac{n-4}{2}, \frac{n-6}{2})$-card, so (a) requires $d_T(x) = 2$.

Our tree now has the structure $\hat{A} - x - z - z' - x' - y' - B$ plus leaves $\hat{z}$ adjacent to $z$ and $\hat{x}'$ adjacent to $x'$. Here $A$ is a subtree with $(n-6)/2$ vertices and $B$ is a subtree with $(n-8)/2$ vertices, and the degrees of $(x, z, z', x', y')$ are $(2, 3, 2, 3, 2)$. Let $y$ be the neighbor of $x$ in $A$, and let $w'$ be the neighbor of $y'$ in $B$.

The centroids of connected cards lie in $\{x, z, z', x'\}$. In a $(\frac{n-4}{2}, \frac{n-6}{2})$-card, the centroid is a 1-burl and must be $z$ or $x'$. In order to have a $(\frac{n-4}{2}, \frac{n-6}{2})$-card with a 2, 2-spear, we must have $d_T(y) \geq 3$ (centroid $z$) or $d_T(w') \geq 3$ (centroid $x'$).

Neither $x$ nor $x'$ can be the centroid of a $(\frac{n-4}{2}, \frac{n-4}{2})$-card, since $A$ and $B \cup \{y'\}$ have only $(n-6)/2$ vertices. Hence every $(\frac{n-4}{2}, \frac{n-4}{2})$-card has centroid $z$ or $z'$. Since all of $\{x, z', y'\}$ have degree 2 in $T$, (c1) fails for all $(\frac{n-4}{2}, \frac{n-4}{2})$-cards, and (c2) must hold. Consider a $(\frac{n-4}{2}, \frac{n-4}{2})$-card with centroid $z'$ obtained by deleting the leaves $\hat{z}$ and $\hat{x}'$ and one more leaf from $Y$.
whose deletion does not reduce \( y \) to degree 2 (such a leaf exists when \( n/2 > 6 \), so \( n \geq 14 \)). This card has no 3+ vertex within distance 2 of \( z' \). Hence (c2) requires \( d_T(y) = d_T(w') = 2 \), which contradicts the preceding paragraph. \( \square \)

We have now recognized the family of \((n−2)/2, n−2)/2\)-trees having \( a, b \)-spears with \( (a, b) \) lexicographically at most \((2, 2)\).

**Lemma 7.6.** Let \( D \) be the \((n−3)\)-deck of a tree \( T \) that is bicentroidal or is a \((n−2)/2, n−2)/2\)-tree whose centroid has two 2-neighbors. If \( n \geq 16 \) and the tree has an \( a, b \)-spear with \( b \geq 3 \), then every reconstruction from \( D \) is an \((n−2)/2, n−2)/2\)-tree that has a \( 2, 2 \)-spear for some \( b \) at least 3 if and only if all conditions below hold.

(a) \( D \) has 2, \( b' \)-type for some \( b' \) with \( b' \geq 3 \),

(b) \( D \) has no connected card whose centroid is a full vertex,

(c) \( D \) has a \((n−4)/2, n−4)/2\)-card whose centroid has two 3+-neighbors, and

(d) \( D \) has a \((n−4)/2, n−4)/2\)-card with a path of vertices having degrees \((2, 2, 3+, 2, 3+)\) whose second vertex is the centroid.

**Proof.** By Lemma 7.5, we may assume that \( T \) does not have a \( 2, 2 \)-spear.

**Case 1:** \( T \) is a \((n−2)/2, n−2)/2\)-tree that has a \( 2, 2 \)-spear, where \( b \geq 3 \). Both neighbors of the centroid \( z \) have degree 2. Let \( x \) be the neighbor of \( z \) having a 3+-neighbor \( y \), and let \( x' \) be the other neighbor of \( z \). Let \( X \) and \( X' \) be the pieces of \( T \) containing \( x \) and \( x' \), respectively.

Since all centroids of connected cards are in \( \{x, z, x'\} \), the centroid of any card whose centroid has degree at least 3 is \( z \). Thus \( z \) is the centroid of every \((n−4)/2, n−4)/2\)-card. This implies (a). None of \( \{x, z, x'\} \) can be a full vertex, so (b) holds. Deleting three vertices from \( X' \) yields a \((n−4)/2, n−4)/2\)-card with centroid \( x \) satisfying (c). Deleting three vertices from \( X \) (keeping degree at least 3 at \( y \)) yields a \((n−4)/2, n−4)/2\)-card with centroid \( x' \) satisfying (d) (the 3+-vertex in the middle of the required path is \( z \)). This last card requires \((n−2)/2 \geq 7 \), or \( n \geq 16 \).

**Case 2:** \( T \) is a \((n−2)/2, n−2)/2\)-tree that has an \( a, b \)-spear with \( a \geq 3 \). Let \( z \) be the centroid of \( T \), with 2-neighbors \( x \) and \( x' \). In every connected card, the centroid is in \( \{x, z, x'\} \). Since the neighbors of \( x \) and \( x' \) other than \( z \) have degree 2, there is no card in which the centroid has two 3+-neighbors, violating (c).

**Case 3:** \( T \) is a bicentroidal tree. Let \( z \) and \( z' \) be the centroids, with \( Y \) and \( Y' \) the branches containing \( z \) and \( z' \), respectively. By Corollary 3.5, a centroid of a connected card is adjacent to \( z \) or \( z' \). If \( z \) and \( z' \) are both 2-vertices, then no \((n−4)/2, n−4)/2\)-card has a centroid with two 3+-neighbors, violating (c). Hence by Lemma 7.4, we may assume \( z \) is a 1-burl and \( d_T(z') = 2 \). Let \( x' \) be the neighbor of \( z' \) in \( Y' \), and let \( x \) be the non-leaf neighbor of \( z \) in \( Y \).

A \((n−4)/2, n−4)/2\)-card may have centroid \( z \) (delete its leaf neighbor and two from \( Y' \)) or \( z' \) (delete two from \( Y \) and one from \( Y' \)) or \( x' \) (delete three from \( Y \) when \( d_T(x') = 2 \)).
Lemma 7.7. Let \( d_T(x) \geq 3 \). Since \( x' \) is the centroid of any card obtained by deleting three vertices from \( Y \), by (b) \( x' \) must be a 1-burl.

A \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card with centroid \( z \) is obtained by deleting one vertex from \( Y \) and two from \( Y' \), or three from \( Y' \). This can be done without deleting the leaf neighbor of \( x' \) or reducing the degree of \( x \) below 3 if \( d_T(x) \geq 3 \), as long as \( n/2 - 2 \geq 5 \). Hence (a) requires \( d_T(x) = 2 \), and then for the same reason \( d_T(y) = 2 \), where \( y \) is the neighbor of \( x \) other than \( z \), as long as \( n/2 - 2 \geq 6 \).

Finally, we seek a \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card \( C \) satisfying (d). The centroid cannot be \( z \), because \( z \) has no 3\(^+\)-neighbor. We have noted that \( x' \) can be the centroid of a \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card only if \( d_T(x') = 2 \), which has been excluded. Hence \( C \) must have centroid \( z' \). Since \( d_T(y) = 2 \), the required path cannot be \( \langle x', z', z, x, y \rangle \). Hence it must be \( \langle z, z', x', u, v \rangle \), requiring a vertex \( v \) at distance 2 from \( x' \) in \( Y' \) to have degree at least 3. Now by deleting three vertices from \( Y \) while keeping the leaf neighbor of \( z \), we make a \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card with centroid \( x' \) that has a 2, 2-spear, violating (a).

We have now recognized the family of \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-trees having \( a, b \)-spears with \( a \leq 2 \).

**Lemma 7.7.** Let \( D \) be the \( (n-3) \)-deck of a tree \( T \) that is bicentroidal or is a \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-tree whose centroid has two 2-neighbors. If \( n \geq 16 \) and the tree has an \( a, b \)-spear where \( (a, b) \) is lexicographically at least \( (3, 3) \), then every reconstruction from \( D \) is a \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-tree that has a 3, 3-spear if and only if all conditions below hold.

(a) \( D \) has 3, 3-type,
(b) \( D \) has no connected card whose centroid is a full vertex, and
(c) in each \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card whose centroid has a 3\(^+\)-neighbor,
the distance from the centroid to the nearest 3\(^+\)-vertex in the other branch is 2.

**Proof.** **Case 1:** \( T \) is a \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-tree having a 3, 3-spear. Let \( z \) be the centroid of \( T \), with \( \langle y, x, z, x', y' \rangle \) being the path of 2-vertices around the centroid. Since \( T \) has a 3, 3-spear, in \( T \) both \( y \) and \( y' \) have a 3\(^+\)-neighbor.

Let \( X \) and \( X' \) be the pieces containing \( x \) and \( x' \), respectively. As usual, since deleting a vertex outside \( \{ x, z, x' \} \) leaves a component with at least \( (n+4)/2 \) vertices, all connected cards have centroids in \( \{ x, z, x' \} \), which implies (b). Also all \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-cards have a 1-burl as centroid and hence must have centroid \( z \), which implies (a).

Since \( y, x, x', y' \) all have degree 2 in \( T \), in \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-cards where the centroid has a 3\(^+\)-neighbor the centroid is in \( \{ x, x' \} \) and the 3\(^+\)-neighbor is \( z \). Such cards with centroid \( x \) arise by deleting three vertices from \( X' \), leaving the 3\(^+\)-neighbor of \( y \) as a 3\(^+\)-vertex in the piece not containing \( z \). The same holds for such cards with centroid \( x' \), confirming (c).
Case 2: $T$ is a $(\frac{n-2}{2}, \frac{n-2}{2})$-tree having an $a, b$-spear with $(a, b)$ lexicographically at least $(3, 4)$. Let $z$ be the centroid of $T$, with 2-neighbors $x$ and $x'$. In every connected card, the centroid is in $\{x, z, x'\}$. As above, in $(\frac{n-4}{2}, \frac{n-6}{2})$-cards whose centroid has a $3^+$-neighbor, the centroid is $x$ or $x'$ and the neighbor is $z$. We obtain such cards for both $x$ and $x'$ being centroids, by deleting three vertices from the other piece of $T$. For at least one of these choices (centroid $x$ or $x'$), the distance from the centroid to the nearest $3^+$-vertex in the piece not containing $z$ is at least 3, violating (c).

Case 3: $T$ is a bicentroidal tree. Define centroids $z$ and $z'$ and branches $Y$ and $Y'$ as usual, with $x$ the neighbor of $z$ in a second largest component of $T - z$ and $x'$ the neighbor of $z'$ in a second largest component of $T - z'$. By Lemma 7.4, we may assume $d_T(z') = 2$.

If also $d_T(z) = 2$, then let $C$ be a $(\frac{n-4}{2}, \frac{n-6}{2})$-card having a 3, 3-spear, as guaranteed by (a). The centroid of $C$ is a 1-burl in $C$, so it is not $z$ or $z'$ and must be $x$ or $x'$. By symmetry, let it be $x$. Since $C$ arises only by deleting three vertices of $Y'$, in fact $x$ is a 1-burl in $T$. Now by deleting the leaf neighbor of $x$, another leaf in $Y$, and a leaf in $Y'$ that does not reduce the degree of $x'$ to 2, we obtain a $(\frac{n-4}{2}, \frac{n-4}{2})$-card with centroid $z'$ where the centroid has a $3^+$-neighbor in one branch but no $3^+$-vertex at distance 2 in the other branch, violating (c).

Hence we may assume that $z$ is a 1-burl in $T$ and $d_T(z') = 2$. By (a), a $(\frac{n-4}{2}, \frac{n-6}{2})$-card with centroid $z$ requires $d_C(x') = 2$ and requires $x'$ to have a $3^+$-neighbor $y'$. Now form a $(\frac{n-4}{2}, \frac{n-4}{2})$-card by deleting the leaf neighbor of $z$ and two other vertices from $Y$. The centroid is $x'$, and it has a $3^+$-neighbor $y'$, but in the other piece the vertex $z$ at distance 2 from $x'$ is not a $3^+$-vertex. This again violates (c). \[\square\]

We have now recognized $(\frac{n-2}{2}, \frac{n-2}{2})$-trees having $a, b$-spear with $a \leq 2$ or $(a, b) = (3, 3)$.

**Lemma 7.8.** Let $\mathcal{D}$ be the $(n - 3)$-deck of a tree $T$ that is bicentroidal or a $(\frac{n-2}{2}, \frac{n-2}{2})$-tree whose centroid has two 2-neighbors. If $n \geq 18$ and the tree has an $a, b$-spear where $(a, b)$ is lexicographically at least $(3, 4)$, then every reconstruction from $\mathcal{D}$ is a $(\frac{n-2}{2}, \frac{n-2}{2})$-tree that has a 3, b-spear with $b \geq 4$ if and only if all conditions below hold.

(a) $\mathcal{D}$ has 3, $b'$-type for some $b'$ with $b' \geq 4$,

(b) $\mathcal{D}$ has no connected card whose centroid is a full vertex,

(c) $\mathcal{D}$ has a $(\frac{n-4}{2}, \frac{n-6}{2})$-card with a 3$^+$-vertex at distance 3 from the centroid in the big piece, and another with that vertex in the second biggest piece,

(d) $\mathcal{D}$ has a $(\frac{n-4}{2}, \frac{n-4}{2})$-card that has a 1, 2-spear,

(e) $\mathcal{D}$ has a $(\frac{n-4}{2}, \frac{n-4}{2})$-card where the centroid has a 3$^+$-neighbor and in the same piece a 3$^+$-vertex at distance 4, and

(f) $\mathcal{D}$ has no $(\frac{n-4}{2}, \frac{n-4}{2})$ card where the centroid has a 3$^+$-vertex at distance 2 in one piece and at distances 1 and 4 in the other.

**Proof.** Let $T$ be a tree with $(n - 3)$-deck $\mathcal{D}$. 

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Case 1: \( T \) is a \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-tree having a 3, \( b \)-spear with \( b \geq 4 \). The centroid \( z \) is a 1-burl. Let \( X \) and \( X' \) be the nontrivial pieces, with \( X \) being the one having the closer vertex that is not a 2-vertex. Let \( x \) and \( x' \) be the neighbors of \( z \) in \( X \) and \( X' \), and let \( w \) be the vertex in \( X \) at distance 3 from \( z \). Since \( (n-2)/2 > 3 \), we have \( d_T(w) \geq 3 \).

By Corollary 3.5, every connected card has centroid in \( \{ x, z, x' \} \). Since \( d_T(x) = d_T(x') = 2 \) and \( z \) is a 1-burl, (b) follows. The centroid of every \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card is a 1-burl in the card and must be \( z \); this yields (a).

Deleting one vertex from one of \( \{ X, X' \} \) and two vertices from the other while maintaining degree at least 3 at \( w \) yields the cards required for (c); for this, \( (n-2)/2 - 2 \geq 5 \) suffices. Deleting three vertices of \( X' \) yields a \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card with centroid \( x \) having a 1,2-spear, confirming (d). Deleting three vertices of \( X \) while maintaining degree at least 3 at \( w \) yields a \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card with centroid \( x' \) satisfying (e); \( (n-2)/2 - 3 \geq 5 \) suffices.

Suppose that \( T \) has a \( \left( \frac{n-4}{2}, \frac{n-2}{2} \right) \)-card \( C \) whose centroid has a 3\(^+\)-vertex at distance 2 in one piece and 3\(^+\)-vertices at distances 1 and 4 in the other piece. Since \( z \) has no 3\(^+\)-neighbor, the centroid of \( C \) must be \( x \) or \( x' \). Because \( T \) has a 3, \( b \)-spear with \( b \geq 4 \), the distance from \( z \) to a vertex in \( X' \) with degree other than 2 is at least 4. Thus \( x \) has no 3\(^+\)-vertex at distance 4 in the piece containing its 3\(^+\)-neighbor \( z \), and \( x' \) has no 3\(^+\)-vertex at distance 2 in the piece not containing its 3\(^+\)-neighbor \( z \). Hence there is no such card \( C \).

Case 2: \( T \) is a \( \left( \frac{n-2}{2}, \frac{n-2}{2} \right) \)-tree having an \( a', b'' \)-spear with \( b'' \geq a' \geq 4 \). Let \( z \) be the centroid of \( T \), with 2-neighbors \( x \) and \( x' \). In every \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card, the centroid is \( z \). Hence no \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card can have a 3, \( b'' \)-spear. This violates (a).

Case 3: \( T \) is a bicentroidal tree. Define \( z, z', Y, Y' \) as usual. By Lemma 7.4, we may assume \( d_T(z') = 2 \) and that \( z \) is a 1-burl or has degree 2. Let \( x' \) be the neighbor of \( z' \) in \( Y' \).

If \( d_T(z) = 2 \), with \( x \) the neighbor of \( z \) in \( Y \), then the centroid of a \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card can only be \( x \) or \( x' \), since it must be a 1-burl. A \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card with centroid \( x \) or \( x' \) can only be obtained by deleting three vertices from \( Y' \) or from \( Y \), respectively, and in such a card the vertex at distance 3 from the centroid in the big piece is the other of \( \{ x, x' \} \). Hence by (c) both \( x \) and \( x' \) are 3\(^+\)-vertices, and by (b) they must be 1-burls. These \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-cards further imply by (a) that vertices within distance 3 of \( x \) and \( x' \) (other than each other and their leaf neighbors) all have degree 2. This prevents having a \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card with a 3\(^+\)-vertex at distance 3 from the centroid in the second biggest piece, violating (c).

We may therefore assume that \( z \) is a 1-burl. Deleting one vertex from \( Y \) and two from \( Y' \) (while keeping the leaf neighbor of \( z \)) or deleting three vertices from \( Y' \) yields a \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card with centroid \( z \). By (a), the distance from \( z \) to a 3\(^+\)-vertex is at least 3 in each direction. This requires \( x \) and \( x' \) both to have degree 2 in all such cards and hence also in \( T \).

In a \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card \( C \) guaranteed by (e), the centroid has a 3\(^+\)-neighbor. By Corollary 3.5, the centroid is in \( \{ x, z, z', x' \} \). It cannot be \( x \), since \( T - x \) has only one component with at least \( (n - 4)/2 \) vertices. It cannot be \( z \), since \( z \) has no 3\(^+\)-neighbor. Whether it is
Now if the centroid of \( C \) is \( z' \), then (e) requires the vertex at distance 3 from \( z \) in \( Y \) to be a \( 3^+ \)-vertex, but this violates (a) by having \( 3^+ \)-vertices at distance 3 from \( z \) in both directions. Hence the centroid of \( C \) is \( x' \). Now (e) requires another \( 3^+ \)-vertex at distance 3 from \( y' \) in \( Y' \), but (f) forbids it.

We have now recognized \( (\frac{n-2}{2}, \frac{n-2}{2}) \)-trees having \( a, b \)-spear with \( a \leq 3 \).

**Lemma 7.9.** Let \( D \) be the \((n-3)\)-deck of a tree \( T \) that is bicentroidal or is a \( (\frac{n-2}{2}, \frac{n-2}{2}) \)-tree whose centroid has two 2-neighbors. If \( n \geq 16 \) and the tree has an \( a, b \)-spear with \( a \geq 4 \), then every reconstruction from \( D \) is a \( (\frac{n-2}{2}, \frac{n-2}{2}) \)-tree having an \( a, b \)-spear with \( a \geq 4 \) if and only if all conditions below hold.

(a) \( D \) has \( a', b' \)-type for some \( a' \) and \( b' \) with \( b' \geq a' \geq 4 \),

(b) \( D \) has no connected card whose centroid is a full vertex, and

(c) \( D \) has no \( (\frac{n-4}{2}, \frac{n-4}{2}) \)-card in which there is a \( 3^+ \)-vertex at distance 2 from the centroid.

**Proof.** Let \( T \) be a tree with \((n-3)\)-deck \( D \).

**Case 1:** \( T \) is a \( (\frac{n-2}{2}, \frac{n-2}{2}) \)-tree that has an \( a, b \)-spear with \( b \geq a \geq 4 \). Let \( z \) be the centroid of \( T \), with nontrivial pieces \( X \) and \( X' \), where \( \langle y, x, z, x', y' \rangle \) is a path with \( x \in V(X) \) and \( x' \in V(X') \). By Corollary 3.5, the centroid of every connected card is in \( \{x, z, x'\} \), none of which is a full vertex, so (b) holds. Also, all \( (\frac{n-4}{2}, \frac{n-6}{2}) \)-cards have 1-burls as centroidss and hence must have centroid \( z \), so one distance to a non-2-vertex is at least \( \min\{a, (n-6)/2\} \) and the other is at least \( \min\{b, (n-6)/2\} \). Since \( (n-6)/2 \geq 4 \) when \( n \geq 14 \), (a) holds.

Every \( (\frac{n-4}{2}, \frac{n-4}{2}) \)-card has centroid in \( \{x, z, x'\} \), but none of these has a \( 3^+ \)-vertex at distance 2, so (c) holds.

**Case 2:** \( T \) is a bicentroidal tree. Define \( z, z', Y, Y' \) as usual. By Lemma 7.4, we may assume \( d_T(z') = 2 \) and that \( z \) is a 1-burl or has degree 2.

If \( d_T(y) = 2 \), with \( x \) and \( x' \) the neighbors of \( z \) and \( z' \) in \( Y \) and \( Y' \), then the centroid of a \( (\frac{n-4}{2}, \frac{n-6}{2}) \)-card can only be \( x \) or \( x' \), since it must be a 1-burl. By symmetry, let it be \( x \). Deleting a leaf from \( Y' \) and two vertices from \( Y \) while keeping degree at least 3 at \( x \) yields a \( (\frac{n-4}{2}, \frac{n-4}{2}) \)-card with centroid \( z' \) having distance 2 from \( z' \) to a \( 3^+ \)-vertex, violating (c).

Hence we may assume that \( z \) is a 1-burl. If \( T \) has a \( 3^+ \)-vertex \( v \) within distance 3 of \( z \), then we get a \( (\frac{n-4}{2}, \frac{n-6}{2}) \)-card with centroid \( z \) that violates (a) by deleting one vertex from one of \( \{Y, Y'\} \) and two from the other while keeping degree at least 3 at \( v \). This can be done whenever \( n/2 - 1 \geq 7 \).

Hence there is no \( 3^+ \)-vertex within distance 3 of \( z \). Now delete three vertices from \( Y \) while keeping the leaf neighbor of \( z \). This yields a \( (\frac{n-4}{2}, \frac{n-6}{2}) \)-card with centroid \( x' \) that violates (a) when \( n/2 - 3 \geq 3 \) by having a \( 3^+ \)-vertex at distance 2 from the centroid.
We now know the cost of an \( n \)-vertex tree from its \( (n - 3) \)-deck when \( n \) is even (and \( n \geq 20 \)). In addition, we have reconstructed all trees with \( n \) even except bicentroidal trees and \( (\frac{n-2}{2}, \frac{n-2}{2}) \)-trees, and when \( T \) is a \( (\frac{n-2}{2}, \frac{n-2}{2}) \)-tree we know more about the structure of \( T \). This information is available for reconstructing \( T \).

**Theorem 7.10.** For \( n \geq 20 \) with \( n \) even, \( (\frac{n-2}{2}, \frac{n-2}{2}) \)-trees are 3-reconstructible.

**Proof.** Let \( D \) be the \( (n - 3) \)-deck of an \( (\frac{n-2}{2}, \frac{n-2}{2}) \)-tree \( T \). By the recent lemmas, we know that every graph with this deck is an \( (\frac{n-2}{2}, \frac{n-2}{2}) \)-tree. The centroid \( z \) of \( T \) is a 1-burl, so it suffices to determine \( X \) and \( X' \) with their roots.

Every connected card has cost \((n - 4)/2\), and the centroid is always in \( \{x, z, x'\} \). The centroid is \( x \) precisely when three vertices outside \( X \) are deleted, and similarly for \( x' \) and \( X' \). When \( z \) is the centroid of a connected card \( C \), the subcost is \((n - 4)/2\) if \( d_C(z) = 2 \) and \((n - 6)/2\) if \( d_C(z) = 3 \). When a connected card has subcost at most \((n - 8)/2\), the centroid is \( x \) or \( x' \) and is a full vertex. When every connected card has subcost at least \((n - 6)/2\), neither \( x \) nor \( x' \) is a full vertex. We consider four cases for \( x \) and \( x' \).

**Case 1:** \( x \) or \( x' \) is a full vertex. This is the case where some connected card has subcost at most \((n - 8)/2\). The largest piece has \((n - 4)/2\) vertices and is rooted at \( z \). If deleting the vertices of the largest piece in such cards does not always yield the same rooted tree, then both \( x \) and \( x' \) are full vertices, and over these cards we see both \( X \) and \( X' \) with their roots.

Otherwise, exactly one of \( x \) and \( x' \) is a full vertex or \( X \) and \( X' \) are the same as rooted trees. By symmetry, we may assume that \( x \) is a full vertex and these connected cards give us \( X \) when the vertices of the largest piece are deleted. Also, over these cards the largest piece gives the rc3-deck of a rooted tree (with \( z \) prepended at the root), possibly twice. Since we know \( X \), we know its rc3-deck. If we obtain two copies of that rc3-deck, then \( X = X' \) and we have reconstructed \( T \). Otherwise, Lemma 3.12 applies to complete the reconstruction.

**Case 2:** \( x \) and \( x' \) are both 1-burls. Since no connected card has subcost at most \((n - 8)/2\), we are not in Case 1. Since neither \( x \) nor \( x' \) is a full vertex, we can obtain \( (\frac{n-4}{2}, \frac{n-4}{2}) \)-cards in which the centroid and its neighbors all have degree 2 by deleting the leaf neighbors of all of \( \{x, z, x'\} \). If they do not all have leaf neighbors, the we obtain more than one such card by deleting other leaves. Hence this case holds only if there is exactly one \( (\frac{n-4}{2}, \frac{n-4}{2}) \)-card in which the centroid and its neighbors have degree 2, and we then reconstruct \( T \) by giving leaf neighbors to those three vertices.

**Case 3:** One of \( \{x, x'\} \) has degree 2, and the other is a 1-burl. To distinguish the remaining cases, here there is a \( (\frac{n-4}{2}, \frac{n-6}{2}) \)-card in which the centroid has a neighbor that is a 1-burl. That cannot occur when \( d_T(x) = d_T(x') = 2 \), because any connected card with a 2-vertex as centroid is a \( (\frac{n-4}{2}, \frac{n-4}{2}) \)-card.

By symmetry, we may assume that \( x' \) is a 1-burl and \( d_T(x) = 2 \). Let \( y' \) be the non-leaf neighbor of \( x' \) in \( X' \), and let \( Y' \) be \( X' \) minus \( x' \) and its leaf neighbor \( x' \). Since \( d_T(x) = 2 \),
there is a \((\frac{n-4}{2}, \frac{n-4}{2})\)-card whose centroid has two \(3^+\)-neighbors if and only if \(y'\) is a \(3^+\)-vertex and \(x'\) is the centroid. Let \(\mathcal{D}'\) be the multiset of such cards. Cards in \(\mathcal{D}'\) arise only by deleting three vertices from \(X\). The piece with \((n-4)/2\) vertices always contains \(x\), and the piece having \((n-6)/2\) vertices is always \(Y'\). Hence we know which piece is which and know both \(Y'\) and the rc3-deck of \(X\). By Lemma 3.12, we can reconstruct \(X\) and \(T\).

If \(\mathcal{D}' = \emptyset\), then \(d_T(y') = 2\). Let a piece of a \((\frac{n-4}{2}, \frac{n-4}{2})\)-card be furry if both nonleaf vertices within distance 2 of the centroid are 1-burls. Let \(\mathcal{D}''\) be the multiset of \((\frac{n-4}{2}, \frac{n-4}{2})\)-cards having a furry piece. Since \(d_T(x) = d_T(y') = 2\), the center of such a card must be \(x\). The cards in \(\mathcal{D}''\) arise only by deleting three vertices of \(Y'\), and in each such card one piece is \(X - x\). If no card in \(\mathcal{D}''\) has two furry pieces, then we know which piece is \(X - x\) in each such card and collect the other piece from these cards to obtain the rc3-deck of \(Y'\). By Lemma 3.12, we reconstruct \(Y'\) and \(T\).

If some card in \(\mathcal{D}''\) has two furry pieces, then consider the cards in \(\mathcal{D}''\) such that one piece is furry and the other has no 1-burl within distance 2 of the centroid. Like all cards in \(\mathcal{D}''\), these have centroid \(x\). They are obtained by deleting the leaf neighbors of \(z\) and \(x'\) plus one vertex of \(Y'\), and we can tell which piece is which. One piece is \(X - x\), and the other provides the rc1-deck of \(Y'\). By Theorem 2.1, we reconstruct \(Y'\) and \(T\).

**Case 4:** \(d_T(x) = d_T(x') = 2\). This is the remaining case. All connected cards have centroid in \(\{x, z, x'\}\). Now let \(\mathcal{D}'\) be the multiset of \((\frac{n-4}{2}, \frac{n-4}{2})\)-cards in which at least one piece has a \(3^+\)-vertex as root. Since \(d_T(x) = d_T(x') = 2\), these are the cards obtained by deleting three vertices of \(X\) or three vertices of \(X'\), and the centroid is \(x'\) or \(x\).

Again define \(y\) and \(y'\) as above. If no card in \(\mathcal{D}'\) has \(3^+\)-vertices as roots of both pieces, then \(d_T(y) = d_T(y') = 2\). Over all cards in \(\mathcal{D}'\), by taking the centroid together with the piece in which it has a neighbor of degree 2, we see \(X\) and \(X'\) with their roots.

Hence we may assume that \(y\) or \(y'\) has degree at least 3. If in every card in \(\mathcal{D}'\) the centroid has two \(3^+\)-neighbors, then \(y\) and \(y'\) both have degree at least 3 (this statement needs \((n-2)/2 \geq 4\); that is, \(n \geq 10\)). In this case, connected cards having a centroid of degree 2 with exactly one neighbor of degree 3 are obtained by deleting the leaf neighbor of \(z\) and two vertices of \(X\) or two vertices of \(X'\). Over all such cards, we see both \(X\) and \(X'\) using the centroid plus the piece in which the neighbor of the centroid has degree at least 3.

Finally, by symmetry we may assume \(d_T(y) \geq 3\) and \(d_T(y') = 2\). Now the cards in \(\mathcal{D}'\) having both pieces rooted at \(3^+\)-vertices are those arising by deleting three vertices of \(X'\). All have \(X - x\) as one piece, and the other piece gives the rc3-deck of \(X'\). If the rc3-cards of \(X'\) are not all equal, then we know which piece is which and reconstruct by Lemma 3.12. When they are all the same, the common rc3-card of \(X'\) is a rooted broom or rooted path, so we can distinguish \(X - x\) from the rc3-card of \(X'\) unless \(X - x\) is a rooted broom or rooted path. Since \(d_T(y) \geq 3\), this happens only if \(X\) is a star. That also distinguishes \(X - x\) from the piece obtained from \(X'\), since \(d_T(x') = d_T(y') = 2\). Hence we again know \(X - x\) and can
8 Trees with Cost $(n - 1)/2$

When $n$ is odd, the remaining trees all have cost $(n - 1)/2$, but we need to recognize the subcost to facilitate reconstruction. Those with small subcost are easy to reconstruct.

**Theorem 8.1.** For $n \geq 15$ with $n$ odd, $n$-vertex trees with cost $(n - 1)/2$ and subcost at most $(n - 7)/2$ are reconstructible.

**Proof.** By Lemmas 3.6, 4.1, and 4.2, we recognize this family. Define $z, X, x, X'$ as usual in such a tree $T$.

**Case 1:** $c'(T) \leq (n - 9)/2$. We recognize this case by the existence of a $(\frac{n-7}{2}, \frac{n-b}{2})$-card with $b \geq 9$. Such cards have centroid $z$ and arise by deleting three vertices from $X$. The unique largest piece is an rc3-card of $X$. Deleting it always leaves $T - V(X)$, rooted at $z$, and we know which piece is the rc3-card of $X$. By Lemma 3.12, we reconstruct $X$ and $T$.

**Case 2:** $c'(T) = (n - 7)/2$. This case occurs when there is a $(\frac{n-7}{2}, \frac{n-7}{2})$-card but no $(\frac{n-7}{2}, \frac{n-b}{2})$-card with $b \geq 9$. Again these cards arise by deleting three vertices from $X$. The centroid is $z$, which is a 3-burl both in $T$ and in these cards. Thus when $(n - 7)/2 \geq 4$ there are exactly two pieces with $(n - 7)/2$ vertices. In these cards, one large piece is the same and is $X'$. If another does not always appear, then we have determined $X'$ and have the rc3-deck of $X$ to reconstruct $T$ as in Case 1.

If all the $(\frac{n-7}{2}, \frac{n-7}{2})$-cards are the same, then the rc3-cards of $X$ are pairwise isomorphic. By Lemma 2.6, they are a rooted broom or a path. If $X'$ is not a rooted broom or a path, then we know which piece is which and reconstruct as in Case 1.

If $X'$ and the common rc3-card of $X$ are identical, then it doesn’t matter to which we apply Lemma 3.12. If they are not identical, then at least one is a broom. Since we can reconstruct the degree list of $T$, we know when the broom with more than one leaf is the rc3-card of $X$ and can reconstruct $X$ by adding three sibling leaves to that broom. Otherwise we reconstruct as in Case 1. \qed

Recognition of $(\frac{n-1}{2}, \frac{n-5}{2})$-trees uses cards with restricted structure.

**Definition 8.2.** A thin card is a connected card having a largest piece rooted at a 2-vertex whose neighbor in the piece is a 2-burl.

**Lemma 8.3.** For $n \geq 15$, an $n$-vertex tree $T$ with $c(T) = (n - 1)/2$ and $c'(T) \geq (n - 5)/2$ is an $(\frac{n-1}{2}, \frac{n-5}{2})$-tree if and only if $T$ has no balanced cards or all of the following hold:
(a) $T$ has a $(\frac{n-5}{2}, \frac{n-7}{2})$-card and has a $(\frac{n-5}{2}, \frac{n-7}{2})$-card in which the root of the piece with $(n-7)/2$ vertices has degree 2, and

(b) Every bicentroidal card has a centroid with degree 2, and

(c) $T$ either has no thin $(\frac{n-5}{2}, \frac{n-7}{2})$-card or does have a thin $(\frac{n-5}{2}, \frac{n-7}{2})$-card.

Proof. Define $z, x, X, x'$ as usual; $X$ and $X'$ have the same size when $c'(T) = (n-1)/2$.

Case 1: $c'(T) = (n-5)/2$. If $T$ has no balanced cards, then there is nothing to prove, so assume that $T$ has a balanced card, with two components of order $(n-3)/2$. Since $X'$ has only $(n-5)/2$ vertices, a balanced card must contain $z$ and lack both vertices of the burl. Hence it also must lack $x$, and then $x$ must be a 2-vertex so $X - x$ is connected.

We now obtain a $(\frac{n-5}{2}, \frac{n-5}{2})$-card with centroid $z$ by deleting two vertices from $X$ and one leaf from the burl. We obtain a $(\frac{n-5}{2}, \frac{n-7}{2})$-tree with centroid $z$ by deleting three vertices from $X$, and the neighbor of $z$ in the piece with $(n-7)/2$ vertices is the 2-vertex $x$. This requires $(n-7)/2 \geq 2$, or $n \geq 11$. Thus (a) holds.

Since $T - V(X')$ has $(n+5)/2$ vertices, it cannot be reduced to $(n-3)/2$ by deleting three vertices. Hence every bicentroidal card has $x$ and $z$ as centroids. Since the balanced card requires $d_T(x) = 2$, every bicentroidal card has a centroid with degree 2, and (b) holds.

For (c), we obtain a thin $(\frac{n-5}{2}, \frac{n-7}{2})$-card when $T$ has a thin $(\frac{n-5}{2}, \frac{n-5}{2})$-card $C$. Let $u$ be the centroid of $C$, and let $w$ be a 2-burl at distance 2 from $u$ such that their common neighbor $v$ is a 2-vertex. Because it is a 3$+$-vertex whose deletion leaves two components of order at least $(n-5)/2$ and $T$ is a $(\frac{n-5}{2}, \frac{n-5}{2})$-tree, the vertex $u$ must be $z$ in $T$. Since $z$ is a 2-burl in $T$ and a 1-burl in $C$, the card $C'$ arises from $T$ by deleting two vertices of $X$ and a leaf from the burl of $T$.

Now $v \in \{x, x'\}$. If $v = x$, then $w \in V(X)$ and we obtain a thin $(\frac{n-5}{2}, \frac{n-7}{2})$-card by deleting the same two vertices of $X$ and one leaf from $X'$. If $v = x'$, then $w \in V(X')$ and nothing was deleted from $X'$ to form $C$; obtain a thin $(\frac{n-5}{2}, \frac{n-7}{2})$-card by deleting three vertices of $X$. Thus (c) holds.

Case 2: $c'(T) = (n-3)/2$. Now $z$ is a 1-burl. Deleting $z$, its leaf neighbor, and a leaf of $X$ yields a balanced card. Hence we may assume that (a), (b), and (c) all hold.

From (a), let $C$ be a $(\frac{n-5}{2}, \frac{n-7}{2})$-card with centroid $u$ whose neighbor in the piece with $(n-7)/2$ vertices has degree 2. The only choice for $u$ is $x$, since otherwise too many vertices must be deleted to get the pieces down to the desired sizes. In a $(\frac{n-5}{2}, \frac{n-7}{2})$-card, the centroid is a 2-burl, so $x$ is a 2-burl. Now deleting one vertex from $X$ and two vertices from $X'$ other than the leaf neighbor of $z$ yields a bicentroidal tree with centroids $z$ and $x$, both having degree at least 3. Hence (b) fails.

Case 3: $c'(T) = (n-1)/2$. Now $d_T(z) = 2$. Deleting $z$ and one leaf from each of $X$ and $X'$ yields a balanced card. Hence we may assume that (a), (b), and (c) all hold.

As in Case 2, (a) provides a $(\frac{n-5}{2}, \frac{n-7}{2})$-card $C$ with centroid $u$ whose neighbor in the piece with $(n-7)/2$ vertices has degree 2. Since $u$ is a 2-burl in $C$, vertex $u$ must be $x$ or
x', not z; by symmetry let it be x. The piece of C with \((n - 7)/2\) vertices is contained in X, since x is a 2-burl in C, and C arises by deleting three vertices outside X.

Hence the part of X outside the burl of x is too small to allow x to be the centroid of a \((\frac{n-5}{2}, \frac{n-5}{2})\)-card C' provided by (a). Also z cannot be the centroid of C', since the centroid has degree 3 in a \((\frac{2-n}{2}, \frac{n-5}{2})\)-card. The remaining possibility is x'. Since X' has \((n - 1)/2\) vertices and the centroid of C' is a 1-burl, C' must arise by deleting three vertices in X, and x' is a 1-burl in T. With \((n - 1)/2 - 3 \geq 4\), we can delete three vertices of X while leaving x as a 2-burl, and then C' is a thin \((\frac{n-5}{2}, \frac{n-5}{2})\)-card.

We have shown that z and x' are 1-burls in T, while x is a 2-burl. Now (c) requires T to have a thin \((\frac{n-5}{2}, \frac{n-7}{2})\)-card. The centroid of this card is a 2-burl, so it cannot be x' or z and must be x. Since X - x has only \((n - 7)/2\) vertices outside the burl of x, the neighbor of x in the large piece must be z. However, since x' is a 1-burl in T, we cannot complete a thin \((\frac{n-5}{2}, \frac{n-5}{2})\)-card.

\[\square\]

### Theorem 8.4.
For \(n \geq 19\) with \(n\) odd, \(n\)-vertex \((\frac{n-1}{2}, \frac{n-5}{2})\)-trees are reconstructible.

**Proof.** By Lemma 8.3 and earlier results, we can recognize from the deck that all reconstructions are \((\frac{n-1}{2}, \frac{n-5}{2})\)-trees. We determine the reconstruction T uniquely. Label z, x, X, x', X' as usual. The centroid z is a 2-burl. Every \((\frac{n-5}{2}, \frac{n-7}{2})\)-card arises by deleting three vertices of X, leaving z as centroid; this tells us whether the vertices of the burl in T are adjacent. There are balanced cards if and only if \(d_T(x) = 2\), so we can tell whether x is a 3°-vertex.

**Case 1:** \(d_T(x) \geq 3\). In this case, x and z are the centroids of all bicentroidal cards, which are obtained by deleting one vertex of X and two vertices outside X.

There are no bicentroidal cards with two centroids of degree 2 if and only if x is a full vertex. The bicentroidal cards having one centroid of degree 2 are then obtained by deleting the burl of T and one vertex of X; the centroid of degree 2 is z. Over all such cards, the branch containing the centroid of degree at least 3 gives us the rcl-deck of X, and we can reconstruct X by Lemma 2.1. The other branch is X' with z prepended at the root.

If \(d_T(x) \geq 3\) and x is not a full vertex, then x is a 1-burl. We recognize this by having a bicentroidal card C with two centroids of degree 2, obtained by deleting the leaf neighbor \(\hat{x}\) of x and the burl of T, but this card does not determine which branch is which. Now a bicentroidal card in which one centroid has degree 2 and the other is a 2-burl is obtained only by deleting the leaf neighbor of x and two vertices of X'. We obtain X from the branch whose root (which is x) has degree 2 by adding a leaf neighbor to that root. We then return to C and know which branch in C is \(X - \hat{x}\), so we know which centroid receives one leaf neighbor and which receives the 2-burl.

**Case 2:** \(d_T(x) = 2\). Deleting at least two vertices from X yields unicentroidal cards with centroid z, while deleting at least two vertices outside X yields bicentroidal cards. All \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards are obtained by deleting two vertices from X and one from the burl,

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yielding centroid \( z \) as a 1-burl. In every such card, one of the large pieces is \( X' \). If exactly one piece with \((n - 5)/2\) vertices is always present, then that piece is \( X' \). We then also know the numbers of leaves in \( X', T, \) and \( X \) (since we already know the burl). The large pieces in these cards after removing \( X' \) form the rc2-deck of \( X \) (or two copies of it if the burl has two leaves), and by Theorem 2.3 we can reconstruct \( X \).

Hence we may assume that all \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards have the same two pieces with \((n - 5)/2\) vertices. This requires that the rc2-cards of \( X \) are pairwise isomorphic. Hence \( X \) is a rooted broom or \( P_{(n-1)/2}'' \). Since the other piece is always \( X' \), we know \( X' \) if it is neither a rooted broom nor \( P_{(n-1)/2}'' \). Suppose otherwise.

Now consider the balanced cards of \( T \). Since every balanced card consists of two components with \((n - 3)/2\) vertices, the balanced cards are formed by deleting \( x \) and two vertices outside \( X \). There are multiple such cards, but every one has \( X - \{x\} \) has a component. This is a broom or is \( S(1, 2, (n - 11)/2) \) (when the rooted \( X \) is \( P_{(n-1)/2}'' \)), and this common component is distinguishable from the other components. Hence we now know \( X \) and its rc2-cards, so we can return to the \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards and determine \( X' \).

\[ \text{Lemma 8.5.} \text{ For } n \geq 11, \text{ if } D \text{ is the } (n-3)\text{-deck of a tree } T \text{ with cost } (n-1)/2 \text{ and subcost at least } (n-3)/2, \text{ then the deck determines whether } T \text{ has subcost } (n-3)/2 \text{ or } (n-1)/2. \]

**Proof.** We consider cases according to \( D \). The cases will be exhaustive and determine \( c'(T) \), so that every reconstruction from \( D \) has the same subcost. For \( j > 1 \), the hypothesis of Case \( j \) includes the property that Case \( i \) fails when \( i < j \). Define \( z, x, X, x', X' \) as usual.

**Case 1:** \( D \) has no \((\frac{n-5}{2}, \frac{n-5}{2})\)-card. If \( T \) is a \((\frac{n-1}{2}, \frac{n-3}{2})\)-tree, then deleting two vertices from \( X \) and one from \( X' \) yields a \((\frac{n-5}{2}, \frac{n-5}{2})\)-card. Hence \( T \) must be a \((\frac{n-1}{2}, \frac{n-1}{2})\)-tree.

**Case 2:** In some bicentroidal card \( C \), both centroids are \( 3^+ \)-vertices. If \( T \) is a \((\frac{n-1}{2}, \frac{n-1}{2})\)-tree, then the centroid \( z \) is a 2-vertex. To make a card in which \( z \) is not a centroid, we must delete three vertices from one piece of \( T \), leaving it with \((n - 7)/2\) vertices. Adding \( z \) and its other neighbor \( u \) yields \((n - 3)/2\) vertices, but this will be a branch in a bicentroidal card only if \( d_T(u) = 2 \). Hence if \( T \) has a bicentroidal card with both centroids being \( 3^+ \)-vertices, then \( T \) is a \((\frac{n-1}{2}, \frac{n-3}{2})\)-tree.

**Case 3:** Some connected card has a full vertex. Let an optimal card be a connected card minimizing \( r \), where in a unicentroidal tree \( r \) is the distance from the centroid to a closest full vertex, and in a bicentroidal card \( r \) is the average distance from the two centroids to a closest full vertex. Since \( d_T(z) \leq 3 \), such a full vertex \( v \) cannot be \( z \). Hence \( v \) is in \( X \) or \( X' \), and distance from \( v \) to a centroid is minimized by moving the average position of the centroid(s) away from \( z \) toward \( v \). Hence we may consider an optimal card \( C \) that arises by deleting three vertices outside \( X \) or outside \( X' \), respectively.

If \( T \) is a \((\frac{n-1}{2}, \frac{n-3}{2})\)-tree, then \( z \) is a 1-burl. Since Case 2 does not occur, \( x \) and \( x' \) must be 2-vertices, so \( v \notin \{x, z, z'\} \). Now whether \( v \in X \) or \( v \in X' \), all optimal cards are bicentroidal.
In some of them, the leaf neighbor of \( z \) is not deleted, leaving a 1-burl within distance 2 of the centroid that is nearer \( v \) in the branch not containing \( v \). Therefore, if some optimal card is unicentroidal or \( T \) has no such bicentroidal card, then \( T \) is a \( \left( \frac{n-1}{2}, \frac{n-1}{2} \right) \)-tree.

Hence we may assume that every optimal card is bicentroidal and that some such card has a 3-vertex within distance 2 of the centroid nearer to \( v \) in the branch not containing \( v \). If \( T \) is a \( \left( \frac{n-1}{2}, \frac{n-1}{2} \right) \)-tree, then by symmetry we may suppose \( v \in X \). Now avoiding optimal cards that are unicentroidal requires \( d_T(x) = 2 \). The centroid of an optimal card that is farther from \( v \) is \( x \), and its noncentroidal neighbor in the card is \( z \). These both have degree 2, so the claimed 3\(^+\)-vertex is missing. Thus in this situation \( T \) must be a \( \left( \frac{n-1}{2}, \frac{n-3}{2} \right) \)-tree.

**Case 4:** \( D \) has 1, \( b \)-type. We pause to establish notation for the remaining cases. Since we are not in Case 3, every reconstruction \( T \) has no full vertex, meaning that it has maximum degree 3 and a longest path contains an endpoint of every edge. Hence \( T \) contains a unique path \( \langle w, y, x, z, x', y', w' \rangle \) with non-leaves \( w \) and \( w' \) such \( x \) is the root of a largest piece \( X \) of \( T \) and \( x' \) is the root of the other nontrivial piece \( X' \).

Since we are not in Case 1, \( T \) has a \( \left( \frac{n-1}{2}, \frac{n-2}{2} \right) \)-card, and hence \( D \) has \( a, b \)-type for some \( (a, b) \) with \( 1 \leq a \leq b \). Since we are not in Case 2, \( D \) contains no bicentroidal card whose centroids are both \( 3^+ \)-vertices. Hence if \( T \) is a \( \left( \frac{n-1}{2}, \frac{n-3}{2} \right) \)-tree, then \( x \) and \( x' \) are both \( 2 \)-vertices, and all \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-cards of \( T \) have centroid \( z \). Thus if \( D \) has \( (1, b) \)-type, then \( T \) is a \( \left( \frac{n-1}{2}, \frac{n-1}{2} \right) \)-tree.

**Case 5:** \( D \) has 2, \( 2 \)-type. Suppose that both a \( \left( \frac{n-1}{2}, \frac{n-2}{2} \right) \)-tree \( T \) with centroid \( z \) and a \( \left( \frac{n-1}{2}, \frac{n-1}{2} \right) \)-tree \( \tilde{T} \) with centroid \( \tilde{z} \) have \( (n-3) \)-deck \( D \). We have the path \( \langle w, y, x, z, x', y', w' \rangle \) with pieces \( X \) and \( X' \) in \( T \) and \( \langle \tilde{w}, \tilde{y}, \tilde{x}, \tilde{z}, \tilde{x}', \tilde{y}', \tilde{w}' \rangle \) with pieces \( \tilde{X} \) and \( \tilde{X}' \) in \( \tilde{T} \).

In \( T \), a card with a \( 2, 2 \)-spear requires \( d_T(y) = d_T(y') = 3 \). Deleting their leaf neighbors and one other leaf of \( X \) yields a \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-card \( C \) in \( D \) having an \( i, j \)-spear with \( j \geq i \geq 3 \).

In \( \tilde{T} \), every \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-card has centroid \( \tilde{x} \) or \( \tilde{x}' \), by deleting three vertices from \( \tilde{X}' \) or three vertices from \( \tilde{X} \). If \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-cards having \( 2, 2 \)-spears arise from \( \tilde{T} \) both with centroid \( \tilde{x} \) and with centroid \( \tilde{x}' \), then every \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-card of \( \tilde{T} \) has a 1-burl within distance 2 of the centroid, because \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-cards arise from \( \tilde{T} \) only by deleting vertices in one piece. This contradicts the existence of \( C \) obtained above.

Therefore, by symmetry we may assume that all \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-cards of \( \tilde{T} \) with \( 2, 2 \)-spears have centroid \( \tilde{x} \), with \( d_{\tilde{T}}(w) = d_{\tilde{T}}(x') = 3 \) and \( d_{\tilde{T}}(\tilde{w'}) = 2 \). With \( (n-1)/2 \geq 9 \), there is a \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-card \( C' \) of \( \tilde{T} \) with centroid \( \tilde{x}' \) such that one piece has 1-burls at distance 2 and 4 from the centroid.

Returning to \( T \), the card \( C' \) requires a 1-burl \( v \) at distance 4 from \( z \). Now we form a bicentroidal card. If \( v \in V(X') \), delete three vertices from \( X \); if \( v \in V(X) \), delete one vertex from \( X \) and two from \( X' \). In either case, keep the leaf neighbors of \( \{ y, z, y' \} \). We obtain a bicentroidal card such that each branch has two 1-burls within distance 3 of the root; distance 0 and 2 in the branch containing \( z \), distance 1 and 3 in the branch containing \( v \).
However, \( \tilde{T} \) has no such bicentroidal card. Bicentroidal cards arise with centroids \( \tilde{x} \) and \( \tilde{z} \) by deleting one vertex from \( \tilde{X} \) and two from \( \tilde{X}' \); they arise with centroids \( \tilde{x}' \) and \( \tilde{z} \) by deleting one vertex from \( \tilde{X}' \) and two from \( \tilde{X} \). Since \( d_T(z) = 2 \), in either case the required 1-burls in the branch containing \( \tilde{x}' \) are \( \tilde{x}' \) and \( \tilde{w}' \), but \( \tilde{w}' \) is a 2-vertex in \( \tilde{T} \).

Thus in this case \( T \) and \( \tilde{T} \) cannot both exist, and the deck determines that all reconstructions are \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-trees or all are \( (\frac{n-1}{2}, \frac{n-1}{2}) \)-trees.

**Case 6:** \( D \) has 2, \( b \)-type with \( b \geq 3 \). Let \( C \) be a \( (\frac{n-5}{2}, \frac{n-5}{2}) \)-card having a 2, \( b \)-spear.

If \( T \) is a \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-tree, then \( y \) or \( y' \) is a 1-burl in \( C \), and the nearest 1-burl in the other piece is farther from \( z \). Let a 2-centroid be a centroid of degree 2 in a bicentroidal card. Bicentroidal cards with \( x' \) as a 2-centroid arise by deleting three vertices from \( X \). They arise with \( x \) as a 2-vertex by deleting one vertex from \( X \) and two vertices outside \( X \) (and they can avoid deleting the leaf neighbor of \( z \)). Among bicentroidal cards of these two types (which is which depends on which of \( y \) and \( y' \) is a 1-burl), in one the 2-centroid has two 3-neighbors, and in the other there are 1-burls in the large piece at distance 1 and 3 from the 2-centroid.

If \( T \) is a \( (\frac{n-1}{2}, \frac{n-1}{2}) \)-tree, then by symmetry we may assume that \( x \) is the centroid of some \( (\frac{n-5}{2}, \frac{n-5}{2}) \)-card having a 2, \( b \)-spear. The 1-burl at distance 2 from \( x \) may be \( w \) in \( X \) or \( x' \) in \( X' \). If it is \( w \), then we cannot obtain a bicentroidal card with a 2-centroid having two 3-neighbors, since when \( x \) is a 1-burl no card has \( y \) as the centroid. Hence we may assume that \( x' \) is a 1-burl and \( d_T(y) = d_T(w) = 2 \). Now \( z \) is a 2-centroid in every bicentroidal card. Since \( d_T(z) = 2 \), a bicentroidal card with \( x \) or \( x' \) as a 2-centroid cannot be either of the special types described above. Since \( d_T(w) = 2 \), having \( z \) as a 2-centroid with 1-burls at distance 1 and 3 in the large piece requires \( d_T(w') = 3 \). This leads to a contradiction, since deleting three vertices from \( X \) without deleting the leaf neighbor of \( x \) would yield a \( (\frac{n-5}{2}, \frac{n-5}{2}) \)-card with centroid \( x' \) having a 2, 2-spear.

Hence \( T \) is a \( (\frac{n+1}{2}, \frac{n+3}{2}) \)-tree if \( D \) contains the two types of bicentroidal cards specified, and otherwise \( T \) is a \( (\frac{n-1}{2}, \frac{n-1}{2}) \)-tree.

**Case 7:** \( D \) has \( a, b \)-type with \( a \geq 3 \). If \( T \) is a \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-tree, then \( z \) is a 1-burl and all of \( \{y, x, x', y'\} \) have degree 2. Here \( (\frac{n+5}{2}, \frac{n+5}{2}) \)-cards arise by deleting two vertices from \( X \) and one from \( X' \). All other connected cards are bicentroidal, with centroids \( \{y, x\}, \{x, z\} \), or \( \{z, x'\} \) when the number of vertices deleted from \( X \) is 0, 1, or 3, respectively.

Since all of \( \{y, x, x', y'\} \) have degree 2, the only bicentroidal cards with two 2-centroids in which one has a 3-neighbor arise by deleting three vertices from \( X' \). Here \( x \) has \( z \) as a 3-neighbor; \( y \) may also have a 3-neighbor if \( a = 3 \), but in any case the branch containing \( y \) is the same in all such cards.

If \( T \) is a \( (\frac{n-1}{2}, \frac{n-1}{2}) \)-tree, then the centroid of a \( (\frac{n+5}{2}, \frac{n+5}{2}) \)-card must be \( x \) or \( x' \); we may assume by symmetry that \( x \) is a 1-burl. Since \( D \) has \( a, b \)-type with \( a \geq 3 \), all of \( \{w, y, z, x'\} \) have degree 2. Let \( \hat{x} \) denote the leaf neighbor of \( x \). All \( (\frac{n+5}{2}, \frac{n+5}{2}) \)-cards arise by deleting three vertices from \( X' \) and none from \( X \). All other connected cards are bicentroidal, with
centroids \( \{x, z\} \), \( \{z, x'\} \), or \( \{x', y'\} \) when the number of vertices deleted from \( X \) is 1, 2, or 3, respectively. All the \((n-5, n-5)\)-cards have \( X - \{x, \hat{x}\} \) as one piece. Hence if the \((n-5, n-5)\)-cards do not have a common piece, then \( T \) is a \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-tree.

If the \((n-5, n-5)\)-cards are all the same, then by Lemma 2.6 and the lack of full vertices in \( T \), the \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards must be copies of \( P_{n-4} \) plus a leaf neighbor of the middle vertex. Hence \( b \geq a \geq (n-11)/2 \), and \((n-11)/2 \geq 5 \) when \( n \geq 21 \). Now consider the bicentroidal cards described above. If \( T \) is a \( (\frac{n-1}{2}, \frac{n-1}{2}) \)-tree, then some bicentroidal card has a 1-burl at distance 2 from the nearest centroid. If \( T \) is a \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-tree, then there is no such card. Hence in this case the deck determines whether \( T \) is a \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-tree or a \( (\frac{n-1}{2}, \frac{n-1}{2}) \)-tree.

Finally, suppose that the \((n-5, n-5)\)-cards have a common largest piece but are not the same. If \( T \) is a \( (\frac{n-5}{2}, \frac{n-5}{2}) \)-tree, then Lemma 2.6 and the lack of full vertices again make the common piece a path. Hence \( T \) must be a \( (\frac{n-1}{2}, \frac{n-1}{2}) \)-tree if the common piece is not a path.

In the remaining case, the common piece is a path, and some \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards are not paths. Hence \( a < (n-11)/2 \leq b \). If \( T \) is a \( (\frac{n-1}{2}, \frac{n-1}{2}) \)-tree, then the common piece in the \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards is \( X - \{x, \hat{x}\} \), and some such cards have a 1-burl in \( X' \) at distance \( a \) from the centroid \( x \). Deleting three vertices from \( X \) (but not \( \hat{x} \)) yields a bicentroidal tree with centroids \( x' \) and \( y' \) having 1-burls at distances 2 and \( a - 3 \) from the roots in the two branches. If \( T \) is a \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-tree, because \( z \) must be one of those two 1-burls, and in every bicentroidal card \( z \) is within distance 1 of the root of its branch. Hence in this case also the deck determines whether \( T \) is a \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-tree or a \( (\frac{n-1}{2}, \frac{n-1}{2}) \)-tree.

\textbf{Lemma 8.6.} Let \( D \) be the \((n-3)\)-deck of an \( n \)-vertex tree \( T \) with \( c(T) = (n-1)/2 \) and \( c'(T) \geq (n-3)/2 \) that has \( t \) leaves. If \( D \) has at most \( t - 1 \) balanced cards, then \( T \) is a \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-tree. Also, if \( T \) is a \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-tree, then \( D \) has at least \( t - 1 \) balanced cards if and only if the neighbor of the centroid in the largest piece has degree 2.

\textit{Proof.} Since \( n \) is odd, \( T \) is unicentroidal; define \( z, X, x, X', x' \) as usual.

If \( T \) is a \( (\frac{n-1}{2}, \frac{n-1}{2}) \)-tree, then balanced cards arise by deleting \( z \), one leaf of \( X \) and one leaf of \( X' \). If \( T \) has \( t_1 \) leaves in \( X \) and \( t_2 \) leaves in \( X' \), then the total number of balanced cards obtained this way is at least \( \max\{t_1, 2\} \cdot \max\{t_2, 2\} \), which is at least \( t \).

If \( T \) is a \( (\frac{n-1}{2}, \frac{n-3}{2}) \)-tree, then balanced cards arise by deleting \( z \), its leaf neighbor, and a leaf of \( T \) in \( X \). The number of such balanced cards is the number \( s \) of leaves of \( T \) in \( X \); since \( z \) is a 1-burl, \( s \leq t - 2 \). If \( d_T(x) \geq 3 \), then these are the only balanced cards, since deleting any nonleaf vertex outside \( \{z, x\} \) leaves a component with at least \( (n + 3)/2 \) vertices, and deleting \( x \) leaves at least two components with fewer than \( (n-3)/2 \) vertices.

If \( d_T(x) = 2 \), then we can form additional balanced cards by deleting \( x \) and two vertices from the large component of \( T - x \). For example, we can delete the leaf neighbor of \( z \) and a leaf of \( X' \). This yields \( t - s - 1 \) additional balanced cards. There is at least one more by
deleting $z$, its leaf neighbor, and $x$. Thus when we know that $T$ is a $(\frac{n-1}{2}, \frac{n-3}{2})$-tree, we can determine whether $d_T(x) = 2$ from the number of balanced cards in the deck. \hfill \Box

Recall that for odd $n$ an $(n-3)$-deck $D$ has $(a,b)$-type if $(a,b)$ is the lexicographically least $(i,j)$ such that some $(\frac{n-5}{2}, \frac{n-5}{2})$-card has an $(i,j)$-spear (Definition 7.3).

**Theorem 8.7.** For odd $n$ with $n \geq ???$, $(\frac{n-1}{2}, \frac{n-3}{2})$-trees are reconstructible.

**Proof.** Let $D$ be the $(n-3)$-deck of a $(\frac{n-1}{2}, \frac{n-3}{2})$-tree $T$. By Lemma 8.5 and earlier cases, every reconstruction from $D$ is an $(\frac{n-1}{2}, \frac{n-3}{2})$-tree. The centroid $z$ is a 1-burl. The piece with $(n-1)/2$ vertices is $X$, with root $x$. The piece with $(n-3)/2$ vertices is $X'$, with root $x'$.

**Case A:** $D$ has at most $t-2$ balanced cards. By Lemma 8.6, this requires $d_T(x) \geq 3$. Unicentroidal cards arise by deleting three vertices outside $X$, yielding unique centroid $x$, or by deleting two vertices of $X$ and one of $X'$, yielding unique centroid $z$. Bicentroidal cards arise by deleting one vertex of $X$ and two vertices outside $X$, yielding centroids $x$ and $z$, or by deleting three vertices outside $X'$, yielding centroids $z$ and $x'$.

**Subcase A1:** $D$ has a connected card with a full vertex. Since $T$ is obtained from cards by adding vertices, also $T$ has a full vertex.

For a full vertex $v$ in $X$ (including the case $v = x$), the minimum average distance from $v$ to a centroid in any card containing $v$ as a full vertex occurs in a unicentroidal card with centroid $x$, obtained by deleting three vertices outside $X$. The average distance from $v$ to the centroids in a bicentroidal card with centroids $x$ and $z$ is greater by $1/2$.

For a full vertex $v'$ in $X'$ (including the case of $x'$ itself), the minimum average distance from $v'$ to a centroid in any card containing $v'$ as a full vertex occurs only in bicentroidal cards with centroids $z$ and $x'$, obtained by deleting three vertices outside $X'$.

Let $D'$ be the multiset of connected cards exhibiting the smallest average distance $h$ from a full vertex to the centroids; $D'$ consists of unicentroidal cards (and $h \in \mathbb{N}$) when the full vertex is in $X$, bicentroidal cards (and $h \notin \mathbb{N}$) when the full vertex is in $X'$. In particular, this distance $h$ cannot occur for full vertices in both directions from the centroid in the cards.

When the cards in $D'$ are bicentroidal ($v \in X'$), we see $X'$ as one branch of every card in $D'$. The multiset provided by the other branch is the rc3-deck of the rooted tree $Y$ obtained from $X$ by prepending $z$ with one leaf neighbor. By Lemma 3.12, we reconstruct $Y$ and $T$.

When the cards in $D'$ are unicentroidal ($v \in X$), only one piece has a full vertex at distance $h$ from $x$, and together with $x$ and the burl it induces $X$ in each card in $D'$. Deleting $V(X)$ from these cards gives the rc3-deck of the rooted tree $Y'$ obtained from $X'$ by prepending $z$ with one leaf neighbor. By Lemma 3.12, we reconstruct $Y'$ and $T$.

**Subcase A2:** $D$ has no connected card with a full vertex. Here $T$ also has no full vertex. As in Lemma 8.5, $T$ has maximum degree 3, and a longest path contains an endpoint of every edge. Let $\hat{x}$ and $\hat{z}$ be the leaf neighbors of $x$ and $z$, respectively. With $d_T(x) = 3$, 55
every balanced card is obtained by deleting \( z, \hat{z}, \) and one leaf of \( X; \) hence the number of leaves of \( T \) in \( X \) equals the number of balanced cards, so we know this quantity. Since we also know the number of leaves in \( T, \) this tells us the number of leaves of \( T \) in \( X'. \)

The subtree \( X' \) occurs as a component of every balanced card, so we know which component is \( X' \) unless the balanced cards are all the same. In that case, the trees obtained from \( X \) by deleting one leaf are isomorphic. Since \( T \) has no full vertex and \( x \) has degree 2 in \( X, \) this occurs only when \( X \) is a path. Hence again we know which component of the balanced cards is \( X' \) (even if \( X' \) is also a path). Since we know the number of leaves of \( T \) in \( X', \) we can determine whether \( d_T(x') = 3 \) or \( d_T(x') = 2. \)

First suppose \( d_T(x') = 3; \) let \( \hat{x}' \) be the leaf neighbor of \( x'. \) Let \( D' \) be the set of bicentroidal cards having one centroid of degree 3 and one centroid of degree 2 whose neighbor in its branch is another 2-vertex. Such cards arise with centroids \( z \) and \( x' \) by deleting \( \hat{z}, \hat{x}, \) and another leaf of \( X; \) here the branch whose root has degree 3 is always \( X'. \) They also arise with centroids \( x \) and \( z \) by deleting \( \hat{z}, \hat{x}', \) and a leaf of \( X \) other than \( \hat{x}; \) here the branch whose root has degree 2 is always \( X' - \hat{x}' \) plus \( z \) prepended at the root. There are exactly \( s - 1 \) cards of each type, where \( s \) is the number of leaves of \( X. \) Since \( X' \) is the branch rooted at a 3-vertex in half of these cards, we obtain \( X' \) unless the rc1-cards of \( X - \hat{x} \) are the same, which requires \( X \) to be a path (since there is no full vertex). In that case we also obtain \( X'. \) We now know \( X' \) and the rc1-deck of \( X, \) so by Lemma 2.1 we can reconstruct \( X \) and \( T. \)

Finally, suppose \( d_T(x') = 2, \) and let \( y \) be the nonleaf neighbor of \( x \) other than \( z. \) If \( T \) has unicentroidal cards whose centroid has two 3-neighbors, then \( d_T(y) = 3, \) and such cards have centroid \( x \) and arise by deleting three vertices of \( X'. \) One piece is always \( X - \{x, \hat{x}\}, \) and the other piece (with \( z \) and \( \hat{z} \) deleted) provides the rc3-deck of \( X'. \) We know which piece is which unless those rc3-cards are the same, in which case they are paths and again we know \( X. \) Knowing which branch is which, by Lemma 3.12 we reconstruct \( X' \) and \( T. \)

If there is no such unicentroidal card, then \( d_T(y) = 2. \) Now consider the bicentroidal cards in which one centroid has degree 2 and the other is a 1-burl with a 1-burl neighbor. Since \( d_T(y) = d_T(x') = 2, \) such cards arise only by deleting three vertices from \( X \) while not deleting \( \hat{x} \) or \( \hat{z}. \) The branch whose root has degree 2 is \( X', \) and the rooted trees forming the other branch (after deleting \( \{x, z, \hat{x}, \hat{z}\} \)) comprise the rc3-deck of \( X - \{x, \hat{x}\}. \) By Lemma 3.12, we reconstruct \( X \) and \( T. \)

**Case B:** \( D \) has at least \( t - 1 \) balanced cards. By Lemma 8.6, \( d_T(x) = 2. \) All unicentroidal cards are \( \{z - \hat{z}, \frac{z - \hat{z}}{2}\} \)-cards with centroid \( z \) and arise by deleting two vertices from \( X \) and one from \( X'. \) All other cards are bicentroidal, with centroids \( \{y, x\}, \{x, z\}, \) or \( \{z, x'\}, \) where \( y \) is the nonleaf neighbor of \( x \) other than \( z. \) Let \( y' \) be the nonleaf neighbor of \( x' \) other than \( z. \)

**Subcase B1:** \( D \) has a connected card with a full vertex. This case is similar to Subcase A1. Since \( T \) is obtained from cards by adding vertices, also \( T \) has a full vertex.

For a full vertex \( v \) in \( X, \) the minimum distance from \( v \) to a centroid in any card having \( v \)
as a full vertex occurs only in bicentroidal cards with centroids \( y \) and \( x \), obtained by deleting three vertices outside \( X \). For a full vertex \( v' \) in \( X' \) (including the case \( v' = x' \)), the minimum distance from \( v' \) to a centroid in any card having \( v' \) as a full vertex occurs only in bicentroidal cards with centroids \( z \) and \( x' \), which arise by deleting three vertices outside \( X' \).

Let \( D' \) be the multiset of bicentroidal cards exhibiting the smallest distance \( h \) from a full vertex \( v \) to a centroid occurs, and suppose \( (n - 3)/2 \geq 3 \). By the observations above, this distance \( h \) cannot occur in a single card for full vertices in both \( X \) and \( X' \).

We have \( v \in X' \) if and only if in some cards of \( D' \) the centroid in the branch not containing \( v \) has degree 3 (it is \( z \)). Such cards all have \( X' \) as the branch containing \( v \). Deleting \( z \) and its leaf neighbor from the other branch in these cards yields the rc3-deck of \( X \). By Lemma 3.12, we reconstruct \( X \) and \( T \).

If no card in \( D' \) has a 3-vertex as the centroid in the branch not containing \( v \), then \( v \in X \), and the branch containing \( v \) is \( X - x \) in each such card. We can tell which branch it is, because the other branch has no such full vertex (otherwise we would use the prior case). In the cards of \( D' \) where the neighbor of the root of the opposite branch has degree 3 (it is \( z \)), beyond it we see rc3-cards of \( X' \). By Lemma 3.12, we reconstruct \( X' \) and \( T \).

**Subcase B2:** \( D \) has no connected card with a full vertex. As in Definition 7.3, \( D \) has \( a, b \)-type when \((a, b)\) is the lexicographically least \((i, j)\) such that \( T \) has a \((\frac{n-5}{2}, \frac{n-5}{2})\)-card whose vertices not of degree 2 that are closest to the centroid have distance \( i \) from the centroid in one big piece and distance \( j \) in the other.

Since no connected card has a full vertex, \( T \) is a caterpillar with maximum degree 3, as in Subcase A2; every vertex is a 1-burl, a 2-vertex, or a leaf. By examining the \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards, we find \( D \) to be of \( a, b \)-type for some \((a, b)\) with \( 1 \leq a \leq b \). Let \( D' \) be the set of \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards having an \( a, b \)-spear. There may be \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards outside \( D' \) having \((i, j)\)-spears with \((i, j)\) lexicographically larger than \((a, b)\).

**First case:** \( a = 1 \). Since \( d_T(x) = 2 \) in Case B, here \( d_T(x') = 3 \). Hence the only bicentroidal cards with 1-burls as both centroids have centroids \( z \) and \( x' \) and arise by deleting three vertices from \( X \). In each such card, one branch is \( X' \), and the other becomes an rc3-card of \( X \) after deleting \( z \) and \( z' \). We recognize which branch is which unless all rc3-cards of \( X \) are the same. Since there is no full vertex, that occurs only when \( X \) is a path, in which case we still know \( X' \). Using the rc3-deck of \( X \), by Lemma 3.12 we reconstruct \( X \) and \( T \).

**Second case:** \( 2 \leq a = b \). First suppose that the ends of the spear are 1-burls \( u \) and \( u' \) with leaf neighbors \( \hat{u} \) and \( \hat{u}' \), respectively. Since no card has a full vertex, this is equivalent to \( b \leq (n - 9)/2 \). A \((\frac{n-5}{2}, \frac{n-5}{2})\)-card having an \( i, j \)-spear with \( i, j > a \) is obtained by deleting \( \hat{u} \) and \( \hat{u}' \) and one more leaf of \( X \). In these cards one piece is always \( X' - \hat{u}' \), and the other nontrivial piece is an rc1-card of \( X - \hat{u} \). We can tell them apart unless the rc1-cards of \( X - \hat{u} \) are all the same, but then those cards are paths. Hence we still know \( X' - \hat{u}' \) (even if it is a path), we can reconstruct \( X - \hat{u} \), and we can restore \( \hat{u} \) and \( \hat{u}' \) to reconstruct \( T \).
The other possibility is $a = b = (n - 5)/2$, in which case every $(n-5, n-5)$-card is a path plus a leaf neighbor $\hat{z}$ of the centroid $z$. Now the rooted $X'$ is $\hat{P}_{(n-3)/2}$ or $\hat{P}'_{(n-3)/2}$, and the rooted $X$ is in $\{\hat{P}_{(n-1)/2}, \hat{P'}_{(n-1)/2}, \hat{P''}_{(n-1)/2}\}$. The bicentroidal cards in which a centroid has a 1-burl neighbor in its own branch arise by deleting three vertices of $X'$, and the branch not containing this centroid is $X - x$. Knowing $X$, the number of unicentroidal cards tells us which of the two choices for $X'$ occurs.

**Third case:** $2 = a < b$. Here exactly one of $\{y, y'\}$ is a 1-burl, and the other is a 2-vertex. Suppose first that in some bicentroidal card $C$, both centroids are 2-vertices, and one centroid $u$ has 3-vertices at distances 1 and 3 in its own branch and a non-2-vertex at distance $b - 1$ in the other branch.

If $C$ exists, then $u \neq z$, since $z$ has no 3-neighbor. Suppose $d_T(y) = 3$ and $d_T(y') = 2$. With $d_T(y') = 2$, the centroids cannot be $\{z, x'\}$. They cannot be $\{y, x\}$ with $u = x$ (since $d_T(y') = 2$) or $u = y$ (since bicentroidal cards with centroids $\{y, x\}$ contain all of $X$). Finally, they cannot be $\{z, x\}$ with $u = x$ because the distance from $x$ to a non-2-vertex in the opposite branch would be at least $b + 1$.

Hence if $C$ exists, then $d_T(y') = 3$ and $d_T(y) = 2$. Now let $\mathcal{D}'$ be the multiset of bicentroidal cards where one centroid is a 2-vertex with two 3-neighbors. This centroid must be $x'$, since $d_T(y) = 2$. The cards in $\mathcal{D}'$ arise by deleting three vertices of $X$. All have $X'$ as a branch, and the other branch has root $z$ of degree 3, so we know which branch is which. When we ignore $z$ and its leaf neighbor in the other branch, the cards in $\mathcal{D}'$ give us the rc3-deck of $X$. By Lemma 3.12 we reconstruct $X$ and $T$.

Since $C$ does exist when $d_T(y) = 2$ and $d_T(y') = 3$, we recognize $d_T(y) = 3$ and $d_T(y') = 2$ by the nonexistence of $C$. Let $\mathcal{D}''$ be the multiset of bicentroidal cards in which one centroid is a 3-vertex with closest non-2-vertices at distance 2 in its branch and distance $b$ in the opposite branch. Such cards occur with centroids $z$ and $x'$. They don’t occur with centroids $y$ and $x$ since with $b \geq 3$ the 1-burl $z$ is too close to $y$, and they don’t occur with centroids $x$ and $z$ since then $z$ does not have a 3-vertex at distance 2 in its branch.

Since the centroids are $z$ and $x'$ in all cards in $\mathcal{D}''$, these are the cards arising by deleting three vertices of $X - \hat{y}$, and the branch rooted at the centroid of degree 2 is always $X'$. After deleting $\{z, \hat{z}, x, \hat{y}\}$, the other branch provides an rc3-card of $X - \{x, \hat{y}\}$ with root $y$. By Lemma 3.12, we reconstruct this rooted tree and $T$.

**Fourth case:** $3 \leq a < b$. Consider bicentroidal cards having a centroid with degree 2 that has a 3-neighbor in its own branch. Since $a > 2$, such a card arises only by deleting three vertices from $X'$, and the centroids are $y$ and $x$. Let $\mathcal{D}'$ be the multiset of these cards.

If no card in $\mathcal{D}'$ has 1-burls neighboring both centroids, then the cards in $\mathcal{D}'$ arise by deleting three vertices of $X'$ but not the leaf neighbor $\hat{z}$ of $z$. For each such card, the branch whose root does not have a 3-neighbor is $X - x$. Over all these cards, the other branches, with $\{x, z, \hat{z}\}$ deleted, form the rc3-deck of $X'$. Knowing which branch is which, by Lemma 3.12

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we reconstruct $X'$ and $T$.

Hence we may assume that $y$ has a 3-neighbor $w$, so $a = 3$. Since $b > a$, the path $\langle x', y', w' \rangle$ in $X'$ consists of 2-vertices in $T$. Now let $\mathcal{D}'$ be the multiset of bicentroidal cards in which one centroid is a 1-burl having another 1-burl at distance 3 in its own branch. The cards in $\mathcal{D}'$ arise by deleting three vertices of $X$ but not the leaf neighbor $\tilde{w}$ of $w$; they exist if $(n - 1)/2 \geq 8$. Furthermore, we know which branch is which because the other centroid $x'$ has degree 2. Hence we see $X'$ as that branch in each such card and obtain the rc3-deck of $X - \{x, y, \tilde{w}\}$. By Lemma 3.12, we reconstruct $X$ and $T$. \qed

**Definition 8.8.** When a centroid of a tree $T$ has degree 2, the **spar** of the tree is the maximal path containing that centroid whose internal vertices all have degree 2. We call it a $p, b$-spar when the spar has $p$ internal vertices and $b$ is the maximum of the numbers of vertices in the two components left by deleting the internal vertices of the spar. The deck of a tree is of $p, b$-class if every connected card has a spar, $p$ is the minimum number of internal vertices in a spar, and $b$ is the maximum $i$ such that some connected card has a $p, i$-spar.

The motivation for the terms “spar” and “spear” is that both English words refer to long straight objects, possibly of wood. In our spears, a 1-burl distinguishes an internal point, but in spars the internal vertices all have degree 2. Hence we classify spars by the distribution of vertices outside the two ends rather than by internal sublength.

**Lemma 8.9.** Let $T$ be an $n$-vertex tree in which all centroids and their neighbors have degree 2. If the $(n - 3)$-deck of $T$ is of $p, b$-class, then we can determine the parameters of the spar in $T$. In fact, $T$ has a $p, b$-spar, except that when $p + b = n - 6$ it is also possible that $T$ has a $p, b + 1$-spar, and we can recognize this case.

**Proof.** By Corollary 3.5, every connected card of $T$ has a centroid with degree 2 and hence has a spar, so the class of the deck is well defined. Let $a = n - 3 - p - b$. In a card $C$ having a $p, b$-spar, deleting the internal vertices of the spar leaves components with $a$ and $b$ vertices. Since $b \geq a$ by definition, we call those components the “small end” and “big end” of $C$, respectively.

Reconstructing $T$ from $C$ cannot attach vertices to internal vertices of the spar, since there would then be a card having a spar with fewer than $p$ internal vertices. Similarly, if $b > a > 3$, then reconstructing $T$ from $C$ cannot attach a vertex to the big end of $C$, since deleting an extra vertex from the small end would yield a card having a $p, b + 1$-spar. If $b = a > 3$, then the same argument applies to either end, so that case cannot exist. Thus $T$ has a $p, b$-spar if $p + b < n - 6$.

If $p + b = n - 6$, then $a = 3$. Let $W$ and $W'$ be the subtrees obtained by deleting the centroid(s) of $T$, and let $W$ be the one containing the small end of $C$. Let $j$ be the number of vertices deleted from $W$ in forming $C$ ($3 - j$ are deleted from $W'$).
If \( j = 0 \), then the card formed by instead deleting all three vertices on the small end has a \( p - 1, b + 3 \)-spar, contradicting the choice of \( C \). If \( j = 1 \), then instead deleting two more vertices from the small end yields a card with a \( p, b + 2 \)-spar, again contradicting the choice of \( C \). If \( j = 2 \), then \( T \) has a \( p, b + 1 \)-spar. If \( j = 3 \), then \( T \) has a \( p, b \)-spar. Hence \( T \) has a \( p, b + 1 \)-spar or a \( p, b \)-spar. To distinguish between these cases, note that when \( j = 2 \), deleting three vertices from \( W \) (and none from \( W' \)) yields a card having a \( p + 1, b + 1 \)-spar, but when \( j = 3 \) there is no such card of \( T \).

It is not possible to have \( p + b = n - 5 \), because that requires \( a = 2 \), in which case \( C \) would actually have a \( p + 1, b \)-spar. If \( p + b = n - 4 \), then \( a = 1 \). If fewer than three vertices were deleted from \( W \) to form \( C \), then deleting one more yields a card having a \( p - 1, b + 1 \)-spar, again contradicting the choice of \( C \). Hence in this case also \( T \) has a \( p, b \)-spar.

\[ \Box \]

**Theorem 8.10.** For odd \( n \) with \( n \geq ???? \), \( \left( \frac{n-1}{2}, \frac{n-1}{2} \right) \)-trees are reconstructible.

**Proof.** Given the \( (n - 3) \)-deck \( D \) of a \( \left( \frac{n-1}{2}, \frac{n-1}{2} \right) \)-tree \( T \), we know that every reconstruction is a \( \left( \frac{n-1}{2}, \frac{n-1}{2} \right) \)-tree, because we have shown that all other \( n \)-vertex trees are reconstructible when \( n \) is odd. Define \( z, x, X, x', X' \) as usual.

Since \( d_T(z) = 2 \), unicentroidal cards can arise only by deleting three vertices from \( X \) or three vertices from \( X' \). The centroid is then \( x' \) or \( x \), respectively, if and only if that is a \( 3^+ \)-vertex, but if it has degree 2 then the card is bicentroidal. Deleting one vertex from one piece and two from the other always yields a bicentroidal card in which \( z \) is one centroid and the other centroid is the root of the piece that lost one vertex.

**Case A:** Some connected card is unicentroidal. This case is \( \max\{d_T(x), d_T(x')\} \geq 3 \).

**Subcase A1:** \( x \) and \( x' \) are both full vertices. This case occurs when no bicentroidal card has both centroids of degree 2, because there is no such bicentroidal card if and only if neither \( x \) nor \( x' \) can be reduced to degree 2 by deleting one vertex from the piece containing it. Deleting three vertices from \( X \) or from \( X' \) now yields unicentroidal cards, all being \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-cards with \( b \geq 7 \). When three vertices are deleted from \( X \) to form \( C \), deleting the vertices of the largest piece of \( C \) leaves \( X' \). When the vertices are deleted from \( X' \), we similarly obtain \( X \). Over all \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-cards with \( b \geq 7 \), we obtain the two pieces of \( T \).

**Subcase A2:** Exactly one neighbor of \( z \) is a full vertex. By symmetry, let \( x \) be this full vertex; \( x' \) is not full. With the previous case excluded, we recognize this case by the existence of \( \left( \frac{n-5}{2}, \frac{n-5}{2} \right) \)-cards with \( b \geq 7 \), obtained by deleting three vertices from \( X' \). As above, these cards give us \( X \). The large pieces, after deleting the root \( z \), form the rc3-deck of \( X' \). Since we know which piece is which, by Lemma 3.12 we can reconstruct \( X' \) and \( T \).

**Subcase A3:** Each of \( x \) and \( x' \) is a 1-burl. After excluding the prior cases, we recognize this case by the existence of a bicentroidal card having a centroid with two 3-neighbors. As usual, let \( y \) and \( y' \) be the non-leaf neighbors of \( x \) and \( x' \), so that \( \langle y, x, z, x', y' \rangle \) is a path of non-leaf vertices. Let \( x \) and \( x' \) be the leaf neighbors of \( x \) and \( x' \).
The unicentroidal cards are \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards obtained by deleting three vertices from one piece of \(T\); they have centroid \(x'\) when three vertices are deleted from \(X\), centroid \(x\) when three vertices are deleted from \(X'\). In each such card, one piece is \(X' - \{x', \hat{x}'\}\) or \(X - \{x, \hat{x}\}\). The other piece has root \(z\), and deleting the root leaves an rc3-card of \(X\) or \(X'\), respectively. If we are unable to identify the two pieces that exhaust the unicentroidal cards, then all the rc3-cards of \(X\) or \(X'\) are the same. This cannot occur when \(x\) and \(x'\) are 1-burls.

**Subcase A4:** \(x\) is a 1-burl and \(d_T(x') = 2\). For \(\max\{d_T(x), d_T(x')\} \geq 3\), we may now assume by symmetry that \(x\) is a 1-burl and \(d_T(x') = 2\). Now every unicentroidal card is a \((\frac{n-5}{2}, \frac{n-5}{2})\)-card with centroid \(x\), obtained by deleting three vertices from \(X'\). One piece is \(X - \{x, \hat{x}\}\), where \(\hat{x}\) is the leaf neighbor of \(x\); call this rooted tree \(Y\). The other nontrivial pieces form the rc3-deck of the rooted tree \(Y'\) formed from \(X'\) by prepending \(z\) at the root.

If the \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards are not all the same, then \(Y\) is the piece that appears in each card. We then know the rc3-deck of \(Y'\) and can reconstruct \(Y'\) and \(T\) by Lemma 3.12.

If the \((\frac{n-5}{2}, \frac{n-5}{2})\)-cards are all the same, then we need to distinguish which piece is \(Y\) and which is the common rc3-card of \(Y'\). By Lemma 2.6, the common rc3-card is a path or broom. If \(Y\) is not a path or broom, then we know which piece is \(Y\) and still apply Lemma 3.12. If \(Y\) is a path or broom, then its rc3-cards are all the same.

Consider now bicentroidal cards having a 1-burl at distance 2 from a centroid and no closer. Such cards are obtained by deleting three vertices from \(Y\) and have centroids \(x'\) and \(y'\). Since the rc3-cards of \(Y\) are all the same, these cards are isomorphic. The branch containing \(x'\) has the 1-burl \(x\) at distance 2 from its root, and the other branch is \(Y' - \{z, x'\}\). In order to distinguish the branches (and therefore also obtain \(Y\) from the earlier cards), it suffices that in \(Y' - \{z, x'\}\), the closest 1-burl to the root \(y'\) cannot be within distance 2. Since the common rc3-card of \(Y'\) is a path or a broom, this follows from \((n-3)/2 \geq 8\).

**Case B:** All connected cards are bicentroidal. As noted, both neighbors of \(z\) are 2-vertices. All connected cards have a centroid within distance 1 of \(z\), so all have spars and \(D\) is of \(p, b\)-class for some \((p, b)\) with \(p + b \leq n - 4\). By Lemma 8.9, \(T\) has a \(p, b\)-spar, except possibly a \(p, b + 1\)-spar when \(p + b = n - 6\), and we know whether the exception occurs. Let \(a = n - 3 - p - b\). Let \(x\) and \(x'\) be the neighbors of \(z\) on the paths to the outside subtrees with \(a\) and \(b\) vertices, respectively. Let \(X\) and \(X'\) be the pieces of \(T\) rooted at \(x\) and \(x'\), respectively.

Let \(C\) be a card having a \(p, b\)-spar. The exceptional case occurs when \(a = 3\) and the end of \(C\) contained in \(X\) consists of a 1-burl with two leaf neighbors. As noted in the proof of Lemma 8.9, we recognize this case within \(a = 3\) by the existence of a card having a \(p + 1, b + 1\)-spar. Such a card arises by deleting three vertices from \(X\), and we see all of \(X'\). Now the rc3-cards of \(X\) are all paths. Since we know which piece is which in the cards with these spar (we can assume \(T\) is not a path), Lemma 3.12 applies, using counts of small spiders to determine \(X\). We can recognize this case and obtain this piece.
In the remaining case, $T$ has a $p, b$-spar. In every connected card, the spar has at least $p$ internal vertices, and both outside subtrees have at most $b$ vertices. Let $\mathcal{D}'$ be the multiset of connected cards where an outside subtree has exactly $b$ vertices (including the endpoint of the spar). Every card in $\mathcal{D}'$ arises by deleting three vertices from the subtree outside the other end of the spar.

Such cards may have longer spars than $T$, but all have $b$ vertices at one end and at most $a$ vertices at the other end. This distinguishes the branches when $a < b$. The branch with $b$ vertices outside the spar is the rooted tree $X' - x'$. The multiset consisting of the opposite branch in these cards, after deleting the root $x'$ and its neighbor $z$, is the rc3-deck of $X$. Since the branches are distinguished, Lemma 3.12 applies to reconstruct $X$ and $T$.

If $b = a \geq 6$, then again $T$ has a $p, b$-spar, but now in the cards whose larger outside subtree has exactly $b$ vertices we see always $X' - x'$ or $X - x$ as a branch. Whether they are different or not, we obtain $T$. As usual, there is confusion only if the rc3-cards of $X$ or $X'$ are all the same. However, since we also have both $X$ and $X'$ as candidate branches, we can recognize when the piece with $b$ vertices outside the spar is a broom. \hfill \Box

\section{Bicentroidal Trees}

It remains to consider bicentroidal trees, where the number $n$ of vertices is even, the trees have cost $n/2$, and all cards in the $(n - 3)$-deck are unicentroidal. By Sections 3–5, we can recognize the decks of bicentroidal trees. In order to reconstruct such a tree from the deck, we will consider cases according to the status of the two centroids.

\textbf{Remark 9.1.} As usual for a bicentroidal tree $T$, in this section let $z$ and $z'$ be the centroids, and let $Y$ and $Y'$ be the two branches of $T$, with $z \in V(Y)$ and $z' \in V(Y')$. Let $x$ be the neighbor of $z$ in a largest component of $Y - z$, and let $x'$ be the neighbor of $z'$ in a largest component of $Y' - z'$. When $v$ is a 1-burl, let $\hat{v}$ denote the leaf neighbor of $v$.

For all $v \in V(T)$, there is a component with at least $n/2$ vertices in $T - v$; hence every connected $(n - 3)$-card of $T$ has cost $(n - 6)/2$ or $(n - 4)/2$. Furthermore, every card with cost $(n - 6)/2$ has centroid $z$ or $z'$. Other vertices can be centroids only for cards with cost $(n - 4)/2$: possibly $x$ when $d_T(z) = 2$, and possibly $x'$ when $d_T(z') = 2$. No other vertex of $T$ can be a centroid of a connected card.

We begin by showing 3-reconstructibility in an easy case.

\textbf{Theorem 9.2.} For $n \geq 2$, an $n$-vertex bicentroidal tree is reconstructible if at least one centroid is a full vertex.

\textit{Proof.} Let $\mathcal{D}$ be the $(n - 3)$-deck of such a tree $T$. Using prior results, we recognize from the deck that $T$ is bicentroidal. Let $v$ be a centroid of $T$. 

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If $v$ is a full vertex, then $v$ cannot become a centroid of degree 2 in a card in $D$, because deleting the burl at $v$ to reach degree 2 (and possibly deleting one other vertex) would leave a piece with at least $(n - 2)/2$ vertices. If $v$ is not full, then deleting one vertex from its branch ($v$ if $v$ is a 1-burl) and two vertices from the other branch yields a $(n-4, n-4)$-card whose centroid has degree 2 (it is $v$). Hence both centroids of $T$ are full if and only if no connected card has a centroid of degree 2.

If we delete three vertices from the branch containing $v$, then we obtain a connected card with cost $(n-6)/2$ whose centroid is the centroid of $T$ other than $v$, retaining all its neighbors from $T$. As noted in Remark 9.1, only centroids of $T$ can be centroids of connected cards with cost $(n-6)/2$. Hence exactly one centroid of $T$ is full if and only if some connected card has a centroid of degree 2 and another with cost $(n-6)/2$ has a full vertex as centroid.

Having recognized these two cases, we proceed to reconstruction.

**Case 1:** Both centroids are full vertices. By Remark 9.1, every card has centroid in $\{z, z'\}$. Cards with cost $(n-6)/2$ arise when and only when three vertices are deleted from one branch. What remains of that branch is a largest piece of the card, the centroid is the root of the other branch of $T$. Letting $z$ be a $(b-2)/2$-burl and $z'$ be a $(b'-2)/2$-burl, we thus have $(n-6, n-6)$-cards and $(n-6, n-b')$-cards in $\mathcal{D}$, with $b, b' \geq 6$. If $b, b' \geq 8$, then the largest pieces in these cards are unique, and deleting them leaves $Y'$ or $Y$, respectively. Over all cards with cost $(n-6)/2$, we thus obtain both $Y'$ and $Y$, whether $Y = Y'$ or not.

If (by symmetry) $b \geq 8$ and $b' = 6$, then we still obtain $Y$ by deleting the largest piece in any $(n-6, n-6)$-card, in which the next largest piece is an rc3-card of $Y'$. Since the pieces are distinguished by their size, by Lemma 3.12 we reconstruct $Y'$ and $T$.

In the remaining case, every connected card with cost $(n-6)/2$ is a $(n-6, n-6)$-card, and both $z$ and $z'$ are 2-burls. Consider the $(n-4, n-8)$-cards in which the root of the biggest piece has degree 2. Such cards arise by deleting both vertices of the burl at $z$ or $z'$ and a leaf of the opposite branch not in the burl of the other centroid. For each such card, in the largest piece we see one branch (without its burl), and in the next-largest piece (when $(n-8)/2 > 2$) we see the configuration of the burl from the other branch (two leaves or $P_2$). Hence using all such cards we reconstruct $T$.

**Case 2:** One centroid is a full vertex and one is not. By symmetry, assume that $z$ is full and $z'$ is not. As in Case 1, every $(n-6, n-6)$-card with $b \geq 6$ has centroid $z$ as a $(b-2)/2$-burl and arises by deleting three vertices of $Y'$. When $b \geq 8$, the largest piece is unique and is an rc3-card of $Y'$; deleting it yields the branch $Y$. Over all such cards we obtain the rc3-deck of $Y'$. By Lemma 3.12 we reconstruct $Y'$ and $T$.

In the remaining case, $b = 6$ and $z$ is a 2-burl. We still obtain $Y$ and an rc3-card of $Y'$ from any $(n-6, n-6)$-card, but the rc3-card also has $(n-6)/2$ vertices. We can be confused which subtree is $Y$ only if the rc3-cards of $Y'$ are all the same, making the $(n-6, n-6)$-cards identical. Otherwise, we know $Y$ and can reconstruct $Y'$ by Lemma 3.12.
If \( b = 6 \) and the \((\frac{n-6}{2}, \frac{n-6}{2})\)-cards are identical, then by Lemma 2.6 the common rc3-card of \( Y' \) is a rooted broom or path. Since \( z' \) is not a full vertex, this requires \( d_T(z') = 2 \). It also prevents the situation where \( x' \) is a 2-vertex and has a 2-burl other than \( z \) within distance 2, since \( n/2 - 3 \geq 8 \). This means that in any connected card with cost \((n - 4)/2\) having a largest piece rooted at a 2-vertex with a 2-burl neighbor, the centroid is \( x' \), that root is \( z' \), and the 2-burl is \( z \). Such a card is obtained only by deleting three vertices of \( Y' \), and we know which piece contains \( z \) and the rest of \( Y' \). Deleting the rest of \( Y' \) from such a card leaves \( Y' \), and then we can return to the original \((\frac{n-6}{2}, \frac{n-6}{2})\)-cards to see \( Y' \).

\( \square \)

**Lemma 9.3.** When \( D \) is the \((n - 3)\)-deck of a bicentroidal tree in which neither centroid is a full vertex, in every reconstruction the set of centroids satisfies the same case among the following: two 1-burls, one 1-burl and one 2-vertex, or two 2-vertices.

*Proof.* Let \( T \) be a reconstruction from \( D \), with centroids \( z \) and \( z' \). Since neither centroid is full, we have the path \( \langle x, z, z', x' \rangle \) and branches \( Y \) and \( Y' \) as usual. Let \( \rho(D) \) be the minimum, over all \((\frac{n-4}{2}, \frac{n-6}{2})\)-cards, of the minimum distance from the centroid to a \( 3^+ \)-vertex in the largest piece. A \((\frac{n-4}{2}, \frac{n-6}{2})\)-card where this minimum distance is \( j \) is \( j \)-distant.

**Case 1:** \( \rho(D) = 1 \). The centroid of any \((\frac{n-4}{2}, \frac{n-6}{2})\)-card is a 1-burl. If it is \( x \) or \( x' \), then by Remark 9.1 its neighbor in the piece with \((n - 4)/2\) vertices has degree 2, so the card is not 1-distant. Hence the centroid of a 1-distant card must be \( z \) or \( z' \), making it a 1-burl.

Now that it must have centroid \( z \) or \( z' \), a 1-distant card exists with the centroid having two \( 3^+ \)-neighbors if and only if (0) both \( z \) and \( z' \) are 1-burls and at least one of \( \{x, x'\} \) is a \( 3^+ \)-vertex. Otherwise, in every 1-distant card the centroid has a 2-neighbor. In this case, either (1) the degrees along \( \langle x, z, z', x' \rangle \) are \((2, 3, 3, 2)\), or (2) \( \{z, z'\} \) has exactly one \( 2 \)-vertex and one 1-burl, and the 1-burl has a \( 3^+ \)-neighbor. In (2), deleting one vertex from the branch whose root has degree 2 and two vertices from the other branch yields a \((\frac{n-4}{2}, \frac{n-4}{2})\)-card in which one piece is rooted at a 1-burl with a 1-burl neighbor. In (1), there is no such card; no \((\frac{n-4}{2}, \frac{n-4}{2})\)-card has centroid \( x \) or \( x' \) since only one piece of \( T - x \) or \( T - x' \) has at least \((n - 4)/2\) vertices, and deleting the leaf \( \hat{z} \) or \( \hat{z'} \) to make a \((\frac{n-4}{2}, \frac{n-4}{2})\)-card with centroid \( z \) or \( z' \) yields a 2-vertex within distance 1 of the root in each piece.

Hence in all situations having a 1-distant card we can distinguish among (0), (1), and (2), determining whether both centroids or exactly one centroid of \( T \) are 1-burls.

**Case 2:** \( \rho(D) = 2 \). If \( z \) and \( z' \) are both 1-burls, then deleting two vertices from \( Y - \{z, \hat{z}\} \) and one from \( Y' - \{z', \hat{z'}\} \) yields a 1-distant card, which is forbidden. If \( z \) and \( z' \) are both 2-vertices, then the centroid of a 2-distant card must be \( x \) or \( x' \), and the vertex at distance 2 in the piece with \((n - 4)/2\) vertices is \( z' \) or \( z \), whose degree is too small. Hence if \( \rho(D) = 2 \), then exactly one of \( \{z, z'\} \) is a 1-burl.

**Case 3:** \( \rho(D) \geq 3 \). Here the two vertices nearest the centroid in the largest piece of a \((\frac{n-4}{2}, \frac{n-6}{2})\)-card are 2-vertices (also in the full tree, since \((n - 4)/2 \geq 4 \)). Neither centroid
Thus we obtain $y_3$ from $x$ piece). Therefore, to obtain a card like we just found from $\tilde{x}$, the deck has $(n - 4, n - 6)$-cards. Since $T$ also has deck $D$, having $(n - 4, n - 6)$-cards makes $x$ or $x'$ a 1-burl; by symmetry, we may let $x'$ be a 1-burl in $T$. With $D$ having no 1-distant card, $d_T(x') = 2$. With $D$ having no 2-distant card, $d_T(x) = d_T(y) = 2$, where $y'$ is the nonleaf neighbor of $x'$ other than $z'$. We now split this case.

Subcase 3a: $\rho(D) = 3$. When $x'$ is the centroid of a card in $T$, the largest piece contains $z$ and $z'$. Hence a 3-distant card requires $d_T(x) \geq 3$. If $x$ is a full vertex in $T$, then deleting three vertices from $Y'$ in $T$ yields a $(n - 4, n - 6)$-card with $b \geq 8$. Since $\tilde{x}$ and $x'$ are 2-vertices in $\tilde{T}$, the deck $D$ has no such card. Hence $x$ is a 1-burl in $T$.

Now $x$ and $x'$ are 1-burls in $T$. Hence $T$ has $(n - 4, n - 4)$-cards with 1-burls (and no larger burl) at distance 1 and 2 in opposite directions from the centroid. Such cards in the deck of $\tilde{T}$ require the nonleaf neighbor $\tilde{y}$ of $\tilde{x}$ other than $\tilde{z}$ to be a 1-burl in $\tilde{T}$. Since $n/2 \geq 8$, deleting three vertices from $\tilde{Y}$ in $\tilde{T}$ (not the leaf neighbor of $\tilde{y}$) now yields a $(n - 4, n - 6)$-card whose piece with $(n - 6)/2$ vertices has a 1-burl at distance 3 from the centroid.

A $(n - 4, n - 6)$-card arises from $T$ by deleting three vertices from one branch, and its centroid is $x'$ or $x$. The piece with $(n - 6)/2$ vertices does not contain the centroids of $T$. Therefore, to obtain a card like we just found from $\tilde{T}$, in $T$ there must be a 1-burl at distance 3 from $x$ or $x'$ in the direction away from $z$ and $z'$ (the intervening vertices are 2-vertices). Thus we obtain $(n - 4, n - 6)$-cards from $T$ such that in both pieces the nearest 3+-vertices to the centroid are 1-burls at distance 3 from the centroid.

Such cards must also occur in the deck of $\tilde{T}$, so $\tilde{T}$ has 1-burls at distance 3 in both directions from the 1-burl $\tilde{z}'$, since in $\tilde{T}$ the only possible centroid of a $(n - 4, n - 6)$-card is $\tilde{z}'$.

We now have in $T$ a 1-burl at $x'$ or $x$ with nearest 1-burls at distance 3 in both directions; by symmetry, let it be $x'$. In $\tilde{T}$ we have a 1-burl at $\tilde{z}'$ and nearest 1-burls at distance 3 in both directions. By deleting the leaf neighbor of $\tilde{z}'$ and two vertices of $\tilde{Y}$ (other than the leaf neighbor of $\tilde{y}$) from $\tilde{T}$, we obtain a $(n - 4, n - 4)$-card such that in each piece the nearest 3+-vertex is distance 3 from the centroid. However, $T$ has no such card; all $(n - 4, n - 4)$-cards of $T$ have centroid at $z$ or $z'$, and the distance in such a card from the centroid to the nearest 3+-vertex in $Y'$ is 1, 2, 4, 5, or larger.

Subcase 3b: $\rho(D) > 3$. With no 3-distant card, $d_T(x) = 2$. Every $(n - 4, n - 6)$-card of $T$ arises by deleting three vertices of the branch in $T$ containing $x$ and $z$ (yielding the large piece). The piece with $(n - 6)/2$ vertices in these cards is the same, rooted at the neighbor $y'$ of $x'$; call it $W$.

On the other hand, $(n - 4, n - 6)$-cards arise from $\tilde{T}$ by deleting three vertices of $\tilde{Y}$ or by
deleting two vertices from \( \tilde{Y} \) and one from \( \tilde{Y}' \). In one case the piece with \((n - 6)/2\) vertices is contain in \( \tilde{Y} \); in the other it is in \( \tilde{Y}' \). That piece must always be \( W \) due to the cards from \( T \), so in \( \tilde{T} \) the rc3-cards of the branch \( \tilde{Y} \) and the rcl-cards of the subtree obtained from \( \tilde{Y}' \) by deleting \( z' \) and its leaf neighbor all equal \( W \). By Lemma 2.6, \( W \) is a path or a broom.

The tree \( \tilde{T} \) now consists of the 1-burl \( z' \), its leaf neighbor, and two copies of \( W \) rooted at the neighbors of \( z' \) in \( \tilde{Y} \) and \( \tilde{Y}' \), with \( \tilde{Y}' \) completed by adding one vertex beyond the copy of \( W \) and \( \tilde{Y} \) completed by adding three vertices beyond \( W \). Let \( C \) be a connected card obtained from \( \tilde{T} \) by deleting the leaf neighbor of \( z' \) and two vertices of \( \tilde{Y} \), leaving the copy of \( W \) rooted at \( \tilde{z} \) in \( \tilde{Y} \). Note that \( C \) consists of \( \tilde{z} \), two copies of \( W \) rooted at its neighbors, and one vertex beyond \( W \) in each direction.

In order to obtain \( C \) as a card from \( T \), we must delete the leaf neighbor of \( x' \) and have \( x' \) become the root of a copy of \( W \). This means that deleting a leaf from \( W \) and prepending a vertex at the root can yield another copy of \( W \). Hence \( W \) must be a path.

Deleting three vertices from \( Y' \) in \( T \) yields a connected card \( C' \) with a vertex \( v \) such that \( C' - v \) has components of orders \((n - 12)/2\), \((n + 2)/2\), and 1. Viewing \( C \) as a card of \( T \), the role of \( v \) is played by the 1-burl \( x' \). To obtain \( C \) as a card of \( \tilde{T} \), it cannot be that \( \tilde{z}' \) plays the role of \( v \), because the component of \( \tilde{T} - \tilde{z}' \) with \((n - 4)/2\) vertices cannot be cut down to \((n - 12)/2\) by deleting three vertices. To find another \( 3^+ \)-vertex in \( \tilde{T} \) to serve as \( v \), we must move to within one vertex of the end of \( W \), but the sizes of the components when we delete a vertex out there (even after deleting three vertices to form the card) are too unbalanced.

Hence we cannot have both \( T \) and \( \tilde{T} \) as reconstructions, and in all reconstructions the set of centroids has the same type: both 2-vertices or just one 2-vertex. \( \square \)

**Theorem 9.4.** Bicentroidal trees with at least 20 vertices are 3-reconstructible.

**Proof.** Let \( D \) be the deck of a bicentroidal tree \( T \) with \( n \) vertices. By Theorem 9.2 and earlier results, we may assume that neither centroid is a full vertex, so we have the usual path \( \langle x, z, z', x' \rangle \) and branches \( Y \) and \( Y' \). By Lemma 9.3, we know that all reconstructions have \( \{z, z'\} \) in the same status: two 1-burls, two 2-vertices, or one vertex of each type.

**Case 1:** Both centroids are 1-burls. Here every connected card has cost \((n - 4)/2\) and centroid \( z \) or \( z' \). The \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-cards are obtained by deleting two vertices from one branch and deleting the leaf neighbor of the centroid in the opposite branch, which becomes the centroid of the card. Thus \( x \) and \( x' \) are both \( 3^+ \)-vertices if and only if no \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card has 2-vertices as the roots of both pieces. If such a card does exist, then one of \( \{x, x'\} \) is a \( 3^+ \)-vertex if and only if in some \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card the roots of both pieces are \( 3^+ \)-vertices. Hence we can tell whether none, one, or both of \( x \) and \( x' \) have degree 2.

**Subcase 1a:** \( d_T(x), d_T(x') \geq 3 \). Every \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card arises by deleting the leaf neighbor of a centroid and two vertices from the opposite branch, so in every \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card
with one piece rooted at a 2-vertex the other piece is rooted at \( x \) or \( x' \) and is \( Y - \{ z, \hat{z} \} \) or \( Y' - \{ z', \hat{z}' \} \). Over all such cards we obtain both \( Y \) and \( Y' \), rooted; they may be equal.

**Subcase 1b:** \( d_T(x) = d_T(x') = 2 \). In the \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-cards where exactly one piece has root of degree 2, the only centroid of \( T \) that lost its leaf neighbor is the centroid of the card. The rest of its branch in \( T \) remains. Thus the piece whose root has degree 2 is \( Y - \{ z, \hat{z} \} \) or \( Y' - \{ z', \hat{z}' \} \). Over all such cards we obtain both \( Y \) and \( Y' \), rooted; they may be equal.

**Subcase 1c:** \( d_T(x) = 2 \) and \( d_T(x') \geq 3 \), by symmetry. Let \( D' \) be the multiset of \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-cards where both pieces are rooted at \( 3^+ \)-vertices. Since \( d_T(x) = 2 \), each card in \( D' \) has centroid \( z' \), and \( z \) being a 1-burl implies that the card arises by deleting \( \hat{z}' \) and two vertices of \( Y \) (while keeping \( \hat{z} \)). Let \( y' \) be the nonleaf neighbor of \( x' \) other than \( z' \). If \( d_T(x') > 3 \) or \( x' \) is a 1-burl with \( d_T(y') \geq 3 \), then we know which piece is rooted at \( x' \). Obtain \( Y' \) from that piece by adding \( z' \) and \( \hat{z}' \); this tells us the number of leaves of \( Y' \) in \( T \). Now, over all such cards, deleting the leaf neighbor of the root of the other piece gives the rc2-deck of \( Y - \hat{z} \). Since we know the number of leaves, we can reconstruct \( Y \) by Theorem 2.3.

In the remaining case, in all cards of \( D' \) the roots of both pieces are 1-burls, and their nonleaf neighbors other than the centroid have degree 2. In particular, \( x' \) is a 1-burl and \( d_T(y') = 2 \). Now let \( D'' \) be the multiset of \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-cards where the roots of the pieces have degrees 2 and 3, and the nonleaf neighbor of the root with degree 3 is a \( 3^+ \)-vertex. Since \( d_T(y') = 2 \), the centroid of every card in \( D'' \) is \( z \), and these cards arise by deleting \( \hat{z} \) and two vertices from \( Y' \) not in \( \{ z', \hat{x}' \} \). The piece rooted at a 2-vertex is always \( Y - \{ z, \hat{z} \} \). Hence we know \( Y \) and its number of leaves. The other piece provides the rc2-deck of \( Y' - \{ z', \hat{x}' \} \). Knowing its number of leaves, by Theorem 2.3 we can reconstruct it and \( T \).

**Case 2:** Both centroids have degree 2. This case is essentially the same as Case B of Theorem 8.10, using the branches \( Y \) and \( Y' \) of \( T \) in a bicentroidal tree instead of the pieces \( X \) and \( X' \) in a \( \left( \frac{n-1}{2}, \frac{n-1}{2} \right) \)-tree.

**Case 3:** One centroid is a 1-burl and one is a 2-vertex. We may assume that \( z' \) is a 1-burl and \( z \) is a 2-vertex in \( T \). Every connected card has centroid in \( \{ x, z, z' \} \), where \( x \) is the nonleaf neighbor of \( z \) other than \( z' \). Hence \( T \) has a \( \left( \frac{n-4}{2}, \frac{n-4}{2} \right) \)-card with the roots of both pieces being \( 3^+ \) vertices if and only if \( d_T(x) \geq 3 \).

**Subcase 3a:** \( d_T(x) \geq 3 \). If \( T \) has any connected card whose centroid is a full vertex, then that centroid is \( x \), and the card arises by deleting three vertices of \( Y' \). Such cards are \( \left( \frac{n-4}{2}, \frac{n-b}{2} \right) \)-cards with \( b \geq 8 \), so the largest piece is unique The largest piece is an rc3-card of \( Y' \) plus \( z \) prepended at the root. The next largest piece combines with \( z \), \( x \), and the burl to form \( Y \). Hence we obtain \( Y \) and the rc3-deck of \( Y' \), and we know which is which. By Lemma 3.12, we reconstruct \( Y' \) and \( T \).

Hence we may assume that no connected card has a full vertex as centroid, so \( x \) is a 1-burl. Let \( D' \) be the set of \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-cards in which the root of the unique largest piece is a \( 3^+ \)-vertex. Since \( z \) is the root of the largest piece in any \( \left( \frac{n-4}{2}, \frac{n-6}{2} \right) \)-card with three vertices
or one vertex deleted from $Y'$, every card in $\mathcal{D}'$ has centroid $z'$ and arises by deleting three vertices from $Y$; the root of the largest piece is $x'$. In every such card we see $Y'$, consisting of the largest piece plus $\{z', \hat{z}'\}$, and the remaining piece gives us the rc3-deck of $Y$. Since we know which is which, by Lemma 3.12 we reconstruct $Y$ and $T$.

Hence we may assume $\mathcal{D}' = \emptyset$; now $x'$ is a 2-vertex. Let $y'$ be the nonleaf neighbor of $x'$ other than $z'$. If there is a $(\frac{n-4}{2}, \frac{n-6}{2})$-card in which a piece has a $3^+$-vertex as a root, then this root must be $y'$. The centroid is $y'$ and the other piece is rooted at the 2-vertex $z'$, so we know which piece is which. Such cards arise only by deleting three vertices of $Y'$. In each such card, we see $Y'$, and over all such cards we obtain the rc3-deck of $Y'$, rooted at $z'$. Since we know which piece is which, by Lemma 3.12 we reconstruct $Y$ and $T$.

Hence we may assume $d_T(y') = 2$. Let $y'$ be the neighbor of $x'$ other than $z'$. The path $\langle y', x', z', x' \rangle$ has vertices with degrees $(2, 3, 2, 3, 2)$ in order. If $y'$ is not a 1-burl, then all $(\frac{n-4}{2}, \frac{n-6}{2})$-cards in which the centroid has a 1-burl at distance 2 in one piece and a vertex that is not a 1-burl at distance 2 in the other piece have centroid $z'$ and arise by deleting $\hat{z}'$ and two vertices of $Y$ (other than $\hat{x}$). Furthermore, we know which piece is which. In every such card, we see $Y' - z'$, and the other pieces give us the rc2-deck of $Y' - \hat{x}$. Since we know the distribution of leaves by knowing $Y'$, we can reconstruct $Y$ by Theorem 2.3.

Hence we may assume that $y'$ is a 1-burl. Let $w$ be the neighbor of $y'$ other than $x$. Since $(\frac{n-4}{2}, \frac{n-6}{2})$-cards may have centroid $x$ or $z'$, and both vertices at distance 2 from $z'$ are 1-burls, there is a $(\frac{n-4}{2}, \frac{n-6}{2})$-card having a full vertex at distance 2 from the centroid if and only if $w$ is a full vertex. In that case, such cards arise only by deleting three vertices of $Y'$, and we know which piece is which. We see $Y$ in each such card, and the unique largest piece of these cards gives us the rc3-deck of $Y'$ (with $z$ prepended as a root). By Lemma 3.12, we reconstruct $Y'$ and $T$.

Hence $w$ is a 1-burl or a 2-vertex. Any $(\frac{n-4}{2}, \frac{n-6}{2})$-card with both pieces rooted at 1-burls has centroid $z$. If in some such card the vertices at distance 3 from the centroid in both pieces are 1-burls, then $w$ is a 1-burl (we are assuming $n/2 \geq 7$); otherwise, $w$ is a 2-vertex.

If $w$ is a 1-burl, then consider all $(\frac{n-4}{2}, \frac{n-6}{2})$-cards in which every $3^+$-vertex has distance at least 3 from the centroid. A $(\frac{n-4}{2}, \frac{n-6}{2})$-card with centroid $x$ deletes no vertex of $Y$, so $w$ still has degree 3 in it and distance 2 from $x$. Hence every card considered here has centroid $z'$ and is obtained by deleting the leaves $\hat{x}$ and $\hat{y}'$ and one additional leaf of $Y$ other than $\hat{x}$. The piece rooted at $x'$ has $(n - 6)/2$ vertices, and the piece rooted at $z$ has $(n - 4)/2$ vertices, so we know which is which. The piece with $(n - 6)/2$ vertices is the same in all these cards, while the pieces with $(n - 4)/2$ vertices comprise the rc1-deck of $Y - \hat{x}$. Hence we know $Y'$ and can reconstruct $Y$ (by Theorem 2.1) to complete $T$.

Finally, suppose that $w$ is a 2-vertex. We have the path $\langle w, y', x', z', x', y' \rangle$ with degrees $(2, 2, 3, 2, 3, 2, 3)$. Consider $(\frac{n-4}{2}, \frac{n-4}{2})$-cards in which the nearest 3-vertices to the centroid are at distance 1 and 3 in one piece and distance greater than 3 in the other piece. Since $d_T(w) = 2$, such cards must have $z$ as centroid and arise by deleting $\hat{x}$ and two vertices of $Y'$.
not including \( z' \) or \( y' \). Since one piece is rooted at a 2-vertex and the other at a 3-vertex, we can tell which piece is which. The piece rooted at the 2-vertex is \( Y - \{ z, \hat{x} \} \) in every such card, and the pieces rooted at the 3-vertex comprise the rc2-card of \( Y' - \{ z', \hat{z}', x', y', \hat{y}' \} \). Since we know the distribution of leaves, we can reconstruct \( Y' \) to complete \( T \).

**Subcase 3b:** \( d_T(x) = 2 \). We have noted that this occurs when no \( (n-4, n-4) \)-card has both pieces rooted at 3\(^+\)-vertices. Since \( d_T(x) = 2 \), \( (n-4, n-6) \)-card has centroid \( z' \).

If the deck has a \( (n-4, n-6) \)-card in which the piece having \( (n-4)/2 \) vertices has a 3\(^+\)-vertex within distance 2 of the centroid, then that piece is \( Y' - \{ z', \hat{z}' \} \), since \( d_T(x) = d_T(z) = 2 \). Every such card arises by deleting three vertices in \( Y \). The common piece with \( (n-4)/2 \) vertices gives us \( Y' \), and the pieces with \( (n-6)/2 \) vertices comprise the rc3-deck of \( Y \). Knowing \( Y' \), we know the distribution of leaves and can reconstruct \( Y \) to complete \( T \), unless \( Y \) is one of the exceptions in Theorem 2.9. In that case, deleting three vertices from \( Y' \) yields \( (n-4, n-4) \)-cards with centroid \( x \) that show us which of the exceptions is \( Y \).

Hence we may assume that every \( (n-4, n-6) \)-card has no 3\(^+\)-vertex within distance 2 of the centroid \( z' \). Again let \( y \) be the nonleaf neighbor of \( x \) other than \( z \). If there is a \( (n-4, n-6) \)-card having a full vertex neighboring the centroid, then the centroid is \( x \), the full vertex is \( y \), and the card arises by deleting three vertices of \( Y' \). Hence in all such cards the piece rooted at the full vertex is \( Y - \{ z, x \} \), and the other pieces in these cards comprise the rc3-deck of \( Y' \) with \( z \) prepended at the root. Knowing \( Y \), we know the distribution of leaves and can reconstruct \( Y' \) to complete \( T \), since \( d_T(z') = 3 \).

If there is no such card, then \( y \) is a 1-burl or a 2-vertex. If there is a \( (n-4, n-4) \)-card having a 1-burl as the root of one piece and a 1-burl at distance 2 in the other piece, then the centroid is \( x \) or \( z \) and \( y \) is a 1-burl. If there is no such \( (n-4, n-4) \)-card, then \( y \) is a 2-vertex.

Consider first the case where \( y \) is a 2-vertex. Now every \( (n-4, n-4) \)-card in which the centroid has a 1-burl neighbor has centroid \( z \) and arises by deleting one vertex of \( Y' \) and two vertices of \( Y' \) (keeping \( z' \)). We know which branch is which, since the one arising from \( Y \) has a 2-vertex as the root. We have \( ss' \) such cards, where \( s \) is the number of leaves of \( T \) in \( Y \) and \( s' \) is the number of rc2-cards of \( Y' - \{ z', \hat{z}' \} \). The cards give us \( s \) copies of the rc2-deck of \( Y' \) and \( s' \) copies of the rc1-deck of \( Y \), but we do not yet know \( s \) and \( s' \). If we can determine \( s \) and \( s' \), then we can reconstruct \( Y' \) by Lemma 2.1. We then know the distribution of leaves and can reconstruct \( X' \) by Theorem 2.3.

When we group rc2-cards of \( Y' \) into sets of identical cards, the size of each set is a multiple of the true value \( s \). It cannot be bigger than \( s \) itself, because recognizing that as the truth would require \( Y \) having at least \( 2s \) leaves of \( T \). The number of leaves of \( T \) we see in \( Y \) in these bicentroidal cards is always \( s - 1 \) or \( s \). That can get us up to \( 2s - 1 \) only when \( s = 1 \). In this remaining case, \( Y \) is a path, and we can find the card obtained by deleting three vertices from the end of \( Y \) (it is the one having a 1-burl closest to the distant leaf).

Hence we may assume that \( y \) is a 1-burl. Consider again the \( (n-4, n-6) \)-cards; the centroid
is $z'$. Let $w'$ be the neighbor of $y'$ other than $x'$. Thus $T$ contains the path $⟨y, x, z, z', x', y', w'⟩$, in which $y$ and $z'$ are 1-burls and the others except $w'$ have degree 2. If some $(\frac{n-4}{2}, \frac{n-6}{2})$-card has a 1-burl at distance 3 from the centroid in the piece with $(n - 6)/2$ vertices and a vertex that is not a 1-burl at distance 3 from the centroid in the piece with $(n - 4)/2$ vertices, then the card arises by deleting three vertices from $Y$ (keeping $\hat{y}$). In all such cards, the piece with $(n - 4)/2$ vertices is $Y' - \{z', \hat{z}'\}$, and the pieces with $(n - 6)/2$ vertices comprise the $rc3$-deck of $Y - \hat{y}$, which we reconstruct as usual.

Hence we may assume that $w'$ is a 1-burl. Now consider $(\frac{n-4}{2}, \frac{n-4}{2})$-cards in which the centroid has a 1-burl at distance 2 in one piece and no $3^+$ vertex within distance 4 in the other piece. A $(\frac{n-4}{2}, \frac{n-4}{2})$-card with centroid $x$ arises only by deleting three vertices from $Y'$, so it would still have a 1-burl neighbor at $y$. A $(\frac{n-4}{2}, \frac{n-4}{2})$-card with centroid $z$ has no 1-burl neighbor at distance 2. Hence the centroid of any such card is $z$. Vertex $y$ is the only 1-burl at distance 2 from $z$ in $T$, so $\hat{y}$ must remain. The leaf neighbors of 1-burls $z'$ and $w'$ at distances 1 and 4 from $z$ in the other piece must be deleted. The third deleted vertex must be from $Y$ to make $z$ the centroid, and we know which piece is which. In every such card we see $Y' - \{z', \hat{w}'\}$ as the piece with no 1-burl close to $z$, and the pieces having a 1-burl adjacent to the root comprise the $rc1$-deck of $Y - \{z, \hat{y}\}$ when $\hat{y}$ is deleted. Hence we can obtain both $Y'$ and $Y$ to complete $T$ in this last case.

□

References


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