1. **Stirling numbers and Eulerian numbers.**

   a) \( k!S(n, k) = \sum_{i=0}^{k} \binom{n-i}{k-i} A(n, i) \). The left side counts ordered partitions of \([n]\) into \(k\) blocks. As in Gessel’s proof of Woritzky’s Identity, partitions become barred permutations by writing the elements in increasing order in each successive block, followed by a bar. All these blocks are nonempty, so we obtain precisely the barred permutations with \(k\) nonconsecutive bars, with each segment between bars in increasing order.

   The right side counts the barred permutations with \(k\) nonconsecutive bars by the number of runs, \(i\). Each run must be followed by a bar, and \(k-i\) additional bars are inserted into distinct locations among the remaining \(n-i\) positions following elements. All the barred permutation with \(k\) nonconsecutive bars arise in this way, so the two sides count the same set. (The \(k-i\) inserted bars correspond to the smallest element of a block exceeding the largest element of the preceding block.)

   \[ \sum_{k=0}^{n} k!S(n, k) x^{n-k} = \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{n-i}{k-i} A(n, i) x^{n-k} \]

   we obtain

   \[ \sum_{k=0}^{n} k!S(n, k) x^{n-k} = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{n-i}{k-i} A(n, i) x^{n-k} = \sum_{i=0}^{n} A(n, i) \sum_{k=0}^{i} \binom{n-i}{k-i} x^{n-k} = \sum_{i=0}^{n} A(n, i) (1 + x)^{n-i} \]

   When \(n = 0\), both sides equal 1. For \(n > 0\), we can start the sums at 1. Setting \(x = 1\) on the left counts all the ordered partitions of \([n]\). On the right this produces \(\sum_{i=1}^{n} A(n, i) 2^{n-i}\). Setting \(j = n + 1 - i\) converts this to \(\sum_{i=1}^{n} A(n, n+1-j) 2^{j-1}\), and \(A(n, n+1-j) = A(n, j)\).

2. **Special Stirling permutations.** A **Stirling permutation** is an arrangement of two copies of the elements of \([n]\) such that for all \(i\), the entries between the two copies of \(i\) exceed \(i\). A **skyline** is a Stirling permutation with the additional property that no strictly increasing triple has its last two entries consecutive in the arrangement. Let \(a_n\) be the number of skylines of length \(2n\), and let \(A(x) = \sum_{n \geq 1} a_n x^n / n!\).

   a) \( A'(x) = e^{2x} A(x) \), and \(A(x) = e^{(e^{2x} - 1)/2}\). In a skyline, consider the three segments formed by the positions of 1; each number greater than 1 has both copies in one segment. The first segment satisfies both specified properties. Since the other two segments follow a 1, they have no ascents and are weakly decreasing. Conversely, any distribution of the values in \([n]\) into three segments that satisfy these constraints forms a skyline. Thus \(a_n = \sum_{i+j+k=n-1} (-1)^{i+j+k} A(i, j, k)\).

   Shifting the index and using the multiplication rule for EGFs yields \(A(x) = \sum_{i+j+k=n-1} (-1)^{i+j+k} (i+j+k)! / i! j! k! \), and hence \(A'(x) = e^{2x} A(x)\). With the convention that there is one skyline of length 0, we want \(A(0) = 1\), and hence the solution of the differential equation is \(A(x) = e^{(e^{2x} - 1)/2}\).

   b) \( a_n = \sum_{k=0}^{n} 2^{n-k} S(n, k) \), from part (a).

   \[ A(x) = e^{(e^{2x} - 1)/2} = \sum_{k=0}^{\infty} \left( \frac{(e^{2x} - 1)/2}{k!} \right)^k \]

   \[ = \sum_{k=0}^{\infty} \frac{2^k}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} e^{2x(k-i)} = \sum_{k=0}^{\infty} \frac{2^k}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{1}{2^n} \]

   \[ = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} 2^{n-k} \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-1)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} 2^{n-k} S(n, k) \]

   c) **Combinatorial proof of part (b).** Let \(B_n\) be the set of objects consisting of a partition of \([n]\) with a subset of the non-minimal elements of each block marked. There are \(2^{n-k} S(n, k)\) such objects in which the partition has \(k\) blocks. We map \(B_n\) bijectively into the set of skylines of length \(2n\).

   Note first that a skyline can be broken into segments starting at its left-to-right minima \(m_1, \ldots, m_k\) so that the \(i\)th such segment has the form \(m_1 m_2 \cdots m_i \beta_i\), since the second copy of \(m_i\) cannot come earlier and cannot follow the next left-to-right minimum. Furthermore, the second copy of each element in \(\alpha_i\) must also be in \(\alpha_i\), and similarly for \(\beta_i\), to avoid surrounding a smaller element. Finally, since \(m_i\) is the least entry in this segment, \(\alpha_i\) and \(\beta_i\) are nonincreasing.

   Given a marked partition of \([n]\), index the blocks in decreasing order of their least elements. From the \(i\)th block, form the \(i\)th segment of a skyline by using the same least element, letting \(\alpha_i\) consist of both copies of the marked elements of the block, and letting \(\beta_i\) consist of both copies of the unmarked elements of the block. The steps of the construction are reversible, so the map is a bijection.

3. **Variation of Euler’s Identity for partitions into distinct parts.**

   \[ \prod_{i=1}^{\infty} (1 + x^i) = 1 + \sum_{k=1}^{\infty} \frac{x^{(k+1)/2} y^k}{(1-x)(1-x^2) \cdots (1-x^k)} \]
5. \[ \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{x^n}{k} = n \sum_{k=0}^{n} (-1)^{k} \frac{1}{k!} (x^n)^{k} \]

The probability that a random permutation of \( a_n \) has no cycles of length 2 is about \( e^{-\frac{1}{2}} \). Dividing the formula for \( a_n \) by \( n! \) yields a sum that is a truncation of the multiplicative series for \( e^{-\frac{1}{2}} \). The analogous computation yields \( e^{-\frac{1}{2}} \) as the approximate probability for large \( n \).

b) Use of the Exponential Formula to compute \( a_n \). There are \( n-1 \) cycles of length 2 on the labels. Cycles are formed to have length 2 or other lengths.

By the Exponential Formula, the Exponential Generating Function (EGF) under this restriction on power series by taking initial sums of coefficients, \( a_n = \lambda^n/n! \), where \( \lambda \) is the number of ways to pick \( k \) elements and pair them into cycles. By the rule, this is \( a_n = \prod_{k=1}^n (k-1/2) \).

4. The number of permutations of \( [n] \) with no cycles of length 2.

The number of permutations of \( [n] \) with no cycles of length 2 is \( a_n \). We use PIE in the universe \( \mathcal{G}_n \) for \( i, j \in [n] \).

For \( i < j \), let \( A_{ij} \) be the set of permutations outside the set \( \mathcal{G}(S) \). Then \( g(S) = 0 \) if \( g(S) \) avoids them all. The desired value is \( \sum_{S \subseteq [n]} (-1)^{|S|} g(S) \). The number of nonempty subsets is \( 2^n - 1 \).

The number of elements outside the set \( \mathcal{G}(S) \) with \( |S| = k \) is \( \sum_{j=0}^k (-1)^{k-j} \binom{n}{j} \binom{x^n}{k} \).

5. The number of cycles of length 2.

The number of cycles of length 2 on \( [n] \) is \( \sum_{i=1}^{n-1} \binom{n}{i} \binom{x^n}{i} \).

6. Distinguishable \( k \) beads on \( n \) necklaces with \( k \) colors.

For \( k = 3 \), the three colors are identical, so we use the standard pattern inventory. The three flips used to count positions as the cycle order in the standard pattern inventory. The cycle index for three beads in each of the \( k \) necklaces is \( \frac{1}{k!} \frac{1}{k!} \frac{1}{k!} \) for \( k = 3 \).

For \( k = 4 \), we have the number with no two colors used the same color twice. The number of necklaces with \( k \) colors is \( \binom{4}{k} \).

For \( k = 5 \), we have the number with no two colors used the same color twice.

a) Proof 1 (PIE). We use PIE in the universe \( \mathcal{G}_n \) for \( i, j \in [n] \).

For \( i < j \), let \( A_{ij} \) be the set of elements in \( \mathcal{G}(S) \). Then \( g(S) = 0 \) if \( g(S) \) avoids them all. The desired value is \( \sum_{S \subseteq [n]} (-1)^{|S|} g(S) \). The number of nonempty subsets is \( 2^n - 1 \).

b) Proof 2 (OCPS). Using the Binomial Theorem for the second step.

The coefficient \( x^n \) on the left counts partitions of \( n \) into \( k \) distinct parts with at least one card from each suit. For the right side, we pick \( k \) suits to contribute two cards and pick one card from each of the \( k \) suits to contribute two cards and pick one card from each suit.

When specified suits are omitted, we choose \( k+1 \) cards from the remaining \( n \) cards. Hence the summand here is exactly the summand in the standard inclusion-exclusion computation to count the selections in the \( k \) suit omitting no suits.
around opposite edges have structure $x_2^3$. The 60/300-degree rotations have structure $x_6$, the 120/240-degree rotations have structure $x_3^2$, and the 180-degree rotation has structure $x_2^2$. Hence the cycle structure is

$$Z_{D_6} = \frac{1}{12} (x_1^6 + 2x_6 + 2x_3^2 + 4x_2^3 + 3x_1^2x_2^2)$$

For the pattern inventory with colors red, white, and blue, we set $x_j = r^j + w^j + b^j$ for each $x_j$. For the classes with no two colors used the same number of times, we may assume that the usages of the three colors red/white/blue are in decreasing order and then multiply by 6.

Hence the usage multiplicities correspond to the partitions of 6 into (at least two) distinct parts. These are $5+1$, $4+2$, and $3+2+1$ (6 is forbidden, since then two colors are used 0 times). Therefore, we seek the coefficients of $r^3w$, $r^4w^2$, and $r^3w^2b$ in the pattern inventory. In each case we list only the nonzero contributions from the terms in the cycle index.

$$[r^5w]Z_{D_6} = \frac{1}{12} \left( \binom{6}{5,1,0} + 2 \cdot 0 + 2 \cdot 0 + 4 \cdot 0 + 3 \cdot 2 \right) = \frac{6 + 6}{12} = 1$$

$$[r^4w^2]Z_{D_6} = \frac{1}{12} \left( \binom{6}{4,2,0} + 2 \cdot 0 + 2 \cdot 0 + 4 \cdot \binom{3}{2} + 3(1 + 2) \right) = \frac{15 + 12 + 9}{12} = 3$$

$$[r^3w^2b]Z_{D_6} = \frac{1}{12} \left( \binom{6}{3,2,1} + 2 \cdot 0 + 2 \cdot 0 + 4 \cdot 0 + 3 \cdot 2 \cdot 2 \right) = \frac{60 + 12}{12} = 6$$

The final entry is the most delicate. We seek

$$[r^3w^2b](r + w + b)^2(r^2 + w^2 + b^2)^2.$$ We must get $b$ from one of the linear factors. Hence we cannot get a contribution to $w^2$ from the linear factors and must get $w^2$ from a quadratic factor. Hence we contribute to $r^3$ from one linear factor and one quadratic factor. This means we use one cross term from the product of the linear factors and one from the product of the quadratic factors.

Multiplying by 6 for the choices of highest and next-highest colors gives the answers 6, 18, and 36 for a total of 60.

c) For $n = 9$, there are 94 distinguishable necklaces using three colors three times each. With nine rotations and nine flips, the cycle structure is

$$Z_{D_9} = \frac{1}{18} (x_1^9 + 6x_9 + 2x_3^3 + 9x_1x_2^3)$$

Setting $x_j = a^j + b^j + c^j$ for all $j$, we extract the coefficient of $a^3b^3c^3$. From $x_1^9$, the contribution is $\binom{9}{3,3,3}$, which equals $14 \cdot 120$. From $2x_3^3$, the contribution is 12. The other terms make no contribution. We compute

$$\frac{1}{18}(14 \cdot 120 + 12) = \frac{1}{9}(14 \cdot 20 + 2) = 94.$$