1 Introduction

Definition. Let $G$ be a multigraph. The (edge) deck of $G$ is the multiset $\{G - e : e \in E(G)\}$; the members of the deck are cards. A multigraph $G$ is edge-reconstructible if every multigraph with the same deck as $G$ is isomorphic to $G$. Likewise, a multigraph property (or parameter) is edge-reconstructible if its occurrence (or value) for a particular multigraph can be determined from the deck. A class of multigraphs is edge-recognizable if the property of membership in that class is edge-reconstructible. A multigraph is simple if it has no loops or multiedges. Given a multigraph $M$, we will call a simple graph $G$ the underlying simple graph of $M$ if $G$ is obtained by replacing all multiedges of $M$ with edges of multiplicity 1.

The Edge-Reconstruction Conjecture states that every simple graph with more than three edges is edge-reconstructible. The Edge-Reconstruction Conjecture is implied by the Reconstruction Conjecture, which asserts that every simple graph with more than two vertices is reconstructible from its deck of vertex-deleted subgraphs (cards are obtained by removing vertices rather than edges). A survey of results on edge-reconstruction appears in [4]. Surveys on the original vertex analogue appear in [1, 2, 3].

The subject of edge-reconstructibility of nonsimple multigraphs does not appear to have received prior study. The topic was broached by grad student Yancey in the Structure of Graphs course taught by Professor Kostochka in Spring 2010. The following conjecture was eventually made.

Conjecture 1. Every nonsimple multigraph with more than two edges is edge-reconstructible.

Multigraphs in which every multiedge has the same multiplicity $m$ have the same vertex-deleted subgraphs as their underlying simple graphs, except each edge is replaced by a multiedge of multiplicity $m$, so the Reconstruction Conjecture on nonsimple multigraphs is more difficult than the Reconstruction Conjecture on simple graphs. However, the Edge-Reconstruction Conjecture may be easier for nonsimple multigraphs than for simple graphs. This summer, I showed several properties, parameters, and classes of loopless nonsimple multigraphs are edge-reconstructible. Some of the proofs are similar to those for analogous results for simple graphs, while others rely heavily on the multigraphs being nonsimple.

2 Results

With more than two edges, a multigraph is nonsimple if and only if it has a nonsimple card, so the desired class is edge-recognizable. Furthermore, multigraphs with multiple edges including at least one loop are edge-recognizable since at least one card in their deck must contain a loop. Therefore
Proof. Let $G$ be the multigraph with at least three edges and no isolated vertices, and $G$ will always refer to the underlying simple graph of $M$.

For $m \geq 1$ we will refer to a multiedge of multiplicity $m$ as an $m$-edge. We will use the term edge when referring to a 1-edge or an individual edge making up part of a larger multiedge, and we will refer to multiedges of multiplicity at least 2 as nontrivial multiedges.

The underlying simple graph $G$ of $M$ is an edge-constructible property ($G$ will be the underlying simple graph of any card with the maximum number of 1-edges) and therefore classes of multigraphs defined by underlying simple graph structure are edge-recognizable. For a multigraph $Q$ with fewer edges than $M$, counting arguments show that the parameter $S_Q(M)$ is edge-reconstructible, where $S_Q(M)$ counts the times $Q$ appears as a subgraph of $M$.

For a set $X = \{Q_1, \ldots, Q_k\}$ of multigraphs let $S_X^M$ denote the number of times $Q_i$ appears in $M$ not as a subgraph of any $Q_j$ for $i \neq j$.

**Lemma 1.** Let $X = \{Q_1, \ldots, Q_k\}$ be a set of multigraphs each with fewer edges than $M$. Suppose it can be verified from the deck of $M$ that for any (not necessarily distinct) $a, b, c$, if a copy of $Q_a$ is contained in the intersection of a copy of $Q_b$ and a copy of $Q_c$, then for some $d$ (potentially one of $a, b, c$), those copies of $Q_d$ and $Q_e$ are contained in some copy of $Q_d$. Then the parameter $S_X^M$ is edge-reconstructible.

**Proof.** Without loss of generality assume that $Q_1, \ldots, Q_k$ are ordered first by decreasing number of vertices, then by decreasing number of edges. Then $S_{Q_1}^M = S_{Q_1}(M)$. Now let $j > 1$ and assume that for each $i < j$ the value of $S_{Q_i}^M$ has been computed. Then $S_{Q_j}^M = S_{Q_j}(M) - (\sum_{i=1}^{j-1} S_{Q_i}^M S_{Q_j}(Q_i))$.

**Corollary 1.** For a multigraph $Q$ with fewer edges than $M$, the parameter $S_Q(M)$ is edge-reconstructible, where $S_Q(M)$ counts the times $Q$ appears as an induced subgraph of $M$.

**Proof.** Suppose $Q$ is a multigraph with fewer edges than $M$. If $Q$ has the same number of vertices as $M$ then $S_Q^M = 0$, so assume $Q$ has fewer vertices than $M$. Let $X$ be the set of all subgraphs of $M$ with the same number of vertices as $Q$. Then the lemma applies, and $S_Q^M = S_Q(M)$.

**Corollary 2.** The multiset of multiplicities of the multiedges of $M$ is edge-reconstructible.

**Proof.** We can calculate $S_Q^M$ for any multiedge $Q$ if $M$ has at least 3 vertices, and otherwise $M$ is a multiedge with multiplicity equal to the number of cards in the deck.

**Corollary 3.** The multiset consisting of, for each vertex $v$ of $M$, the multiset of multiplicities of multiedges incident to $v$ is edge-reconstructible.

**Proof.** Let $A$ be the desired multiset. If $G$ is a star then $M$ can be drawn by letting its multiedges all share a common vertex. Otherwise, let $X$ be the set of all subgraphs of $M$ whose underlying simple graphs are stars containing at least 3 vertices. Then the lemma applies and accounts for all elements of $A$ except for the singletons containing the multiplicity of a multiedge reaching a leaf of $G$. If $e$ is an $m$-edge, and $m$ appears $k$ times total in the non-singleton multisets of $A$, then $\{m\}$ should appear $2S_e^M - k$ times in $A$. 

2
**Corollary 4.** If $M$ is disconnected, then $M$ is edge-reconstructible.

**Proof.** Let $X$ be the set of all connected subgraphs of $M$. Then the lemma applies, and $M$ is the multigraph whose multiset of components contains precisely, for each $Q \in X$, $S^X_Q(M)$ copies of $Q$.

Using the above tools, we can obtain the following theorems, whose proofs are omitted for the sake of brevity.

**Theorem 1.** If $G$ is a tree, then $M$ is edge-reconstructible.

**Theorem 2.** If $G$ has a cut vertex but no leaf, then $M$ is edge-reconstructible.

**Theorem 3.** If $G$ has an edge $e$ such that $G - e$ contains only one pair of nonadjacent vertices $u, v$ such that adding the edge $uv$ to $G - e$ gives $G$, then $M$ is edge-reconstructible.

**Corollary 5.** If $G$ is regular, edge-transitive, or contains adjacent vertices both of degree 2 and no leaves, then $M$ is edge-reconstructible.

**Theorem 4.** Suppose the orbit $B$ of some edge of $G$ under $\text{Aut}(G)$ is such that no multiedge in $M$ corresponding to an edge in $B$ has multiplicity $m$, but at least one has multiplicity $m + 1$. If $B$ is not a single edge whose corresponding multiedge in $M$ has multiplicity 2 and is the only nontrivial multiedge in $M$, then $M$ is edge-reconstructible.

**Corollary 6.** If $M$ contains no 1-edge, then $M$ is edge-reconstructible.

**Theorem 5.** Let $m$ be the maximum multiplicity of a multiedge in $M$. Let $d$ be the minimum degree of any vertex intersecting an $m$-edge. If some vertex in $M$ has degree $d$ and intersects an $m$-edge but no 1-edge, then $M$ is edge-reconstructible.

**Theorem 6.** If some vertex of minimum degree in $M$ is adjacent to an $(m + 1)$-edge but no $m$-edge, then $M$ is edge-reconstructible.

3 Acknowledgements

Many thanks to Doug West for running REGS and assisting with this project.

The author acknowledges support from National Science Foundation grant DMS 08-38434 "EMSW21-MCTP: Research Experience for Graduate Students”.

References


