The Golden Partition Conjecture
posed by Marcin Peczarski

Let $P(\leq, S)$ be a partial order on the set $S$. An extension of $\leq$ is a partial order $\leq^*$ on $S$ such that $u \leq v$ implies $u \leq^* v$, for all $u, v \in S$. A linear extension of $P$ is an extension of $\leq$ that forms a linear order (a.k.a a total order). The number of linear extensions of $P$ is denoted $e(P)$.

Conjecture (The Golden Partition Conjecture). For any finite poset $P$ that is not a chain, there exist two consecutive comparisons such that regardless of their results the inequality

$$e(P) \geq e(P_1) + e(P_2)$$

holds, where $P_1$ and $P_2$ are the posets obtained from $P$ after the first comparison and after both comparisons, respectively.

The Golden Partition Conjecture is motivated by conjectures whose root is a famous sorting problem. Imagine that a finite set $S$ with cardinality $n$ has a hidden total order $\prec$ that we would like to uncover by making comparisons between pairs of elements from $S$. Comparing a pair $u, v \in S$ means uncovering either $u \prec v$ or $v \prec u$. How many comparisons are needed in the worst case? The classical answer is $\Theta(n \log_2 n)$ comparisons are needed and ‘merge sort’, for example, gives an algorithm to achieve this bound.

How many comparisons are needed if some partial information about $\prec$ is already known? Partial information about $\prec$ can be summarized by a poset $P(\leq, S)$. Let $C(P)$ denote the number of comparisons required to find the hidden linear extension $\prec$, in the worst case, starting from partial information $P$? Clearly $C(P) \geq \log_2(e(P))$, since each comparison can reduce the number of linear extensions by a factor of at most 2. The Golden Partition Conjecture gets its name because Peczarski [Pec06] has shown that it implies $C(P) \leq \log_\phi(e(P))$, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618033988$ is the golden ratio. As Peczarski states, “informally this [bound] means that during the sorting process the number of linear extensions can be decreased in every comparison on average by at least the golden ratio $\phi$.” Linial [Lin84] has constructed a sequence of posets that show that, if true, this bound would be tight.

The survey article by Brightwell [Bri99] is highly recommended.

Notes:

1. Peczarski [Pec06, Pec08] has proven the conjecture for semi-orders, width two posets, 6-thin posets, posets containing at most 11 elements. The conjecture implies the famous $1/3 - 2/3$ conjecture. The fraction of $n$-element posets satisfying the conjecture goes to 1 as $n \to \infty$.

2. Brightwell’s survey [Bri99] notes that Fredman proved that $C(P) \leq 2n + \log_2(e(P))$.

References


