Matchings and The Chinese Postman Problem in Odd-Regular Graphs

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Joint work with
Suil O – College of William & Mary

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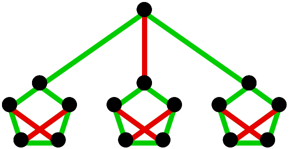
Def. A matching in a graph G is a set of disjoint edges. The matching number $\alpha'(G)$ is the maximum size of a matching in G.



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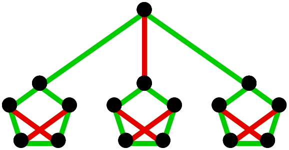

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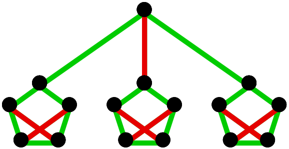

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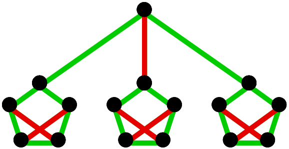

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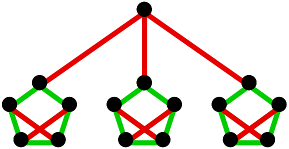

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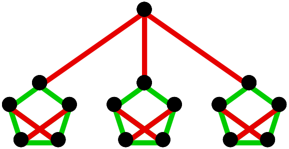

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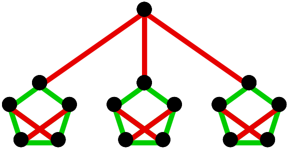

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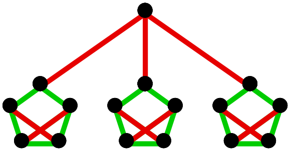

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Thm. (O–West [2011]) Determined $\min \alpha'(G)$ over k-conn. ℓ-regular n-vertex graphs and found those where equality holds. For example, if $G \in \mathcal{F}_{n,r}$ is $(2t+1)$-conn. with $t > 0$, then $\alpha'(G) \geq \frac{n}{2} - \frac{r-t}{2(r+1)^2+t}\frac{n}{2}$.

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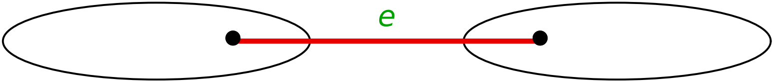

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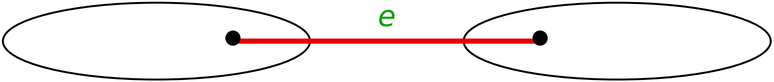

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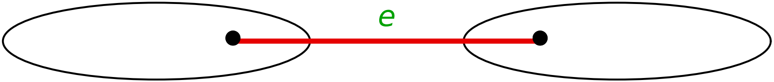

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A parity subgraph H must have odd degree at each v.
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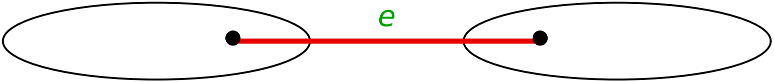

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We seek $(2r+1)$-regular graphs with many cut-edges.

Regular Graphs with Many Cut-Edges

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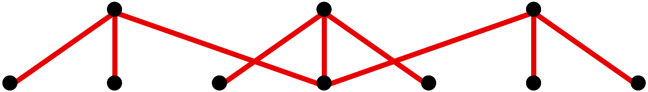

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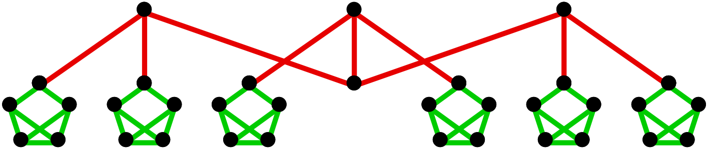

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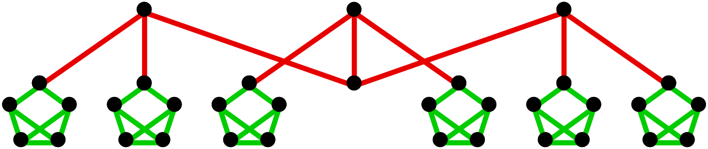

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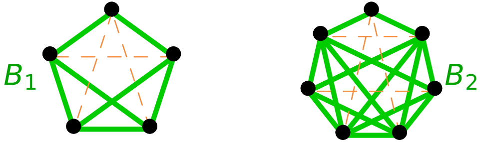

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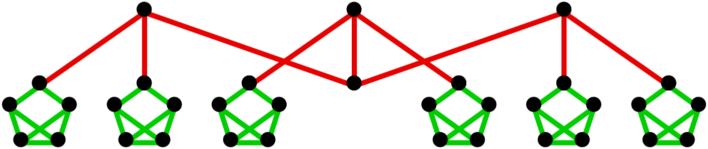

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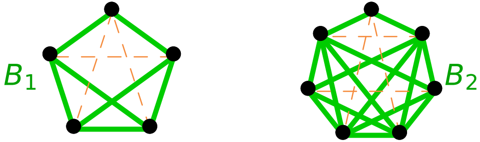

B_r is the smallest such graph.

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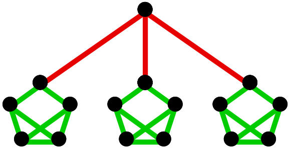

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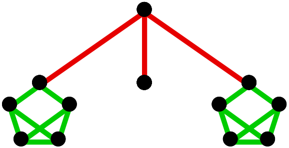

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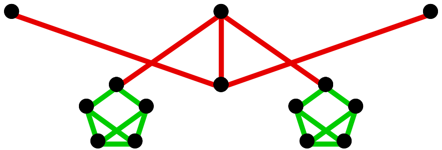

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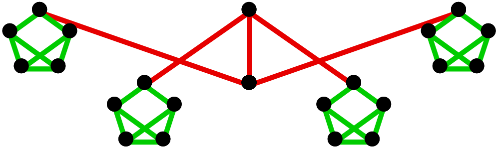

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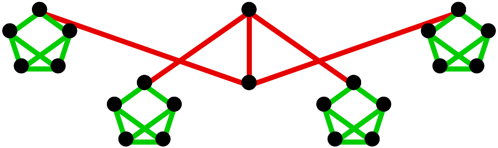

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How do $p(G)$ and the number of cut-edges change?

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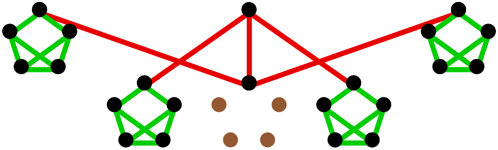

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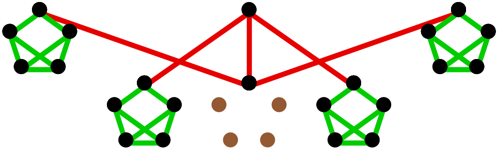

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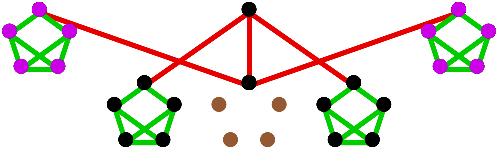

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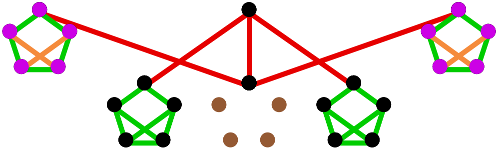

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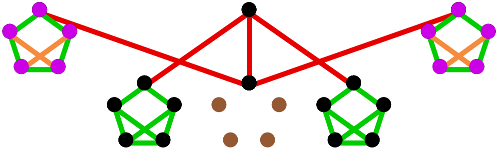

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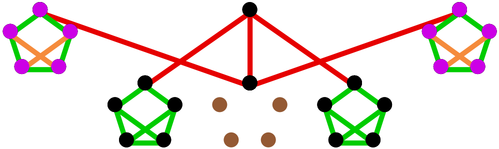

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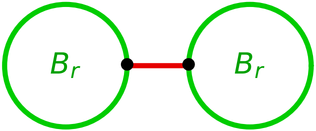
Pf. Basis: $n=4r+6$.



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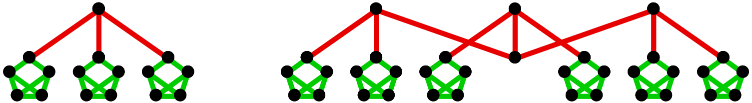

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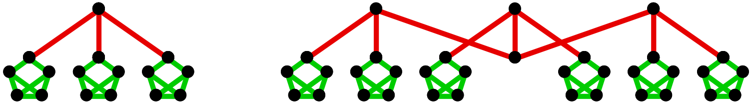

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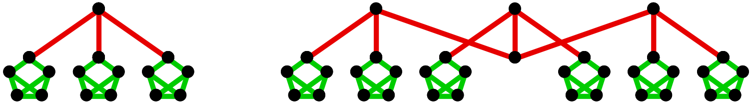

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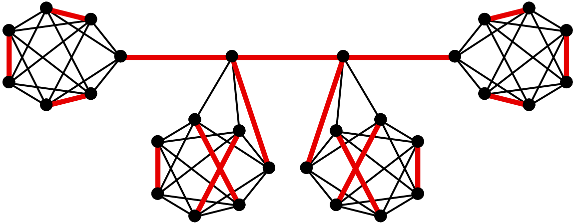
Example achieving equality for $r = 2$ (parity subgraph includes all cut-edges plus a matching on $n-2$ vertices):



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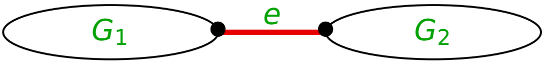

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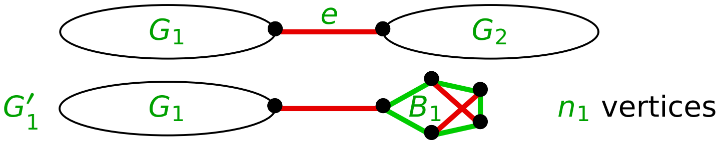

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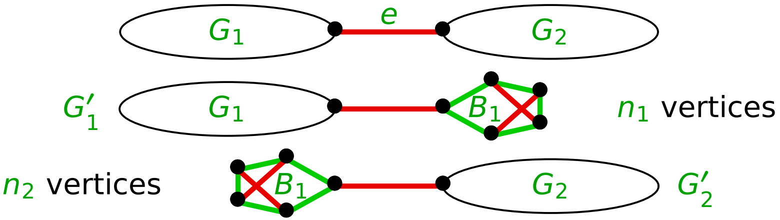

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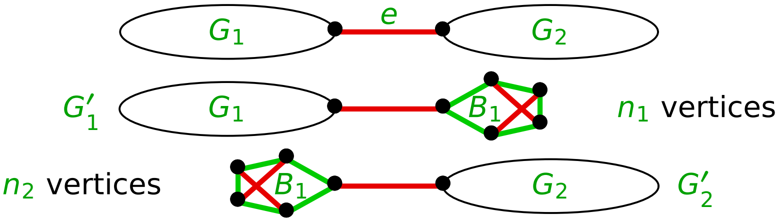

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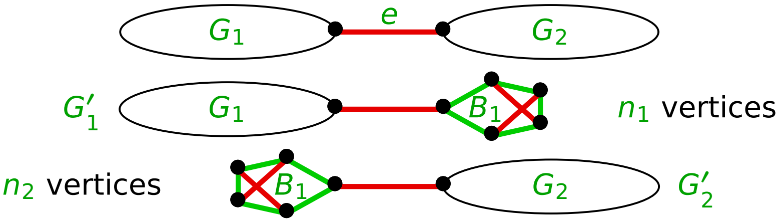

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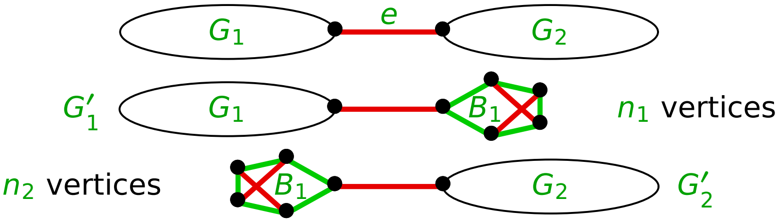

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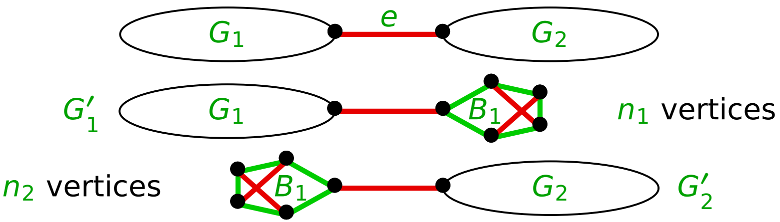

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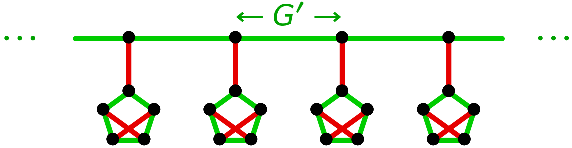

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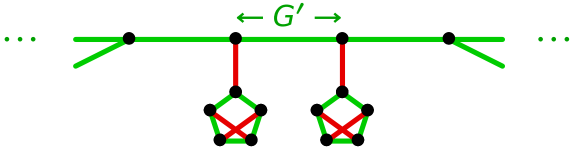

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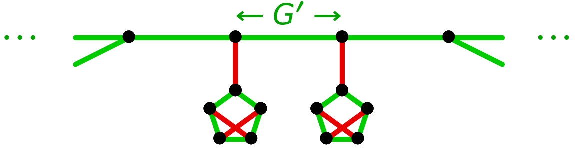

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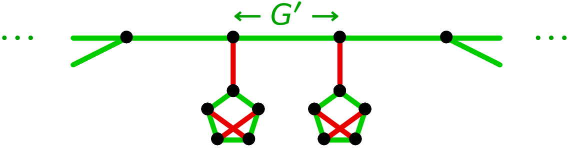

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Idea: Combine a parity subgraph of G'' with the red edges for balloons in G.

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Pf. To cover each edge p times, $|\mathcal{M}| = p(2r+1)$. The total weight over all the matchings is pW; pigeonhole. $\blacksquare$

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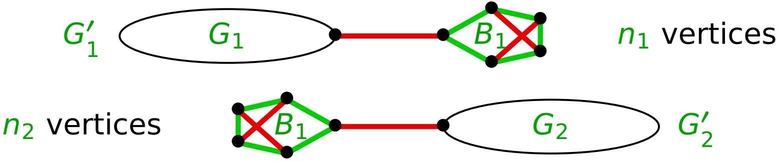

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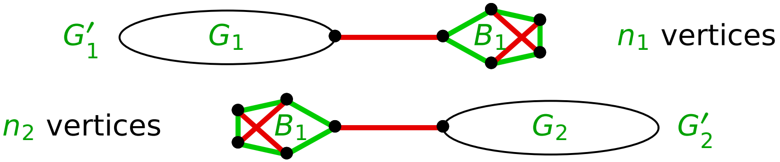

Now G also is constructed by attaching copies of B_1 to the leaves of a tree in $\mathcal{T}_1$, so $G \in \mathcal{H}_1$.

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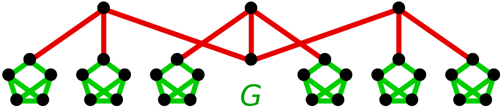

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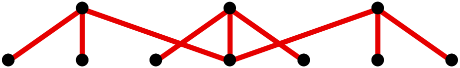

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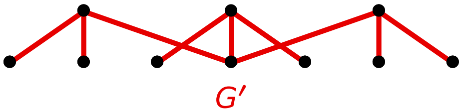

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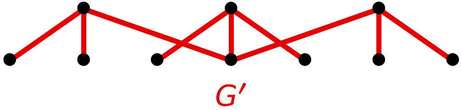

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By vertex degrees, $(2r+1)n' - 2rb(G) = 2m' \ge 2n' - 2$.

$\therefore$ $2rb(G) \le (2r-1)n' + 2$.

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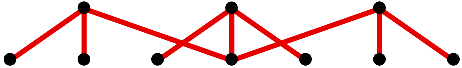

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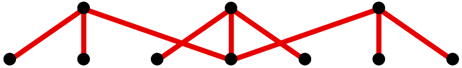

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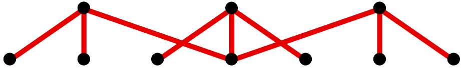

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Pf. More work for cuts with size between 3 and $2r-1$. ■

Game matching number of graphs

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slides available on DBW preprint page

Joint work with
Daniel W. Cranston,
William B. Kinnersley, and Suil O

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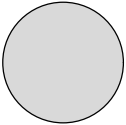

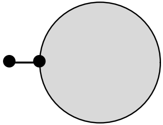

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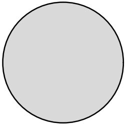

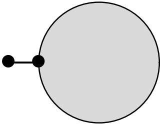

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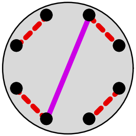

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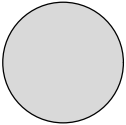

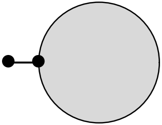

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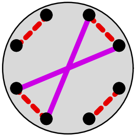

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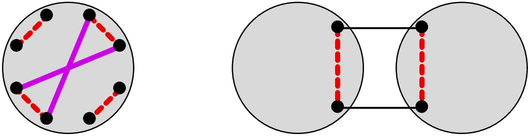

For $(2k-1, 2k)$, use $2K_{2k}$ modified by 2-switch. ■

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Prop. If $uv \in E(G)$, then $\alpha'_g(G) \ge 1 + \hat{\alpha}'_g(G - \{u,v\})$ and $\hat{\alpha}'_g(G) \le 1 + \alpha'_g(G - \{u,v\})$. Equality $\Leftrightarrow$ uv is optimal.

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Always $|\alpha'_g(G) - \hat{\alpha}'_g(G)| \le 1$

Prop. If $uv \in E(G)$, then $\alpha'_g(G) \ge 1 + \hat{\alpha}'_g(G - \{u,v\})$ and $\hat{\alpha}'_g(G) \le 1 + \alpha'_g(G - \{u,v\})$. Equality $\Leftrightarrow$ uv is optimal.

Thm. If $v \in V(G)$, then
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Sharpness: $rK_2 + C_6$, with v in isolated edge. No one wants to start on C_6. (Odd r for α'_g, even r for $\hat{\alpha}'_g$.)

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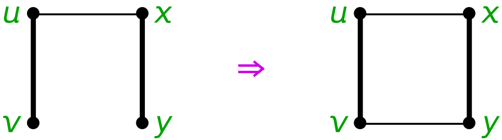

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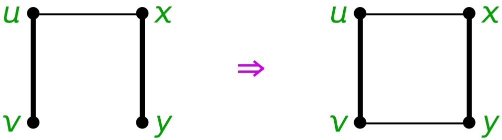

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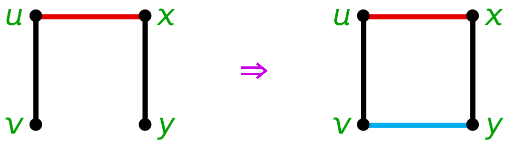

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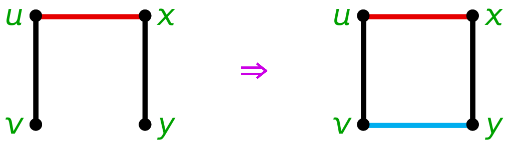

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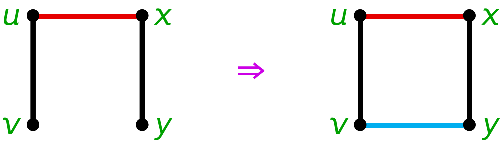

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- The condition is preserved under cartesian product with any graph. Hence Max can force a perfect matching in any $K_{r,r} \square H$, such as hypercubes.

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Ex. $G = K_{3k} \oplus \overline{K}_{n-3k}$: $\alpha'(G) = 3k$ and $\alpha'_g(G) = 2k$.

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Ex. Equality holds when $G = rP_4$ with r even.
Here $\mu(G) = r$ and $\hat{\alpha}'_g(G) = \frac{3}{2}r$.

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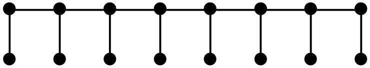

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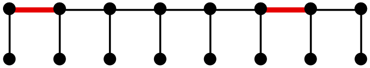

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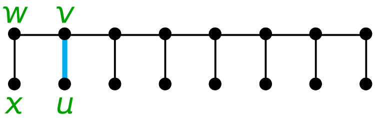

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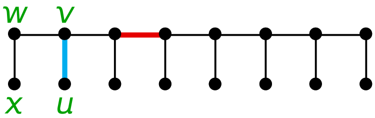

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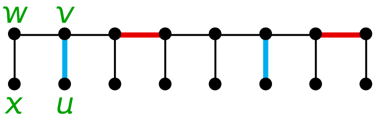

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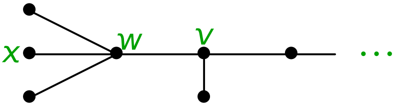

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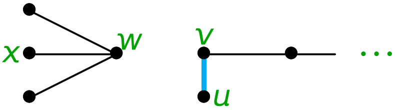

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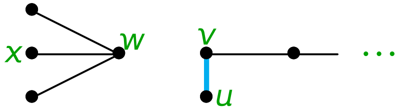

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The response by Min reduces α' by at most 2. Such moves by Min can occur only after moves by Max that guarantee two good moves. Hence $m \geq 4k$. ■

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With $\alpha'(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and $\mu(P_n) = \left\lfloor \frac{n+1}{3} \right\rfloor$, the prior results guarantee only $\frac{3}{8}n \leq \alpha'_g(P_n) \leq \frac{1}{2}n$, roughly.

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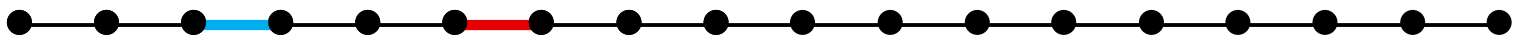

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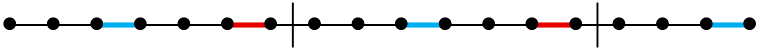

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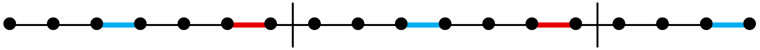

These strategies ensure these bounds against any strategy by the opponent. ■

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Moves by Max (after the first) delete at most $2r-2$ edges; moves by Min delete at most $2r-1$.

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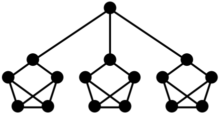
What construction provides an upper bound?

3-regular Examples



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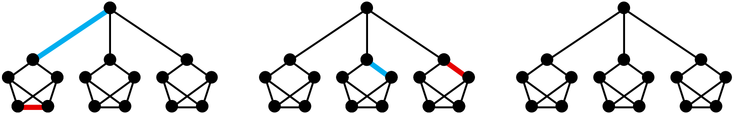

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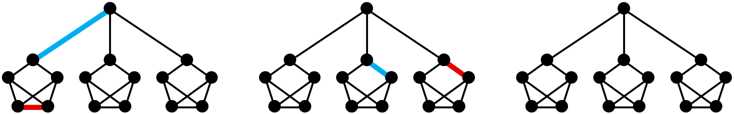

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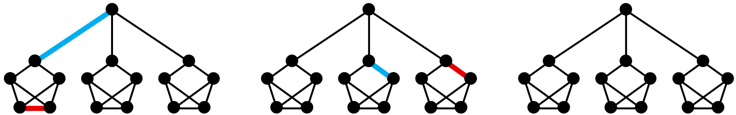

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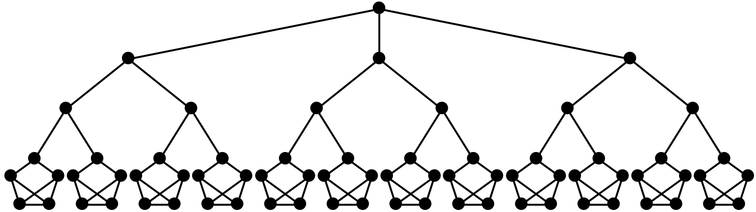

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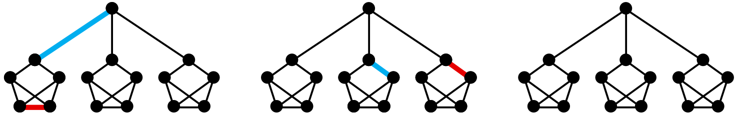

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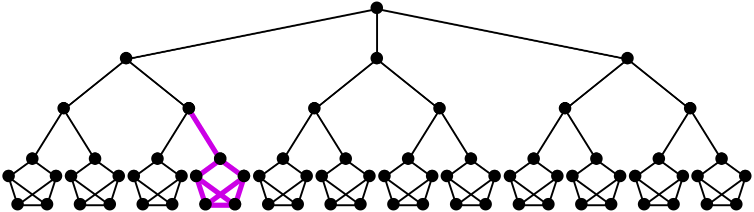

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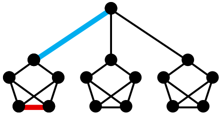

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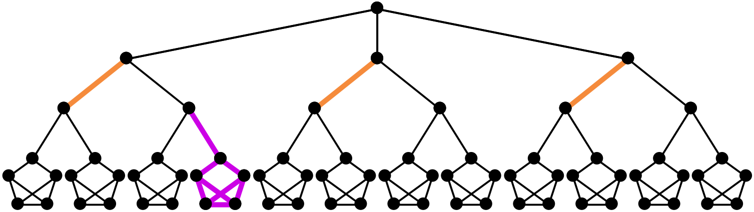

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Together with a maximum matching in the tree above copies of B, fewer than $\frac{7}{18}n$ edges are played.

Open Problems

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Ques. The matching game is the "F-saturation game" with $F = P_3$. What can be proved for other F?