Discrete Applied Mathematics 34 (1991) 279–284
North-Holland

The maximum number of winning 2-sets

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Received 8 June 1989
Revised 27 April 1990

Abstract

West, D.B., The maximum number of winning 2-sets, Discrete Applied Mathematics 34 (1991) 279–284.

The *winning t-set problem* is the problem of choosing nonnegative real numbers $x_1, \dots, x_n$, subject to $x_i \le a$ and $\sum x_i = b$, so as to maximize the number of t-sets I such that $\sum_{i \in I} x_i \ge 1$. For $t=2$, the optimal nontrivial solutions use at most two distinct nonzero values, and the optimal values and multiplicities can be computed quickly from n, a, b. It is conjectured that the phenomenon of using at most two distinct nonzero values persists for $t > 2$.

Keywords. Random variables, subsets.

In this paper, we consider what might be called the *winning t-set problem*. Given parameters n, a, b, t, we are asked to choose nonnegative real numbers $x_1, \dots, x_n$, subject to $x_i \le a$ and $\sum x_i = b$, so as to maximize the number of t-sets I having the property that $\sum_{i \in I} x_i \ge 1$. A t-set of values summing to at least 1 is called a *winning set*. We can view this as a normalized voting problem, with 1 representing a fixed threshold for victory. Then x_i represents the fraction of this threshold held by the ith player, and we seek to maximize the number of winning coalitions of size t.

The problem originated in an investigation by Richard Karp involving sampling without replacement from a discrete distribution. Let X be a random variable that is the sum of the values of t samples taken without replacement from a discrete probability distribution on nonnegative values bounded by a. Let Y be the corresponding random variable for sampling with replacement; Y is the sum of t independent and identically distributed samples from the distribution. Note that X and Y have the same expectation μ. The behavior of Y is fairly well understood. Heoffding [2] extended techniques of Chernoff [1] to show that $\text{Prob}(Y \ge z) \le e^{-2(z-\mu)^2/(ta^2)}$ if $z > \mu$. Hoeffding used the fact that $E(f(X)) \le E(f(Y))$ when f is continuous and convex to prove that the same bound also holds for $\text{Prob}(X \geq z)$. (The argument does not prove $\text{Prob}(Y \geq z) \geq \text{Prob}(X \geq z)$, since the relevant f in that case is a step function, and indeed this inequality need not hold for particular distributions.)

Our combinatorial problem seeks distributions with large tail probabilities for X. We can scale the values of the sample points so that $z = 1$ and $\mu < 1$. Among problems in which the n sample points are equally likely and the values are bounded by a, our problem is equivalent to maximizing $\text{Prob}(X \geq 1)$, since we have $bt/n = E(X)$ and $\text{Prob}(X \geq 1) = f(X)/\binom{n}{t}$, where $f(X)$ is the number of winning t-sets for the multiset of values $X = \{x_1, \ldots, x_n\}$.

We seek the optimal solution as a function of n, t, a, b. As an example, the solution for $(n, t, a, b) = (8, 2, 0.8, 2.4)$ is $X = (0.8, 0.8, 0.2, 0.2, 0.2, 0.2, 0, 0, 0, 0)$. For $t = 2$, we describe the solution by characterizing the potentially optimal configurations. Each such configuration is potentially optimal only in a fixed region of the a, b-plane. For fixed n, 2, a, b, the optimal solution is obtained by comparing a small constant number of values. For $t > 2$, we conjecture that it remains true that there is always an optimal solution with at most two distinct nonzero values, and that furthermore there is an optimal solution with a unique criterion for winning sets, meaning a threshold for the number of nonzero values and a threshold for the number of the higher of the two nonzero values. For each possibility for the criterion, the optimal multiplicities of the values in X are determined by a constrained nonlinear integer optimization problem.

For fixed n, t, the problem is nontrivial only in a rectangular region of the a, b-plane. Note that $b \leq na$ is required for feasibility. If $b \geq n/t$ and there is a feasible solution, then setting each $x_i = b/n$ makes every t-set a winner. If $b < 1$, then no t-set wins. Hence we may assume $1 \leq b < n/t$. If $a < 1/t$, there are no winning sets. If any $x_i > 1$, we obtain at least as many winning sets by reducing x_i to 1 and

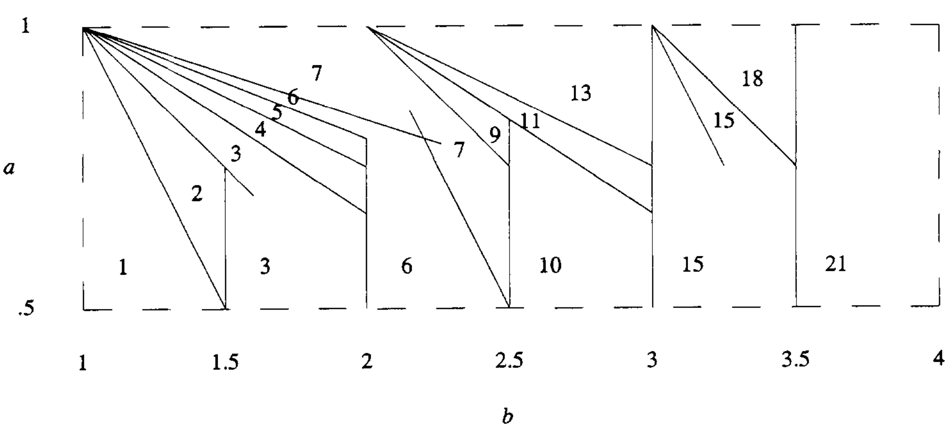

Fig. 1. Regions of constant optimal configurations for $(t, n) = (2, 8)$.

redistributing the excess, so we may assume $1/t \le a \le 1$. Note that $1/t \le a \le 1 \le b < n/t$ implies $b \le na$.

The effect of the theorem we prove is illustrated in Fig. 1 for $(t,n) = (2,8)$. The regions with a vertical left boundary correspond to configurations in which all the values involved in winning pairs are equal; when b reaches $m/2$ we can assign m variables the value b/m, which will be at least 0.5, so there will be $\binom{m}{2}$ winning pairs. The regions fanning out from $(b,a) = (k,1)$ correspond to configurations in which two distinct nonzero values are used, with the larger value having multiplicity k and the smaller value having multiplicity greater than k. The values chosen are functions of a and b, but they always sum to at least 1, so any pair of variables with high value or a high-low pair is a winning pair. The lower boundaries of these regions are determined by feasibility constraints. As $a \to 1$, the high value approaches 1. At $(k,1)$, the high value 1 appears k times, the low value is 0, and there are $\binom{k}{2} + k(n-k)$ winning pairs.

To discuss a feasible solution $X = (x_1, \ldots, x_n)$, we let s denote the number of distinct nonzero values, with $\alpha_1 > \cdots > \alpha_s$ the values and $m_1, \ldots, m_s$ their multiplicities. As above, we let $f(X)$ denote the number of winning t-sets. We say that a collection of indices $i_1 \le \cdots \le i_t$ is a *critical set* if $\alpha_{i_1} + \cdots + \alpha_{i_t} \ge 1$ but replacing any i_j by i_{j+1} destroys this property. By this definition, the collection of critical sets forms an antichain in the componentwise ordering of nondecreasing positive integer sequences of length t with values bounded by s. This partial order is isomorphic to the order known in the combinatorial literature as $L(t, s-1)$. Note that every winning set of indices is dominated in this ordering by some critical set.

The operation merge(i,j) on a feasible set X replaces the values equal to α_i and α_j by $m_i + m_j$ identical values. The new common value is $(m_i\alpha_i + m_j\alpha_j)/(m_i + m_j)$. The resulting set of values is also feasible.

Lemma 1. *In an optimal solution X with the minimum number of distinct nonzero values, every index appears in some critical set, except possibly when there are exactly two distinct nonzero values.*

Proof. Suppose that X has s distinct nonzero values. If some index $i < s$ appears in no winning set, then we can apply merge$(i, i+1)$ to reduce the number of distinct values without losing any winning sets. Hence every index except the largest appears in some winning set. If the largest index appears in no winning set and $s > 2$, then the value $m_s\alpha_s$ given to the elements of least nonzero value can be reallocated to those of value α_{s-1} until they reach α_{s-2} or the small ones reach value 0. (If $s = 2$ and the largest index appears in no winning set, then we can spread $\alpha_2 m_2$ over $n - m_1$ elements in no winning sets or choose to have all but at most one of the elements in no winning set get value 0.)

Now suppose that i appears in a winning set S of indices and that S has exactly k indices that are at least i (note that repetition is allowed). Any winning set S is dominated by some critical set T; if T has no i, then T has at least k indices as large as $i+1$. This means that replacing each i by $i+1$ in S yields another winning set of indices. Since this is true for every winning set containing i, we can apply merge$(i, i+1)$ to obtain a feasible X' with fewer distinct values and at least the same winning t-sets. □

For example, if $a=1/t$ and $b=1+\varepsilon$, then the optimal solution is a single winning t-set and an extra element with value $\varepsilon<1/t$.

Lemma 2. *If $t=2$, then there is always an optimal solution using at most two distinct nonzero values.*

Proof. Let X be an optimal solution with the minimum number s of distinct nonzero values, not counting the possible single element of small value. By induction on s, we claim that the only antichain of critical sets that covers all of $\{1,\dots,s\}$ is $\{(i,s+1-i)\colon 1\le i\le \lceil s/2\rceil\}$. If 1 and s appear in distinct sets, then these are related. Hence $(1,s)$ is a critical set, and neither 1 nor s appears in any other critical set. The other critical sets follow by applying induction on s.

If $s>2$, consider the totals $\beta_1=\alpha_1+\alpha_s$ and $\beta_2=\alpha_2+\alpha_{s-1}$ for the critical sets $(1,s)$ and $(2,s-1)$. We show first that we may assume $\beta_1=\beta_2$. If $\beta_1>\beta_2$, we maintain feasibility by reducing α_1 to α_1' and increasing α_2 to α_2' so that $\alpha_1'+\alpha_s=\alpha_2'+\alpha_{s-1}$. We also require $m_1\alpha_1'+m_2\alpha_2'=m_1\alpha_1+m_2\alpha_2$ to avoid changing the total sum, which determines the values α_1', α_2' uniquely. On the other hand, if $\beta_1<\beta_2$, we reduce α_{s-1} to α_{s-1}' and increase α_s to α_s' so that $\alpha_1+\alpha_s'=\alpha_2+\alpha_{s-1}'$ and $m_s\alpha_s'+m_{s-1}\alpha_{s-1}'=m_s\alpha_s+m_{s-1}\alpha_{s-1}$. In either case, all winning sets in the old solution are also winning sets in the new solution.

With $\beta_1=\beta_2$, we now consider changing a pair of values α_1, α_s to the values α_2, α_{s-1}. The resulting solution has the same total sum and is also feasible, but now the winning sets may change. Compared to the original solution, the element of high value now belongs to m_s-1 fewer winning sets, the element of lower value to m_2 more winning sets, and all sets that contain an even number of these two elements are winning if and only if they were winning originally. Hence if $m_2\ge m_s-1$, then we can increase the number of winning sets by making this change, and continue

Table 1

s	m_1	m_2	f	α_1	α_2
1	$\lfloor 2b\rfloor$	–	$\binom{m_1}{2}$	$b/\lfloor 2b\rfloor$	–
2	$\approx\dfrac{b-n(1-a)}{2a-1}$	$n-m_1$	$\binom{m_1}{2}+m_1m_2$	$1-\alpha_2$	$\dfrac{b-m_1}{m_2-m_1}$
2	$\approx\dfrac{2b-1+a}{6a-2}$	$\left\lfloor\dfrac{b-am_1}{1-a}\right\rfloor$	$\binom{m_1}{2}+m_1m_2$	$1-\alpha_2$	$\dfrac{b-m_1}{m_2-m_1}$

to do so at an ever-increasing rate until we reduce m_1 or m_s to 0, at which point we obtain a solution at least as good with a smaller number of distinct values. Similarly, if $m_2 \le m_s - 1$, then we can make the change in reverse to improve the solution until m_2 or m_{s-1} becomes 0. Note that if $s=3$ this may lead to $m_2 = m_{s-1} = 1$, but then we can complete the job by applying merge(2, 3). □

Theorem 3. *If $t=2$, then the optimal solution for the n, t, a, b problem can be found in constant time. The potential optimal configurations are shown in Table 1. (The choices of m_1 and m_2 result from an integer optimization, as explained further in the proof.)*

Proof. We need only consider solutions with at most two nonzero values. The essence of the proof is that choosing m_1 and whether there are one or two nonzero values determines a canonical candidate for an optimal solution. Examining these as a function of m_1 reduces the number of candidate configurations to a constant. We need only check the number of winning sets to pick the best solution among these.

If there is only one nonzero value, that must be $\alpha \ge 1/2$, and we have $\alpha = b/m_1$ with $\binom{m_1}{2}$ winning sets. Maximizing $\binom{m}{2}$ subject to $b/m \ge 1/2$ is achieved by $m = \lfloor 2b \rfloor$. This is a feasible solution with one nonzero value if and only if $a \ge b/\lfloor 2b \rfloor$; otherwise we set $\alpha = a$ and have an excess element in no winning set. This case does not consider $\alpha = 1$, since that produces more winning pairs.

If there are two nonzero values in the optimal solution X, we may assume that $\alpha_2 < 1/2$, that $m_1 < m_2$, and that $m_1 + m_2 > 2b$. Otherwise, we can apply merge(1, 2) to obtain a solution X' with only one nonzero value and $f(X) = \binom{m_1+m_2}{2} \ge f(X')$. With $\alpha_2 < 1/2$, (1, 2) is the unique critical set and $f(X) = \binom{m_1}{2} + m_1 m_2$. With $m_1 < m_2$, we may assume $\alpha_1 + \alpha_2 = 1$. If not, leave m_1, m_2 unchanged but change α_2 to α and α_1 to $1-\alpha$, where $\alpha = (b - m_1)/(m_2 - m_1)$. This reduces α_1 and does not change the total sum, so feasibility is maintained, as are the winning sets. Note that $\alpha < 1/2$ is equivalent to $m_1 + m_2 > 2b$.

Feasibility requires $a \ge 1 - \alpha = (m_2 - b)/(m_2 - m_1)$, which we can rewrite as $m_2 \le (b - am_1)/(1-a)$ (for $a = 1$, see below). Hence the best solution with two nonzero values is obtained by maximizing $f = \binom{m_1}{2} + m_1 m_2$ subject to $2b < m_1 + m_2 \le n$ and $m_2 \le (b - am_1)/(1-a)$. The feasible region in m_1, m_2 is bounded by the axes and two linear constraints in terms of n, a, b. Only a few points on the boundary need to be examined to find the optimum. Note that these constraints imply $m_1 < b < m_2$.

For fixed m_1, we seek to maximize m_2, so we set $m_2 = \min\{n - m_1, \lfloor (b - am_1)/(1-a) \rfloor\}$. The constraint $m_2 \le n - m_1$ is relevant only if $\lfloor (b-a)/(1-a) \rfloor > n-1$. This happens when a is large. If $a = 1$, we set $m_2 = n - m_1$. For fixed $m_1 + m_2$, i.e., when the constraint $m_1 + m_2 \le n$ is active, we maximize f by maximizing m_1. Hence when $a = 1$ the optimal solution with two distinct values in winning sets is $m_1 = \lfloor b \rfloor < n/2$, $m_2 = n - m_1$, with $\alpha_1 = 1 - \alpha$ and $\alpha_2 = \alpha$ as described above. For example, although $X=\{1,1,1,.25,0,0,0\}$ is optimal for $n,a,b=(7,1,3.25)$, $X'=\{.75,.75,.75,.25,.25,.25,.25\}$ is another optimal solution for these values obtained as above.

In choosing m_1, then, one of the values to consider is $\max\{m_1: \lfloor (b-am_1)/(1-a)\rfloor \geq n-m_1\}$, which is approximately $p=[b-n(1-a)]/(2a-1)$. In this case the value is $\binom{m_1}{2}+m_1(n-m_1)$.

We must also consider the possibility of $m_2=\lfloor (b-am_1)/(1-a)\rfloor < n-m_1$. In this case the value to maximize as a function of m_1 is a quadratic, except for the annoying floor function. Ignoring that integrality requirement, the function would be $(0.5/(1-a))[(1-3a)m_1^2+(2b-1+a)m_1]$, in which the leading term is negative, so the function would be maximized at $q=0.5(2b-1+a)/(3a-1)$. Thus, we must also consider values of m_1 in the neighborhood of q and, if $q<p$, the value just above the boundary p, using the function $\binom{m_1}{2}+m_1\lfloor (b-am_1)/(1-a)\rfloor$. □

The problem of characterizing the optimal configurations for higher t remains. The difficulty in extending this proof for $t=2$ lies in the generalization of Lemma 2, because the variety of antichains in $L(t,s-1)$ that cover $\{1,\ldots,s\}$ is quite large. It is likely that the argument for general t will be somewhat different.

References

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