The Bar Visibility Number of a Graph

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Abstract

The *bar visibility number* of a graph G, denoted $b(G)$, is the minimum t such that G can be represented by assigning each vertex x the set S_x of points in at most t horizontal segments in the plane so that $uv \in E(G)$ if and only if some point of S_u sees some point of S_v via a vertical segment of positive width unobstructed by assigned points. Among our results are the following:

1) Every planar graph has bar visibility number at most 2, which is sharp.
2) $r \le b(K_{m,n}) \le r+1$, where $r = \left\lceil \frac{mn+4}{2m+2n} \right\rceil$.
3) $b(K_n) = \lceil n/6 \rceil$.
4) If G has n vertices, then $b(G) \le \lceil n/6 \rceil + 2$.

1 Introduction

In computational geometry, graphs are used to model visibility relations in the plane. For example, we may say that two vertices of a polygon "see" each other if the segment joining them lies inside the polygon. In the *visibility graph* on the vertex set, vertices are adjacent if they see each other. Similarly, we can define visibility on a set of line segments; two segments see each other if some segment joining them crosses no other segment in the set. Dozens

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of papers have been written concerning the computation and the recognition of visibility graphs and discussing applications to search problems and motion planning.

We consider a simpler model where all visibility is vertical. Let S be a family of pairwise disjoint horizontal segments (henceforth "bars") in the plane. The *bar visibility graph* of S is the graph with vertex set S in which two vertices are adjacent if and only if there is an unobstructed vertical channel (a strip of positive width) joining those bars. Requiring a channel of positive width is realistic, and it permits bars $[(a, y), (x, y)]$ and $[(x, z), (c, z)]$ to block visibility at x without seeing each other.

Tamassia and Tollis [17] and Wismath [21] characterized bar visibility graphs as the planar graphs that are embeddable with all cut-vertices on a common face. Similar questions have been studied for other models. Hutchinson, Shermer, and Vince [11] studied graphs generated by horizontal and vertical visibility of rectangles in the plane (see also [4] and [5]). Hutchinson [10] studied polar visibility of arcs on concentric circles centered at the origin. An introduction to bar visibility and other models appears in [15].

We study a generalization of visibility graphs analogous to a well-studied generalization of interval graphs. The *interval graph* of a family S of intervals on the real line is the graph with vertex set S in which two vertices are adjacent if and only if as intervals they intersect. Bar visibility graphs provide a geometric analogue of interval graphs; visibility replaces intersection as the adjacency relation, and the intervals may be at various heights.

To generalize interval representations, we let a *t-interval* be the set of points in a union of (at most) t intervals on a line. A *t-interval representation* of a graph G assigns t-intervals to vertices so that vertices are adjacent if and only if their assigned t-intervals intersect. The *interval number* $i(G)$ of G is the minimum t such that G has a t-interval representation. A recent nice result and further references on $i(G)$ appear in [2].

Here we similarly generalize the bar visibility model. A *t-bar* is the set of points in a union of (at most) t horizontal bars in the plane. A *t-bar representation* of G is an assignment of t-bars to vertices of G so that vertices are adjacent if and only if some vertical channel of positive width links their t-bars without intersecting any other assigned t-bar. The *bar visibility number* $b(G)$ of a graph G is the minimum t such that G has a t-bar representation. For simplicity, in this paper we abbreviate the term to *visibility number*. Note that bar visibility graphs are the graphs with visibility number equal to 1.

For graphs without large cliques, visibility number tends to be smaller than interval number, because bars can block visibility and because the upper and lower "sides" of a bar can be used independently to establish edges. In Section 2, we observe that it follows easily from the results of [17] and [21] that every planar graph has visibility number at most two, and this is sharp (planar graphs have interval number at most three [16]).

For other families, our lower bounds arise from an easy lemma based on the maximum number of edges in N-vertex planar graphs. Constructions are more difficult; we study complete bipartite graphs, complete graphs, and general n-vertex graphs. The visibility number of the complete bipartite graph $K_{m,n}$ is roughly half its interval number, being either $\left\lceil \frac{mn+4}{2m+2n} \right\rceil$ or one more than this (Section 3), but the complete graph K_n has interval number 1 and visibility number $\lceil n/6 \rceil$ (Section 4).

Every planar graph without cut-vertices is a visibility graph. Thus in some sense visibility number is a measure of how far a graph is from being planar. It is related to other such parameters, such as "thickness" (Section 2) and "splitting number" (Sections 3 and 5). The optimal upper bound on the visibility number of K_n arises from the solution of "Heawood's empire problem" (Section 4).

We conjecture that $b(G) \leq \lceil n/6 \rceil$ when G has n vertices; equality holds for K_n. In support of this conjecture, in Section 5 we show that $b(G) \leq \lceil n/6 \rceil + 2$. We use the result of Lovász [14] that every m-vertex graph decomposes into at most $\lfloor m/2 \rfloor$ paths and cycles. Gallai's conjecture [6] that $\lceil m/2 \rceil$ paths suffice would yield $b(G) \leq \lceil n/6 \rceil + 1$.

2 Planar Graphs and Thickness

The extremal problem for visibility number of planar graphs is solved by expressing an arbitrary planar graph as the union of two bar visibility graphs.

Remark 1 *$b(G \cup H) \leq b(G) + b(H)$.*

Proof. Bar visibility representations of G and H can be placed in disjoint vertical strips to represent $G \cup H$. □

There are two ways to use earlier results to show that all planar graphs have visibility number at most 2. The most immediate uses a result of Wismath [21] showing that every planar graph is a *rectangle-visibility graph*, meaning that vertices can be assigned rectangles so that two vertices are adjacent if and only if the corresponding rectangles can see each other along a horizontal or a vertical strip of positive width.

Remark 2 *If G is a planar graph, then $b(G) \leq 2$.*

Proof. By [21], G has a rectangle-visibility representation. The projections in the horizontal and vertical directions yield two bar visibility graphs whose union is G. □

As mentioned earlier, a graph is a bar visibility graph if and only if it is a planar graph embeddable so that all cut-vertices lie on a single face [17, 21]. The minimal planar graphs not embeddable with every vertex on a single face are K_4 and $K_{2,3}$. Adding a pendant edge at each vertex of such a graph produces a planar graph that is not a bar visibility graph because the cut-vertices cannot lie on a single face. Thus the bound in Remark 2 is sharp.

The characterization of bar visibility graphs also yields another proof of the upper bound via an inductive proof of a technically stronger statement. The *block-cutpoint tree* of a connected graph G is the bipartite graph whose partite sets are the cut-vertices and the blocks of G, with vertex v adjacent to block B if $v \in V(B)$. The block-cutpoint graph was introduced by Harary and Prins [7], and it is an elementary exercise that it is a tree.

Theorem 3 *Every planar graph has a 2-bar representation in which every vertex that is not a cut-vertex is assigned a 1-bar.*

Proof. If H is a disjoint union of planar graphs with at most one cut-vertex in each component, then the characterization in [17, 21] yields $b(H) = 1$. We express a planar graph G as the union of two such graphs G_0 and G_1, and then Remark 1 applies.

We may assume that G is connected. We allocate the blocks of G to G_0 and G_1. Let B_0 be an arbitrary block of G. The distance of each block from B_0 in the block-cutpoint tree $B(G)$ is even; place a block in G_i if its distance from B_0 is congruent to $2i$ modulo 4. Two blocks wind up in the same component of G_0 or G_1 if and only if they share the same cut-vertex as neighbor on their paths to B_0 in $B(G)$. Hence a component of G_0 or G_1 consists of one or more such blocks of G sharing a single cut-vertex. □

Our subsequent lower bounds on visibility number use an easy counting argument based on the number of edges in planar graphs.

Lemma 4 *The visibility number of a graph G with n vertices and e edges is at least $\left\lceil \frac{e+6}{3n} \right\rceil$. If the graph is triangle-free, then $b(G) \geq \left\lceil \frac{e+4}{2n} \right\rceil$.*

Proof. Consider a t-bar representation of G. Let N be the total number of bars used, so $N \leq nt$. In the plane, draw one vertical segment joining each pair of bars that see each other. Now shrink each bar so that it becomes a single point. The added segments remain, covering the edges of G. The result is a simple planar graph G' with N vertices and at least e edges. Since it also has at most $3N - 6$ edges, we have the desired bound.

If G is triangle-free, then the graph G' will also be simple and triangle-free after we in addition contract edges joining points assigned to the same vertex of G. Now G' has at most $2N - 4$ edges, and again these cover all edges of G. □

The *thickness* of a graph G is the minimum number of planar graphs needed to decompose it. Each graph in a decomposition has at most $3n - 6$ edges when G has n vertices, and is bipartite when G is bipartite. Hence the thickness of an n-vertex graph with e edges is at least $\left\lceil \frac{e}{3n-6} \right\rceil$, and it is at least $\left\lceil \frac{e}{2n-4} \right\rceil$ if the graph is bipartite.

These lower bounds on thickness are slightly larger than the lower bounds of Lemma 4 on visibility number. If G has an optimal decomposition using 2-connected planar graphs, then the thickness of G becomes an upper bound on visibility number. This bound need not be optimal, since the lower bound on visibility number is smaller. In general, we can do better for bar visibility representations because the planar pieces can "interact".

For the complete graph K_n, with two exceptions the thickness equals the counting bound [1], which simplifies to $\lceil (n+2)/6 \rceil$. When n is 9 or 10, the thickness is 3 (see [3, 19]), although the general formula suggests 2. For $n \geq 7$, we improve these upper bounds by showing in Section 4 that $b(K_n) = \lceil n/6 \rceil$.

A graph represented with one bar per vertex must be planar, so the upper bound using thickness cannot be improved when $5 \leq n \leq 6$. However, it can be improved when $n = 9$.

Construction 5 $b(K_9) = 2$. Since K_9 is nonplanar, $b(K_9) \geq 2$. Although the thickness of K_9 is 3, we can express $K_9 - ws$ as the union of two planar graphs when ws is an edge in K_9. We can put a representation of one of these above the other and extend bars for w and s as in Figure 1 to obtain the missing visibility for ws. This establishes $b(K_9) = 2$. $\square$

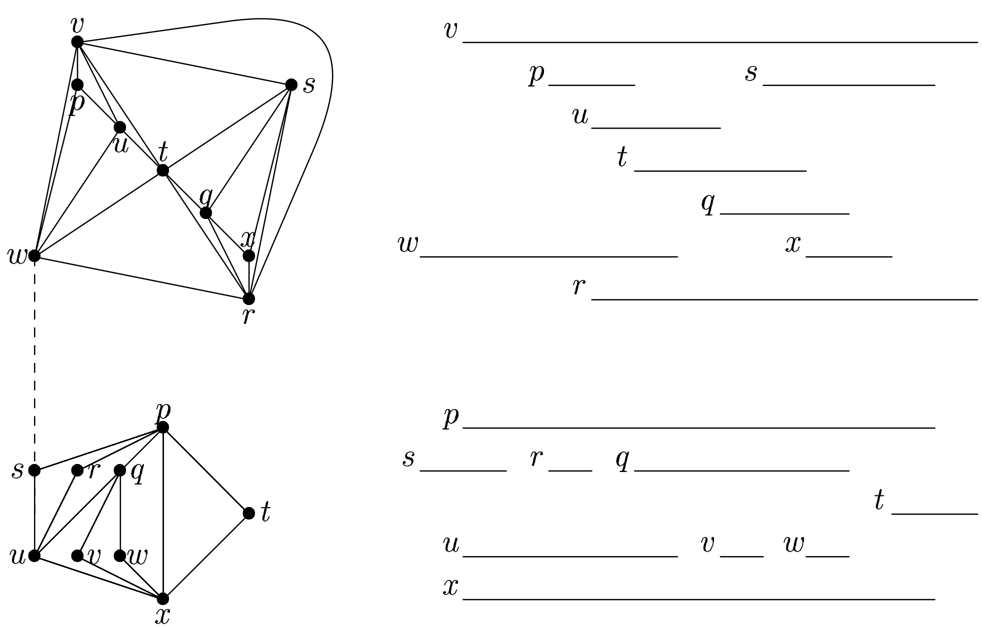

Figure 1. 2-bar representation of K_9.

Thickness also yields an upper bound for visibility number of the complete bipartite graph $K_{m,n}$. The thickness is conjectured to equal the counting bound $\left\lceil \frac{mn}{2m+2n-4} \right\rceil$. For $K_{10,6}$, the lower bound on thickness is 3. Nevertheless, Construction 6 shows that $b(K_{10,6}) = 2$.

In Section 3, we show that the counting bound on $b(K_{m,n})$ can be achieved within 1. Our upper bound $\left\lceil \frac{mn+4}{2m+2n} \right\rceil + 1$ is generally less than the thickness bound.

Construction 6 $b(K_{10,6}) = 2$. Since $(mn+4)/(2m+2n)$ exactly equals 2, achieving the lower bound is quite delicate. Our partite sets are $X = \{1, \ldots, 10\}$ and $Y = \{a, \ldots, f\}$. The construction is shown in Figure 2, with the bars for Y in bold. Neighboring bars on a line meet at their endpoints; there is no visibility through the gap. □

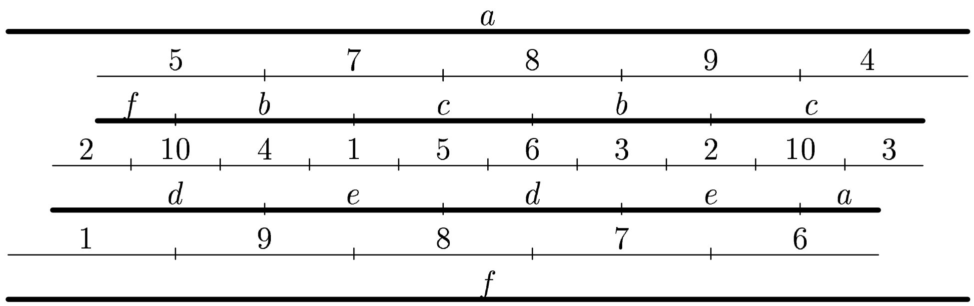

Figure 2. 2-bar representation of $K_{10,6}$.

3 Complete Bipartite Graphs and Splitting Number

Lemma 4 yields $b(K_{m,n}) \geq \left\lceil \frac{mn+4}{2m+2n} \right\rceil$; Construction 6 gives hope for equality. Trotter and Harary [18] proved that $i(K_{m,n}) = \left\lceil \frac{mn+1}{m+n} \right\rceil$. Our lower bound for $b(K_{m,n})$ always equals $\lceil i(K_{m,n})/2 \rceil$ or $\lceil i(K_{m,n})/2 \rceil + 1$. By using the tops and bottoms of bars separately, we prove constructively that $b(K_{m,n})$ is within one of the lower bound from Lemma 4.

Before presenting our construction, we compare $b(K_{m,n})$ with another parameter. Let a *vertex split* of a graph be the replacement of a vertex by two nonadjacent vertices, with each edge incident to the deleted vertex becoming incident instead to exactly one of the new vertices. The *splitting number* $s(G)$ of a graph G is the minimum number of vertex splits needed to turn G into a planar graph. Each vertex of G becomes an independent set in the resulting planar graph G'. If G' is 2-connected, then it has a bar visibility representation using $n(G) + s(G)$ bars. Thus $b(G) \geq 1 + s(G)/n(G)$.

Jackson and Ringel [12] proved that $s(K_{m,n}) = \lceil (m-2)(n-2)/2 \rceil$ when $\min\{m,n\} \geq 2$. Although $1 + \frac{(m-2)(n-2)}{2(m+n)} = \frac{mn+4}{2m+2n}$, this does not yield $b(K_{m,n}) \leq \left\lceil \frac{mn+4}{2m+2n} \right\rceil$, because the independent sets corresponding to vertices of $b(K_{m,n})$ in its optimal split need not have the same size. Indeed, the construction of [12] splits vertices in only one partite set.

We have observed that a bipartite graph G has at most $2N - 4$ edges if it has a t-bar representation using altogether N bars. Since $K_{m,n}$ has mn edges, this means that when $\frac{mn+4}{2m+2n}$ is an integer, achieving $b(K_{m,n}) = \frac{mn+4}{2m+2n}$ requires an $\frac{mn+4}{2m+2n}$-bar representation in which every vertex is assigned exactly $\frac{mn+4}{2m+2n}$ bars, and every face in the planar graph that results from turning the visibilities into edges and shrinking the bars has length exactly 4. It may be that $b(K_{m,n}) = \left\lceil \frac{mn+4}{2m+2n} \right\rceil$ always, but we have not proved this. Instead, we prove that allowing one more bar per vertex provides enough flexibility for a general construction.

Theorem 7 *If* $r = \left\lceil \frac{mn+4}{2m+2n} \right\rceil$, *then* $r \le b(K_{m,n}) \le r+1$.

Proof. We may take $m \ge n$ and let the partite sets be X and Y with $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. As m grows, r increases to $\lceil n/2 \rceil$. When $r = \lceil n/2 \rceil$, we construct an r-bar representation using r vertical strips, where the jth strip consists of one bar each for y_j and y_{n+1-j} with one bar for each vertex of X between them.

We may therefore assume that $r \le \lfloor (n-1)/2 \rfloor$. Let $s = \lfloor (n-1)/2 \rfloor - r$; note that $s \ge 0$. Since $r > n/4$, we have $r > s$. We construct an $(r+1)$-bar representation of $K_{m,n}$.

We start with (up to) $2(r+1)$ rows of bars for vertices of Y as in Figure 3 (in bold, with some labels dropped for clarity). The first row has bars for $y_1, \ldots, y_{\lceil n/2 \rceil}$, the second for $y_{\lceil n/2 \rceil + 1}, \ldots, y_n$, and thereafter these two types alternate. In each row, the ith bar extends from horizontal coordinate $i-1$ to i, except that when n is odd the bars for y_n extend from $(n-3)/2$ to $(n+1)/2$.

We add rows of up to $\lceil n/2 \rceil - 1$ bars for X between successive rows for Y. Consecutive bars in a row share endpoints, so each bar sees only bars for vertices of the opposite partite set in the two neighboring rows.

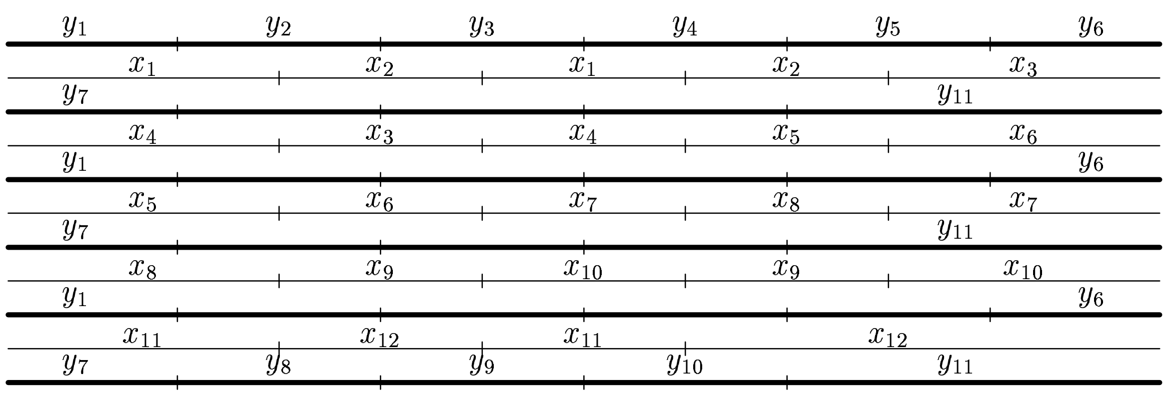

Figure 3. Part of a 4-bar representation of $K_{12,11}$; $r = 3$ and $s = 2$.

Reading from left to right within successive rows for X from top to bottom, we alternate x_1, x_2 until we have s bars for each, then we alternate x_3, x_4, etc. We do this until we reach $x_{2\lfloor m/2 \rfloor}$, using $2s \lfloor m/2 \rfloor$ positions for X. The last of these bars extends out to the right, filling the row to avoid visibilities within Y. Having enough positions available requires $2s \lfloor m/2 \rfloor \le (2r+1)(\lceil n/2 \rceil - 1)$; we prove this. From $\frac{mn+4}{2m+2n} \le r$ we obtain $m(n/2-r) \le rn-2$, and hence

$$2s\left\lfloor\frac{m}{2}\right\rfloor \le ms \le m\left(\frac{n-1}{2}-r\right) \le rn-2-\frac{m}{2} \le 2r\left\lfloor\frac{n-1}{2}\right\rfloor + 2r-2-\frac{m}{2}.$$

Since $r \le \lceil n/2 \rceil \le \lceil m/2 \rceil$, we have $2r-1-m/2 \le \lceil n/2 \rceil$, which yields the desired inequality.

If m is odd, then we add one or two long bars for x_m. If we ended with $x_{2\lfloor m/2 \rfloor}$ after an odd number of rows for X, as in Figure 3, then we put one bar for x_m above the top row for Y and another below the bottom row. If we ended with $x_{2\lfloor m/2 \rfloor}$ after an even number of rows for X, then we have another row for $y_{\lceil n/2 \rceil+1}, \ldots, y_n$ available to add and put one bar for x_m between the bottom two rows for Y. This establishes all visibilities for x_m.

We complete the construction by adding $r+1-s$ more bars for each $x \in \{x_1, \ldots, x_{2\lfloor m/2 \rfloor}\}$. Partition Y into $\lceil n/2 \rceil$ sets: the pairs of the form $\{y_j, y_{j+\lceil n/2 \rceil}\}$, plus the singleton $y_{\lceil n/2 \rceil}$ if n is odd. Via the s bars already placed, x already sees bars for $2s$ of these sets. Since $2s < s+r = \lfloor (n-1)/2 \rfloor$, the s bars for x cannot stretch far enough to cover two neighboring "columns" or the same "column" twice, and hence the $2s$ sets seen by x are distinct.

Since $r+1-s = \lceil n/2 \rceil - 2s$, it suffices to insert the remaining bars for x so that each such bar sees another of these sets. At a place where bars for a needed set appear (or at the right end when the set is $\{y_{\lceil n/2 \rceil}\}$), we shrink the intervening bars for vertices of X and insert small bars for vertices of X that need to add visibility to this set. (When $s = 0$, there are no bars to shrink, and the inserted bars block all the vertical space.) □

4 Complete Graphs and Heawood's Empire Problem

Heawood generalized the Four Color Problem by considering maps where many regions belong to a single "empire". Regions in a single empire must get the same color. Martin Gardner coined the term *m-pire* for an empire consisting of m regions. Jackson and Ringel [12] defined an *m-pire map* to be a map in the plane in which every empire consists of at most m regions. Heawood [9] proved that every m-pire map can be colored with $6m$ colors. He conjectured that the bound is sharp for $m > 1$; this became *Heawood's empire problem.*

Heawood's conjecture is proved by building a map with $6m$ pairwise adjacent m-pires. The adjacency graph is then K_{6m}, which requires $6m$ colors. Heawood did this for $m = 2$, and Herbert Taylor did it for $m = 3$ and $m = 4$. For larger m, it was first done by Jackson and Ringel [12]. Wessel [20] later gave a short proof. The result determines $b(K_n)$.

Theorem 8 *If $n \ge 7$, then $b(K_n) = \lceil n/6 \rceil$.*

Proof. For $n \ge 7$, Lemma 4 yields $b(K_n) \ge \left\lceil \frac{n-1}{6} + \frac{2}{n} \right\rceil = \lceil n/6 \rceil$. For the upper bound, we may assume that n is divisible by 6, because the absence of unwanted visibilities in the complete graph implies that deleting the bars for a vertex in an m-bar representation of K_n yields an m-bar representation of K_{n-1}.

Consider an m-pire map with $6m$ pairwise adjacent m-pires. When we associate a vertex with each region, we obtain a dual graph with at most $6m^2$ vertices. The dual graph is a plane graph, and we may assume that it is 2-connected because the m-pires are pairwise adjacent. That is, a cut-vertex in the dual would correspond to an annular region R that separates one set of regions from another. In cutting a channel through R to establish a common boundary for the two sets that had been separated, we do not change the set of regions neighboring R.

Being a 2-connected plane graph, the dual is a bar visibility graph. Associating the bars arising from each m-pire with one vertex of K_{6m} yields an m-bar representation of K_{6m}. □

The m-pire maps used to prove Heawood's conjecture are quite complex, even in Wessel's proof. A surprisingly simple visibility construction produces a representation using at most $b(K_n)+1$ bars per vertex. It will motivate the construction in the next section.

Construction 9 $b(K_n) \leq \lceil n/6 \rceil + 1$.

Proof. As in Theorem 8, we may assume that n is divisible by 6. Let $n = 6m$. We partition the vertex set into three sets A_1, A_2, A_3, each of size $2m$. A complete graph with $2m$ vertices has a decomposition into m spanning paths, consisting of the m rotations of a zig-zag path when the vertices are placed around a circle (see Figure 4).

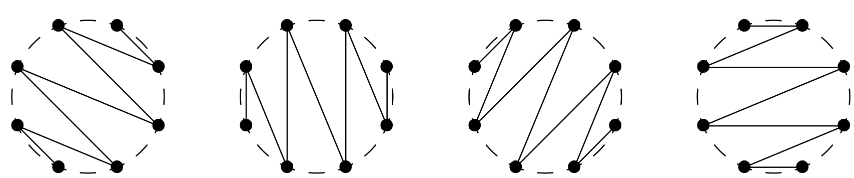

Figure 4. Path decomposition of K_8.

Our representation of K_n has $3m$ modules; each is a bar visibility representation of $P_{2m} \vee 2K_1$ (the *join* $G \vee H$ of graphs G and H is the graph formed from the disjoint union of G and H by adding edges to make each vertex of G adjacent to each vertex of H). We represent the path P_{2m} by a staircase of bars; each sees the bar before and after it. We add one long bar above and one long bar below; each sees the entire path (see Figure 5).

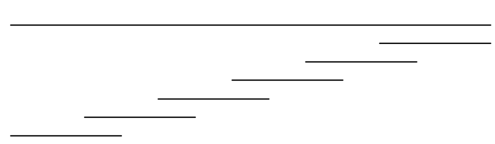

Figure 5. Bar visibility representation of $P_{2m} \vee 2K_1$.

To each path in the decomposition of A_i, we assign two vertices in A_{i+1} (indices modulo 3). This produces m pairwise edge-disjoint copies of $P_{2m} \vee 2K_1$. Their union covers the complete graph on A_i and all edges from A_i to A_{i+1}. Doing this for each A_i yields $3m$ modules whose union represents K_n.

A vertex of A_i appears in each path drawn from A_i, and it appears once as a top or bottom bar in a module for a path in A_{i-1}. Thus each vertex is assigned $m+1$ bars. □

5 n-Vertex Graphs

When bounding $b(G)$, it is tempting to use Remark 1 and express G as a union of bar visibility graphs. Unfortunately, as we have noted, the number of bars per vertex in the resulting representation is at least the thickness of G, which is too large.

Splitting number is more promising. If $H \subseteq G$, then $s(H) \le s(G)$, since any sequence of splits that reduces G to a planar graph also reduces H to a planar graph. Thus it may be possible to prove the conjecture that $b(G) \le \lceil n/6 \rceil$ by using a map with $6m$ pairwise adjacent m-pires (where $m = \lceil n/6 \rceil$), convert it to a splitting of K_{6m}, and delete adjacencies to reach a splitting of G without introducing a need for extra bars.

In addition to the splittings that result from Wessel's construction, one might also start with splittings of K_n studied by Hartsfield, Jackson, and Ringel [8]; they proved that $s(K_n) = \lceil (n-3)(n-4)/6 \rceil$. If their construction split vertices equally often, then it would like Wessel's construction yield an $\lceil n/6 \rceil$-bar representation of K_n.

In both results, the splittings of K_n are hard to describe, and it is not clear how to do the other steps. Instead, we generalize Construction 9 to prove directly that $b(G) \le \lceil n/6 \rceil + 2$.

It is hard to modify Construction 9 directly to delete arbitrary edges. For example, let u, v, w appear consecutively on some path in the decomposition of A_1, and let y, z be the vertices of A_2 whose bars surround this path. Extending the bar for u or w can block v from seeing y or z. Deleting the bar for v and extending those for u and w to the same vertical line can delete all these edges. However, how can we delete vy, vz, uv and keep vw?

If all edges of the path in A_i were present, then we could delete arbitrary edges to y and z by extending the bars for vertices on the path. If t_k is the maximum number of paths needed to partition the edges of a k-vertex graph, we could thus obtain $b(G) \le t_{n/3} + 1$. Gallai [6] conjectured that $t_k = \lceil k/2 \rceil$, which would yield $b(G) \le \lceil n/6 \rceil + 1$.

We do almost as well by using the result of Lovász [14] that every k-vertex graph can be decomposed into $\lfloor k/2 \rfloor$ paths and cycles. Each vertex of odd degree must be an endpoint of some path in such a decomposition. Thus the decomposition must consist entirely of paths when G has at most one vertex of even degree.

Theorem 10 *If G has n vertices, then $b(G) \le \lceil n/6 \rceil + 2$.*

Proof. By adding isolated vertices, we may assume that n is divisible by 6. Let $n = 6m$, and again partition $V(G)$ into sets A_1, A_2, A_3 of size $2m$. To the subgraph $G[A_i]$ induced by A_i, add one vertex w adjacent to all vertices with even degree in $G[A_i]$; call this graph G'_i. Since G'_i has at most one vertex of even degree, G'_i has a decomposition into m paths, since $\lfloor (2m+1)/2 \rfloor = m$.

With each such path P we associate two "special" vertices y and z of A_{i+1}, using different special vertices for different paths. We design a module that establishes the edges of P and the edges of G from A_i to y and z. We use at most one bar for each vertex of A_i and at most two bars each for y and z. Doing this for each i and each P in the decomposition of G'_i produces an $(m+2)$-bar representation of G (see Figure 6), since each vertex serves as a special vertex only once.

Let I_v denote the set assigned to v in such a module. We begin by representing $P \vee \{y, z\}$ as in Figure 5: a staircase plus a bar for y underneath and a bar for z above. The edges on P not involving the added vertex w belong to G, so we do not block these visibilities. Erasing I_w (if $w \in V(P)$) produces a gap that may cause us to break I_y and I_z.

Begining at the upper right end of P, we block visibilities between y and P as needed. When the current vertex v of P is not adjacent to y, we extend the bar for the next lower vertex of P to the right end of the current I_v. If the last vertex v before w is not adjacent to y, then we cannot extend a lower bar to block I_v from I_y; instead, we break I_y and shorten the left end of the right portion to the right endpoint of I_v.

Delete I_w; note that the bars for the neighbors of w on P do not see each other. Block bars in the lower part of P from seeing I_y as needed, in the same manner as before. Visibilities up to I_z are corrected in the same manner, working from the bottom left end of P.

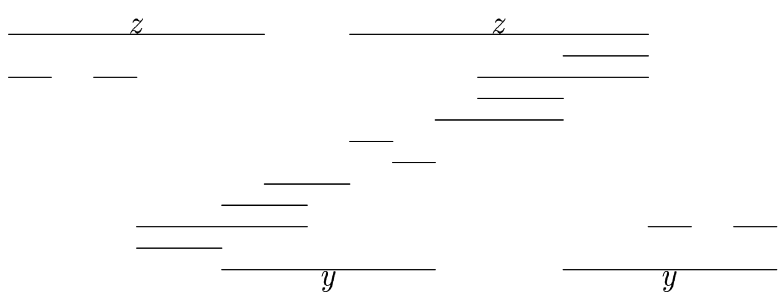

Figure 6. Module for representation of general graph.

For $q \in A_i - V(P)$, we also must consider edges from q to $\{y, z\}$. If $y, z \notin N(q)$, then we add no bar for q in this module. For all $q \in N(y) - N(z)$, we add a bar at the right of P; none of these see each other, and I_y extends to the right to see them all. Similarly, bars for vertices of $A_i - V(P)$ in $N(z) - N(y)$ are added at the left of the bars for P.

Now consider common neighbors of y and z in $A_i - V(P)$. If $w \in V(P)$, then we add bars in the gap between the left and right portions of P where I_w was deleted. Together they fill this gap so that I_y does not see I_z. The left portion of I_y and the right portion of I_z see these bars. If there are no vertices of $V(A) - V(P)$ in $N(y) \cap N(z)$, then we shorten the left portion of I_y and the right portion of z so that they won't see each other. If $w \notin V(P)$, then we did not break I_y or I_z into two bars. We therefore can place another bar for y and z at the right end of the module and use these to establish visibilities for the edges from $(V(A_i) - V(P)) \cap N(y) \cap N(z)$ to $\{y, z\}$, as above.

We have established and/or deleted all the desired adjacencies, using the desired number of bars for each vertex. □

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