# Multiple Vertex Coverings by Specified Induced Subgraphs

Zoltán Füredi[*], Dhruv Mubayi[†], Douglas B. West[‡]
University of Illinois
Urbana, IL, 61801

May 25, 1998; revised December, 1999

**Abstract**

Given graphs $H_1, \ldots, H_k$, let $f(H_1, \ldots, H_k)$ be the minimum order of a graph $G$ such that for each $i$, the induced copies of $H_i$ in $G$ cover $V(G)$. We prove constructively that $f(H_1, H_2) \le 2(n(H_1) + n(H_2) - 2)$; equality holds when $H_1 = \overline{H}_2 = K_n$. We prove that $f(H_1, \overline{K}_n) = n + 2\sqrt{\delta(H_1)n} + O(1)$ as $n \to \infty$. We also determine $f(K_{1,m-1}, \overline{K}_n)$ exactly.

## 1 Introduction

Entringer, Goddard, and Henning [2] determined the minimum order of a simple graph in which every vertex belongs to both a clique of size $m$ and an independent set of size $n$. They obtained a surprisingly simple formula for this value, which they called $f(m, n)$ (an alternative proof using matrix theory appears in [5]).

**Theorem 1.1** [2] *For $m, n \ge 2$, $f(m, n) = \left\lceil (\sqrt{m-1} + \sqrt{n-1}\ )^2 \right\rceil$.*

Theorem 1.1 was motivated by a concept introduced by Chartrand et al. [1] called the *framing number*. A graph $H$ is *homogeneously embeddable* in a graph $G$ if, for all vertices $x \in V(H)$ and $y \in V(G)$, there exists an embedding of $H$ into $G$ as an induced subgraph that maps $x$ to $y$. The *framing number* $fr(H)$ is the minimum order of a graph in which $H$ is homogeneously embeddable. The framing number of a pair of graphs $H_1$ and $H_2$, written $fr(H_1, H_2)$, is the minimum order of a graph $G$ in which both $H_1$ and $H_2$ are homogeneously

* *furedi@math-inst.hu* and *z-furedi@math.uiuc.edu*. Supported in part by the Hungarian National Science Foundation under grant OTKA 016389, and by the National Security Agency under grant MDA904-98-I-0022

† *mubayi@math.gatech.edu*. Supported in part by the National Science Foundation under grant DMS-9970325. Current address: School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160

‡ *west@math.uiuc.edu*

embeddable. Thus $fr(K_m, \overline{K_n}) = f(m,n)$. Various results about the framing number were developed in [1]. The framing number of a pair of cycles is studied in [7].

When the graphs to be homogeneously embedded are vertex-transitive, it matters not which vertex of $H$ is mapped to $y \in V(G)$ as long as $y$ belongs to some induced copy of $H$ in $G$. Determining the framing number for a pair of graphs becomes an extremal graph covering problem. We generalize this variation to more than two graphs.

**Definition 1.2** *A graph is* $(H_1, \ldots, H_k)$-full *if each vertex belongs to induced subgraphs isomorphic to each of* $H_1, \ldots, H_k$. *We use* $f(H_1, \ldots, H_k)$ *to denote the minimum order of an* $(H_1, \ldots, H_k)$*-full graph.*

Equivalently, a graph is $(H_1, \ldots, H_k)$-full if for each $i$, the induced subgraphs isomorphic to $H_i$ cover the vertex set, so we think in terms of multiple coverings of the vertex set.

Because every vertex in a cartesian product belongs to induced subgraphs isomorphic to each factor, we have $f(H_1, \ldots, H_k) \le \prod_i n(H_i)$, where $n(G)$ denotes the order of $G$. In fact, $f(H_1, \ldots, H_k)$ is much smaller. Our constructions in Section 2 yield $f(H_1, \ldots, H_k) \le 2\sum_i (n(H_i) - 1)$. Also, if $k-1$ is a prime power and $n(H_i) < k$ for each $i$, then $f(H_1, \ldots, H_k) \le (k-1)^2$. By Theorem 1.1, the first construction is optimal when $k = 2$ for $H_1 = K_n$ and $H_2 = \overline{K}_n$. We also provide a construction when $H_1$ is arbitrary and $H_2 = \overline{K}_n$ that is asymptotically sharp up to an additive constant.

In Section 3, we prove a general lower bound in terms of the order of $H_2$, the maximum degree of $H_2$, and the minimum degree of $H_1$. In Section 4, we determine $f(K_{1,m-1}, \overline{K}_n)$ exactly (the related parameter $f(K_{m,m}, \overline{K}_n)$ is studied in [4]). In Section 5, we present several open problems.

Since $f(H_1, \ldots, H_k) = f(\overline{H}_1, \ldots, \overline{H}_k)$, all our results yield corresponding results for complementary conditions. We note also that there is an $(H_1, \ldots, H_k)$-full graph for each order exceeding the minimum, since duplicating a vertex in such a graph yields another $(H_1, \ldots, H_k)$-full graph.

We consider only simple graphs, denoting the vertex set and edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. The *order* of $G$ is $n(G) = |V(G)|$. We use $N_G(v)$ for the neighborhood of a vertex $v \in V(G)$ (the set of vertices adjacent to $v$), and we let $N_G[v] = N_G(v) \cup \{v\}$. The degree of $v$ is $d_G(v) = |N_G(v)|$; we may drop the subscript $G$. For $S \subseteq V(G)$, we write $d_S(v)$ for $|N_G(v) \cap S|$. The *independence number* of $G$ is the maximum size of a subset of $V(G)$ consisting of pairwise nonadjacent vertices; it is denoted by $\alpha(G)$. When $S \subseteq V(G)$, we let $N(S) = \bigcup_{v \in S} N(v)$ and let $G[S]$ denote the subgraph induced by $S$.

# 2 General Upper Bounds

Our upper bounds are constructive.

**Theorem 2.1** *If* $H_1, \ldots, H_k$ *are graphs, then* $f(H_1, \ldots, H_k) \le 2\sum_{i=1}^k (n(H_i) - 1)$.

**Proof:** We construct an $(H_1, \ldots, H_k)$-full graph $G$ with $2\sum_{i=1}^k (n(H_i) - 1)$ vertices. For $1 \le r \le k$, let $H_{r+k}$ be a graph isomorphic to $H_r$. For $r \in \{1, \ldots, 2k\}$, distinguish a vertex $u_r$ in $H_r$, and let $N_r = N_{H_r}(u_r)$ and $H'_r = H_r - u_r$. Construct $G$ from the disjoint union $H'_1 + \cdots + H'_{2k}$ by adding, for each $r$, edges making all of $V(H'_r)$ adjacent to all of $N_{r+1} \cup \cdots \cup N_{r+k-1}$, where the indices are taken modulo $2k$.

By construction, $G$ has the desired order. For $v \in V(H'_r)$ and $1 \le j \le k-1$, we have $G[v \cup V(H'_{r+j})] \cong H_{r+j}$ (again taking indices modulo $2k$). Finally, $V(H'_r)$ together with any vertex of $V(H'_{r-1})$ induces a copy of $H_r$ containing $v$. □

Fig. 1 illustrates the construction of Theorem 2.1 in the case $k = 2$; an edge to a circle indicates edges to all vertices in the corresponding set.

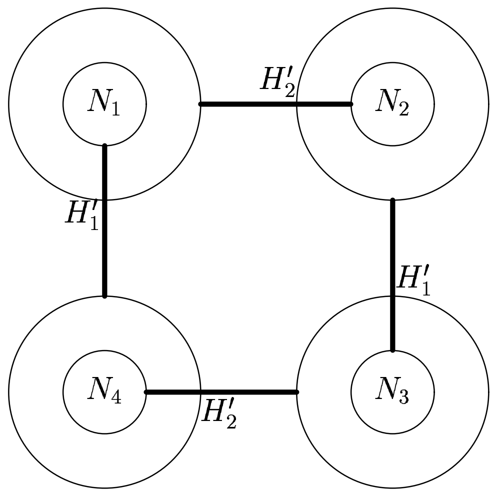


Fig. 1. An $(H_1, H_2)$-full graph

As mentioned earlier, Theorem 2.1 yields sharp upper bounds when $k = 2$ by letting $H_1 = K_n$ and $H_2 = \overline{K}_n$. In general, as pointed out by a referee, the bounds can be off from the optimal by at least a factor of two. To describe the construction that improves Theorem 2.1 in some cases, we use resolvable designs. We phrase the constructions in the language of hypergraphs. A hypergraph $\mathcal{H} = (V, E)$ has vertex set $V$ and edge set $E$ consisting of subsets of $V$. $\mathcal{H}$ is *k-uniform* if every edge has size $k$, and $\mathcal{H}$ is *k-regular* if every vertex lies in exactly $k$ edges. A *matching* $M$ in $\mathcal{H}$ is a set of pairwise disjoint edges; $M$ is *perfect* if the union of its elements is $V$.

A *Steiner system* $S(n, k, 2)$ is an $n$-vertex $k$-uniform hypergraph in which every pair of vertices appears together in exactly one edge. It is *resolvable* if the edges can be partitioned into perfect matchings. Ray-Chaudhuri and Wilson [8] showed that the trivial necessary condition $n \equiv k \pmod{k^2 - k}$ for the existence of a resolvable $S(n, k, 2)$ is also sufficient when $n$ is sufficiently large compared to $k$.

**Theorem 2.2** *If a resolvable Steiner system $S(n, k-1, 2)$ exists and $H_1, \ldots, H_t$ are graphs of order less than $k$, where $t \le (n-1)/(k-2)$, then $f(H_1, \ldots, H_t) \le n$.*

**Proof:** Duplicating vertices cannot decrease $f$, so we may assume that $n(H_i) = k - 1$ for each $i$. Let $V$ and $E$ be the vertex set and edge set of the resolvable Steiner system

$S(n, k-1, 2)$; we construct a graph $G$ on vertex set $V$. For $1 \le i \le t$, consider the $i$th perfect matching $M_i$ consisting of edges $E_1^i, \ldots, E_{n/(k-1)}^i$. For $j = 1, \ldots, n/(k-1)$, add edges within each $E_j^i$ to make a copy of $H_i$.

Since every pair of vertices lies in only one edge of $S(n, k-1, 2)$, this construction is well defined. To see that the construction is $H_i$-full, consider an arbitrary $v \in V$. Exactly one of the $t$ edges containing $v$ belongs to the $i$th matching. This edge forms a copy of $H_i$ containing $v$. □

In the special case when $n = (k-1)^2$, such a resolvable Steiner system is an *affine plane*, denoted $\mathcal{H}_{k-1}$. It is well known (see, [3, page 672] or [9], for example) that an affine plane $\mathcal{H}_{k-1}$ exists when $k-1$ is a power of a prime. This yields the following.

**Corollary 2.3** *If $\mathcal{H}_{k-1}$ exists and $n(H_i) < k$ for each $i$, then $f(H_1, \ldots, H_k) \le (k-1)^2$.*

When $n(H_i) = k-1$ for each $i$, Corollary 2.3 improves the bound in Theorem 2.1 (asymptotically) by a factor of two. When $k = 2$ and $H_2 = \overline{K}_n$, a slightly different construction gives nearly optimal bounds for each $H_1$ as $n \to \infty$. In Theorem 3.1, we shall prove that this construction is asymptotically optimal.

**Theorem 2.4** *If $H$ has order $m$ and positive minimum degree $\delta$, then $f(H, \overline{K}_n) < n + 2\sqrt{\delta n} + 2\delta$ when $n \ge 9\delta(m-\delta-1)^2$.*

**Proof:** Let $x$ be a vertex of minimum degree $\delta$ in $H$. We construct an $(H, \overline{K}_n)$-full graph $G$ in terms of a parameter $r$ that we optimize later. Let $V(G) = U \cup W$, where $U = U_1 \cup \cdots \cup U_r$ and $W = W_1 \cup \cdots \cup W_r$. Let $W$ be an independent set of size $n-1+s$, where $s = \lceil n/(r-1) \rceil$. Let each $W_i$ have size $s-1$ or $s$ (set $|W_r| = s-1$ and put the remaining $n$ vertices equitably into $r-1$ sets). For each $i$, set $G[U_i] \cong H[N(x)]$, and make all of $U_i$ adjacent to all of $W_i$.

Each $U_i \cup w$ with $w \in W_i$ induces $N_H[x]$; we add edges to complete copies of $H$. Let $m' = m - \delta - 1$. For $j \in \{1, 2, 3\}$, let $T_j$ consist of $m'$ vertices, one chosen from each of $U_{(j-1)m'+1}, \ldots, U_{jm'}$. This requires $r \ge 3m'$. Add edges within each $T_j$ so that $G[T_j] \cong H - N[x]$. For each $U_i$ that contains a vertex of $T_j$, add edges from $U_i$ to $T_{j+1}$ (indices modulo 3 here) so that $G[U_i \cup T_{j+1}] \cong H - x$. For $3m'+1 \le i \le r$, add edges from $U_i$ to $T_1$ so that $G[U_i \cup T_1] \cong H - x$. This completes the construction of $G$, as sketched in Fig. 2; dots represent the vertices of $\bigcup T_j$, and arrows suggest the edges from $U_i$ to $T_{j+1}$.

To show that $G$ is $(H, \overline{K}_n)$-full, it suffices to consider $u \in U_i$ and $w \in W_i$. By construction, we have $G[\{w\} \cup U_i \cup T_j] \cong H$ for some $j$. The vertices of $W - W_i$ together with $u$ or $w$ form an independent set of size at least $n + s - 1 - s + 1 = n$.

We now choose $r$ to minimize the order of $G$, which equals $n - 1 + \delta r + \lceil n/(r-1) \rceil$. Calculus suggests the choice $r = \lceil \sqrt{n/\delta}\, \rceil + 1$. This satisfies the requirement that $r \ge 3m'$ when $n \ge 9\delta(m-\delta-1)^2$. With this value of $r$, the order of $G$ is at most $n + \delta(2 + \sqrt{n/\delta}) + \sqrt{\delta n}$, which equals the bound claimed. □

In the optimized construction, each $|W_i|$ is about $r|U_i|$. This reflects the use of $W$ to form the large independent set. When $n$ is smaller than $9\delta(m')^2$, we still obtain an

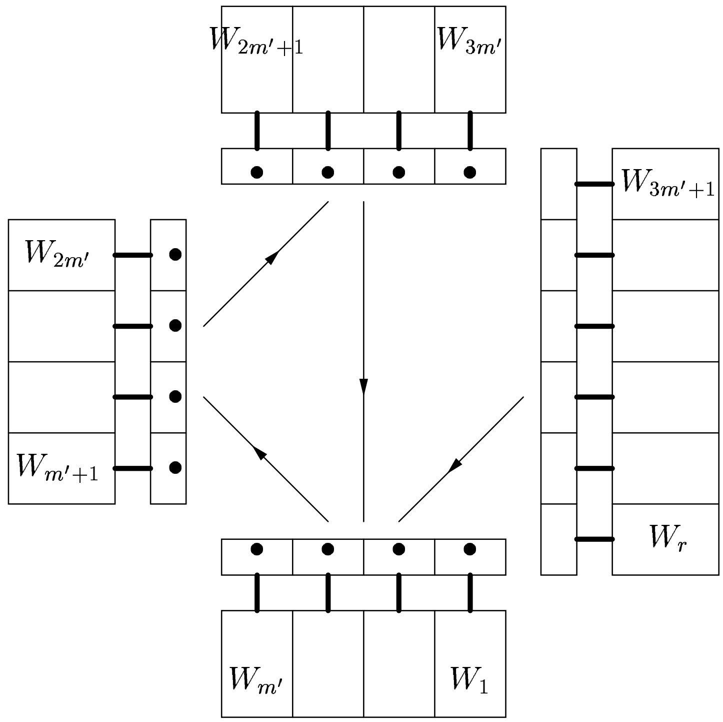


Fig. 2. Structure of an $(H, \overline{K}_n)$-full graph

improvement on Theorem 2.1 by setting $r = 3m'$, where $m' = m - \delta - 1$. The resulting $(H, \overline{K}_n)$-full graph has order $n - 1 + \lceil n/(3m' - 1)\rceil + 3\delta m'$, which is less than $2(n + m)$ when $n$ is bigger than about $3\delta m'$.

## 3 A Lower Bound

In this section we prove a lower bound that holds when the maximum degree of $H_2$ is less than half the minimum degree of $H_1$.

**Theorem 3.1** *Let $H_1$ and $H_2$ be graphs such that $H_1$ has minimum degree $\delta$, and $H_2$ has order $n$ and maximum degree $\Delta$. If $2\Delta < \delta$, then*

$$f(H_1, H_2) \geq n + \left\lceil 2\sqrt{(n + \Delta)(\delta - 2\Delta)}\right\rceil - (\delta - \Delta).$$

**Proof:** Let $G$ be an $(H_1, H_2)$-full graph, and choose $A \subset V(G)$ such that $G[A] \cong H_2$. Let $v$ be a vertex in $V(G) - A$ with the most neighbors in $A$. Since $G$ is $(H_1, H_2)$-full, $v$ belongs to a set $B \subset V(G)$ such that $G[B] \cong H_2$. Let $C = V(G) - (A \cup B)$. Let $k = |A - B|$; we obtain a lower bound on $|C|$ in terms of $k$.

Let $e$ be the number of edges with endpoints in both $C$ and $A \cap B$, and let $d = |N(v) \cap A|$. Our lower bound on $C$ arises from the computation below. The first inequality counts $e$ by the $n - k$ endpoints in $A \cap B$; each lies in a copy of $H_1$ but has at most $2\Delta$ neighbors outside $C$. The second inequality counts $e$ by the endpoints in $C$, using the choice of $v$.

For the third inequality, note that $v$ has at most $\Delta$ neighbors in $B$ and then at most $k$ more in $A-B$.

$$(n-k)(\delta-2\Delta) \le e \le d|C| \le (k+\Delta)|C|.$$

Using the resulting lower bound on $|C|$, we have

$$\begin{aligned} |V(G)| = |A \cup B| + |C| &\ge n + k + \frac{(n-k)(\delta-2\Delta)}{k+\Delta} \\ &= n - (\delta-\Delta) + (k+\Delta) + \frac{(n+\Delta)(\delta-2\Delta)}{k+\Delta}. \end{aligned}$$

This expression is minimized by $k+\Delta = \sqrt{(n+\Delta)(\delta-2\Delta)}$, yielding the desired bound. □

**Corollary 3.2** *If $H_1$ has minimum degree $\delta$, then $f(H_1, \overline{K}_n) = n + 2\sqrt{\delta n} + O(1)$ as $n \to \infty$.*

**Proof:** For $\delta > 0$, the upper bound follows from Theorem 2.4, while the lower bound follows by setting $H_2 = \overline{K}_n$ in Theorem 3.1. Now suppose that $\delta = 0$ and let $m = n(H_1)$. Let $\alpha(G, v)$ denote the maximum size of an independent set containing vertex $v$ in a graph $G$. Let $s = \min_{v \in V(H_1)} \alpha(H_1, v)$.

We claim that $f(H_1, \overline{K}_n) = n - s + m$ for $n \ge s$. For the lower bound, let $u$ be a vertex of $H_1$ such that $s = \alpha(H_1, u)$. Completing an independent $n$-set for a vertex playing the role of $u$ in a copy of $H_1$ requires adding at least $n - s$ vertices to the $m$ vertices of $H_1$. Since $H_1$ has at least one isolated vertex, adding these as isolated vertices yields an $(H_1, \overline{K}_n)$-full graph, thus proving the upper bound also. □

By taking complements, one immediately obtains the following corollary.

**Corollary 3.3** *If $\overline{H_1}$ has minimum degree $\delta$, then $f(H_1, K_n) = n + 2\sqrt{\delta n} + O(1)$ as $n \to \infty$.*

# 4 Stars versus Independent Sets

In this section we determine $f(H_1, H_2)$ when $H_1$ is a star of order $m$ and $H_2$ is an independent set of order $n$. Let $S_m = K_{1,m-1}$. The problem is rather easy when $n < m$.

**Claim 4.1** *For $n < m$, $f(S_m, \overline{K}_n) = n + m - 1$, achieved by $K_{n,m-1}$.*

**Proof:** The center of an $m$-star must lie in an independent $n$-set avoiding its neighbors, so $f(S_m, \overline{K}_n) \ge n + m - 1$ for all $n$. When $n < m$, the graph $K_{n,m-1}$ is $(S_m, \overline{K}_n)$-full. □

The problem behaves much differently when $n \ge m$. First we provide a construction.

**Lemma 4.2** *For $n \ge m \ge 2$,*

$$f(S_m, \overline{K}_n) \le n + \min_k \max\left\{k + \left\lceil \frac{n-1}{k} \right\rceil, 2m-3-k\right\}.$$

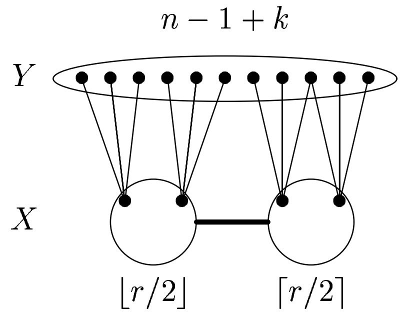


Fig. 3. Construction of an $(S_m, \overline{K}_n)$-full graph.

**Proof:** We define a construction $G$ with parameters $r$ and $k$. Let $V(G)$ be the disjoint union of $X$ and $Y$, where $|X| = r$ and $|Y| = n-1+k$. Let $G[X] = K_{\lceil r/2\rceil,\lfloor r/2\rfloor}$, and let $Y$ be an independent set. Give $k$ neighbors in $Y$ to each vertex in $X$, arranged so that $G$ is bipartite and has no isolated vertices.

With $k \geq 1$, the size chosen for $Y$ ensures that each vertex lies in an independent $n$-set. Keeping $G$ bipartite requires $n-1 \geq k$. This ensures that each vertex of $X$ lies at the center of an induced star of order $k+1+\lfloor r/2\rfloor$. Thus we require

$$r/2 \geq m-1-k. \qquad \text{(A)}$$

Ensuring that the stars cover $Y$ requires

$$(r-1)k \geq n-1. \qquad \text{(B)}$$

Given $n \geq m \geq 2$, we choose $r, k$ to minimize $n-1+k+r$, the order of $G$. Rewrite (A) as $r-1 \geq 2m-3-2k$. Both (A) and (B) impose lower bounds on $r-1$ in terms of $k, m, n$; we set $r-1 = \max\{\lceil (n-1)/k\rceil, 2m-3-2k\}$. This yields the one-variable minimization in the statement of the lemma. □

In fact, the construction of Lemma 4.2 is optimal for all $n \geq m$. We begin the proof of optimality with a lower bound that differs from the upper bound by at most 1.

**Lemma 4.3** *For $n \geq m \geq 2$,*

$$f(S_m, \overline{K}_n) \geq n + \min_d \max\left\{d-1+\left\lceil\frac{n}{d}\right\rceil, 2m-2-d\right\}.$$

**Proof:** We strengthen the general argument of Theorem 3.1. Let $G$ be an $(S_m, \overline{K}_n)$-full graph. Let $d$ be the maximum of $|N(v) \cap T|$ such that $v \in V(G)$ and $T$ is an independent $n$-set in $G$. Let $A$ be an independent $n$-set and $x$ a vertex such that $|N(x) \cap A| = d$.

As in the proof of Theorem 3.1, we choose $B$ to be an independent $n$-set containing $x$, let $C = V(G) - (A \cup B)$, and let $k$ be the size of $A - B$. With $\delta = 1$ and $\Delta = 0$, the argument applied there to the edges joining $C$ and $A \cap B$ yields

$$n - k \leq d|C| \leq k|C|.$$

Since $d \le k$, we obtain $|V(G)| \ge n + d - 1 + \lceil n/d \rceil$.

To complete the proof, we must show that $|V(G)| \ge n + 2m - 2 - d$. As observed in the proof of Claim 4.1, $f(S_m, \overline{K}_n) \ge n + m - 1$ always. Thus we may assume that $d < m - 1$. In proving a lower bound, we may also assume that $G$ is a minimal $(S_m, \overline{K}_n)$-full graph. In particular, if we delete any edge of $G$, then the resulting graph is not $S_m$-full. Let $R_1, \ldots, R_t$ be a collection of induced stars of order at least $m$ that cover $V(G)$. By the minimality of $G$, the vertices that are not centers of these stars form an independent set. We consider two cases.

*Case 1: The centers of* $R_1, \ldots, R_t$ *form an independent set.* In this case, $G$ is a bipartite graph with bipartition $X, Y$, where $X$ is the set of centers of $R_1, \ldots, R_t$ and $Y$ is the set of leaves of $R_1, \ldots, R_t$. By the definition of $d$ and the restriction to $d < m-1$, we have $|Y| < n$. Let $x$ be the center of $R_1$, let $I$ be an independent $n$-set containing $x$, and let $j = |I \cap X|$. Each vertex of $I \cap X$ has at least $m-1$ neighbors in $Y - I$. Since $|Y - I| < n - (n-j) = j$ and there are at least $j(m-1)$ edges from $I \cap X$ to $Y - I$, some $y \in Y - I$ is incident to at least $m-1$ of these edges. This gives $y$ at least $m - 1 > d$ neighbors in $I$, contradicting the choice of $d$. Thus this case cannot occur when $d < m - 1$.

*Case 2. The centers of* $R_1, \ldots, R_t$ *do not form an independent set.* By the minimality of $G$, each edge of $G$ is needed to complete some induced star of order at least $m$ centered at one of its endpoints. We may assume that the centers $x$ of $R_1$ and $y$ of $R_2$ are adjacent and that $R_1$ needs the edge $xy$ to reach order $m$. This implies that $y$ is not adjacent to any leaf of $R_1$. In particular, the $m-2$ or more additional vertices that complete $R_2$ are distinct from those in $R_1$, and $|V(R_1) \cup V(R_2)| \ge 2m - 2$.

Now let $I$ be an independent $n$-set containing $x$. The vertices of $R_1 \cup R_2$ in $I$ are all neighbors of $y$, and hence there are at most $d$ of them. Thus $|V(G)| \ge n - d + 2m - 2$. □

When $d$ in the formula of Lemma 4.3 equals $k$ in the formula of Lemma 4.2, the resulting values differ by at most one. A closer look at the one-variable optimization shows that the lower bound and the upper bound differ by at most one.

**Theorem 4.4** *For $n \ge m \ge 2$, the construction of Lemma 4.2 is optimal.*

**Proof:** We prove that the lower bound of Lemma 4.3 can be improved to match the upper bound of Lemma 4.2.

Choose $A, B, C, d, k$ as in the proof of Lemma 4.3. If $d \le k - 1$ or if there are at most $(d-1)|C|$ edges between $C$ and $A \cap B$, then we obtain $|C| \ge (n-k)/(k-1)$, which yields $|V(G)| \ge n + k - 1 + (n-1)/(k-1)$. Also $2m - 2 - d \ge 2m - 2 - k$. Setting $k' = k - 1$ now yields $|V(G)| \ge n + \max\{k' + \lceil (n-1)/k' \rceil, 2m - 3 - k'\}$. Hence the construction is optimal unless there is another construction satisfying $d = k$ and having more than $(d-1)|C|$ edges between $C$ and $A \cap B$ (thus there is a $z \in C$ with $d_{A \cap B}(z) \ge d$). More precisely, for every independent set $A$ of size $n$, every vertex $x \notin A$ with $d_A(x) = d$, and every independent set $B$ of size $n$ containing $x$, the following holds:

$$B \supseteq A - N(x) \qquad (*)$$

Choose $z \in C$ with $d_{A \cap B}(z) = d$, and let $B'$ be an independent set of size $n$ containing $z$. Letting $(z, A, B')$ play the role of $(x, A, B)$ in $(*)$ implies that $B' \supseteq A - N(z) \supseteq$

$A - B$. On the other hand, letting $(z, B, B')$ play the role of $(x, A, B)$ in $(*)$ implies that $B' \supseteq B - N(z) \supseteq B - A$. This implies that $(A - B) \cup (B - A)$ is an independent set, a contradiction. □

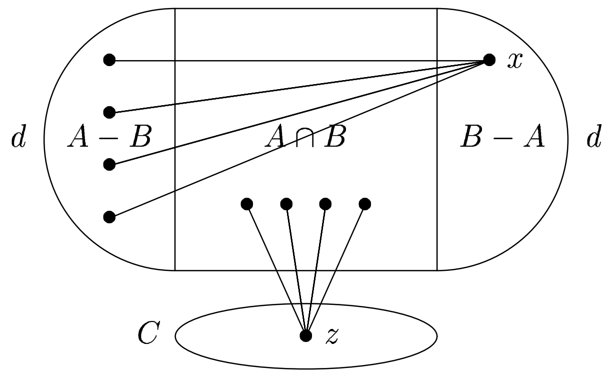


Fig. 4. Final proof of the lower bound.

It is worth noting what the result of the one-variable optimization is in terms of $m$ and $n$. In particular, the construction achieves a lower bound resulting from Theorem 1.1 when $n > 1 + (4/9)(m - 2)^2$.

**Remark 4.5** *If* $n > 1 + (4/9)(m - 2)^2$, *then* $f(S_m, \overline{K}_n) = n + \lceil 2\sqrt{n-1}\,\rceil$.
*If* $m \le n \le 1 + (4/9)(m - 2)^2$, *then* $f(S_m, \overline{K}_n) = n + \lceil \frac{1}{4}(3\beta - \sqrt{\beta^2 - 8})\sqrt{n-1}\,\rceil$, *where* $2m - 3 = \beta\sqrt{n-1}$ *with* $\beta > 3$.

**Proof:** By Theorem 4.4, it suffices to minimize over $k$ in Lemma 4.2. The term $2m-3-k$ is linear. The term $k + \lceil (n-1)/k \rceil$ is minimized when $k = \lceil \sqrt{n-1}\,\rceil$, where it equals $\lceil 2\sqrt{n-1}\,\rceil$. (When $k = \lceil \sqrt{n-1}\,\rceil$, we let $n - 1 = k^2 - r$ with $r < 2k - 1$; both formulas yield $2k - 1$ when $r \ge k$ and $2k$ when $r < k$.)

When $2m - 3 - \lceil \sqrt{n-1}\,\rceil \le \lceil 2\sqrt{n-1}\,\rceil$, the construction yields $f(S_m, \overline{K}_n) \le n + \lceil 2\sqrt{n-1}\,\rceil$. Since every vertex of an induced star belongs to an induced edge, Theorem 1.1 yields $f(S_m, \overline{K}_n) \ge f(K_2, \overline{K}_n) \ge n + \lceil 2\sqrt{n-1}\,\rceil$.

For smaller $n$, the construction is optimized by choosing $x$ so that $x + (n-1)/x = 2m - 3 - x$ and letting $k = \lfloor x \rfloor$. The number of vertices is then $2m - 3 - k$. For large $m$ and $n$, we can approximate the result by ignoring integer parts and defining $\beta$ by $2m - 3 = \beta\sqrt{n-1}$. The solution then occurs at $x = \frac{1}{4}(\beta + \sqrt{\beta^2 - 8}\,)\sqrt{n-1}$, and we invoke Theorem 4.4. □

## 5 Open Problems

We list several open questions. The first is the most immediately appealing, suggested by comparing Theorem 1.1 and Theorem 2.1.

1. Among all choices of an $m$-vertex graph $H_1$ and an $n$-vertex graph $H_2$, is it true that $f(H_1, H_2)$ is maximized when $H_1$ is a clique and $H_2$ is an independent set?

2. Let $G$ be an $S_m$-full graph in which the deletion of any edge produces a graph that is not $S_m$-full. Is it true that $G$ must be triangle-free? [1]

3. Among random graphs, what order is needed so that almost every graph is $(H_1, \ldots, H_k)$-full?

4. Distinguish a root vertex in each of $H_1, \ldots, H_k$. An $(H_1, \ldots, H_k)$-*root-full* graph is an $(H_1, \ldots, H_k)$-full graph in which each vertex appears as the root in some induced copy of each $H_i$. Is it possible to bound the minimum order of such a graph (for arbitrary choice of roots) in terms of $f(H_1, \ldots, H_k)$? (suggested by Fred Galvin)

5. Similarly, one could require induced copies of each $H_i$ so that for each $v \in V(G)$ and $x \in H_i$, some copy of $H_i$ occurs with $v$ playing the role of $x$. The minimum order of such a graph is the framing number $fr(H_1, \ldots, H_k)$. How large can $fr(H_1, \ldots, H_k)$ be as a function of $f(H_1, \ldots, H_k)$? (suggested by Mike Jacobson)

# 6 Acknowledgments

The authors thank the referees for many suggestions, particularly an idea leading to Theorem 2.2.

# References

[1] G. Chartrand, H. Gavlas, and M. Schultz, FRAMED! A graph embedding problem, *Bull. Inst. Combin. Applic.* **4** (1992), 35–50.

[2] R. C. Entringer, W. Goddard, and M.A. Henning, A note on cliques and independent sets, *J. Graph Theory* **24** (1997), 21–23.

[3] Handbook of Combinatorics, MIT Press, 1995.

[4] M. A. Henning, On cliques and bicliques, preprint.

[5] F. Galvin, Another note on cliques and independent sets, preprint.

[6] F. Galvin, Stars and triangles, preprint.

[7] W. Goddard, M. A. Henning, H. Maharaj, Homogeneous embeddings of cycles in graphs, *Graphs and Combinatorics* **15** (1999), 159–173.

[8] D. K. Ray-Chaudhuri, and R. M. Wilson, The existence of resolvable block designs, in *Survey of combinatorial theory* (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971), (North-Holland, Amsterdam, 1973), 361–375.

[9] J. H. Van Lint, R. M. Wilson, *A Course in Combinatorics*, (Cambridge University Press, 1992).

[1] this has recently been proved positively in [6]