COLORING OF TREES WITH MINIMUM SUM OF COLORS

Tao Jiang and Douglas B. West
j-tao@math.uiuc.edu and west@math.uiuc.edu
University of Illinois, Urbana, IL 61801-2975

Abstract. The *chromatic sum* of a graph is the smallest sum of colors among all proper colorings with natural numbers. The *strength* is the minimum number of colors needed to achieve the chromatic sum. We construct for each positive integer k a tree with strength k that has maximum degree only $2k-2$. The result is best possible.

1. INTRODUCTION

A *proper coloring* of the vertices of a graph G is a function $f\colon V(G) \to \mathbb{N}$ such that adjacent vertices receive different labels (colors). The *chromatic number* $\chi(G)$ is the minimum number of colors in a proper coloring of G. The *chromatic sum* $\Sigma(G)$ is a variation introduced by Ewa Kubicka in her dissertation. It is the minimum of $\sum_{v\in V(G)} f(v)$ over proper colorings f of G. A *minimal coloring* of G is a proper coloring of G such that $\sum_v f(v) = \Sigma(G)$.

One might think that a minimal coloring can be obtained by selecting a proper coloring with the minimum number of colors and then giving the largest color class color 1, the next largest color 2, and so on. However, even among trees, which have chromatic number 2, more colors may be needed to obtain a minimal coloring. The *strength* $s(G)$ of a graph G is the minimum number of colors needed to obtain a minimal coloring. Kubicka and Schwenk [4] constructed for every positive integer $k \ge 2$ a tree T_k with strength k. Thus $s(G)$ may be arbitrarily large even when $\chi(G) = 2$ (trivially $s(G) \ge \chi(G)$).

How large can $s(G)$ be in terms of other parameters? When vertices are colored greedily in natural numbers with respect to a vertex ordering $v_1, \ldots, v_n$, the number of colors used is at most $1 + \max_i d^*(v_i)$, where $d^*(v_i)$ counts the neighbors of v_i in $\{v_1, \ldots, v_{i-1}\}$. Always this yields $\chi(G) \le 1 + \Delta(G)$. The best upper bound on $\chi(G)$ that can be obtained in this way is the *Szekeres-Wilf number* $w(G) = 1 + \max_{H\subseteq G} \delta(H)$ (also confusingly called the "coloring number"). Interestingly, the average of these two well-known upper bounds for the chromatic number is an upper bound for the strength $s(G)$.

THEOREM (Hajiabolhassan, Mehrabadi, and Tusserkani [2]) Every graph G has strength at most $\lceil (w(G) + \Delta(G))/2 \rceil$.

We show that this bound is sharp, even for trees. Every nontrivial tree T has Szekeres-Wilf number 2, and thus $s(T) \le 1 + \lceil \Delta(T)/2 \rceil$. In the Kubicka-Schwenk construction [4],

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the tree with strength k has maximum degree about $k^2/2$. To show that the bound above is sharp, we construct for each $k \geq 1$ a tree T_k with strength k and maximum degree $2k-2$. Given a proper coloring f of a tree T, we use Σf to denote $\sum_{v \in V(T)} f(v)$.

2. THE CONSTRUCTION

Linearly order the pairs of natural numbers so that $(h,l) < (i,j)$ if either $h+l < i+j$ or $h+l = i+j$ and $l < j$. With respect to this ordering, we inductively construct for each pair $(i,j) \in \mathbb{N} \times \mathbb{N}$ a rooted tree T_i^j and a coloring f_i^j of T_i^j. In other words, we construct trees in the order T_1^1, T_2^1, T_1^2, $T_3^1, \ldots$. Our desired tree with strength k will be T_k^1. Let $[n] = \{k \in \mathbb{Z}\colon 1 \leq k \leq n\}$.

Construction. Let T_1^1 be a tree of order 1, and let f_1^1 assign color 1 to this single vertex. Consider $(i,j) \neq (1,1)$, and suppose that for each $(h,l) < (i,j)$ we have constructed T_h^l and f_h^l. We construct T_i^j and f_i^j as follows. Let u be the root of T_i^j. For each k such that $1 \leq k \leq i+j-1$ and $k \neq i$, we take two copies of T_k^m, where $m = \lceil (i+j-k)/2 \rceil$, and we let the roots of these $2(i+j-2)$ trees be children of u. The resulting tree is T_i^j (see Fig. 1). Define the coloring f_i^j of T_i^j by assigning i to the root u and using f_k^m on each copy of T_k^m rooted at a child of u. ■

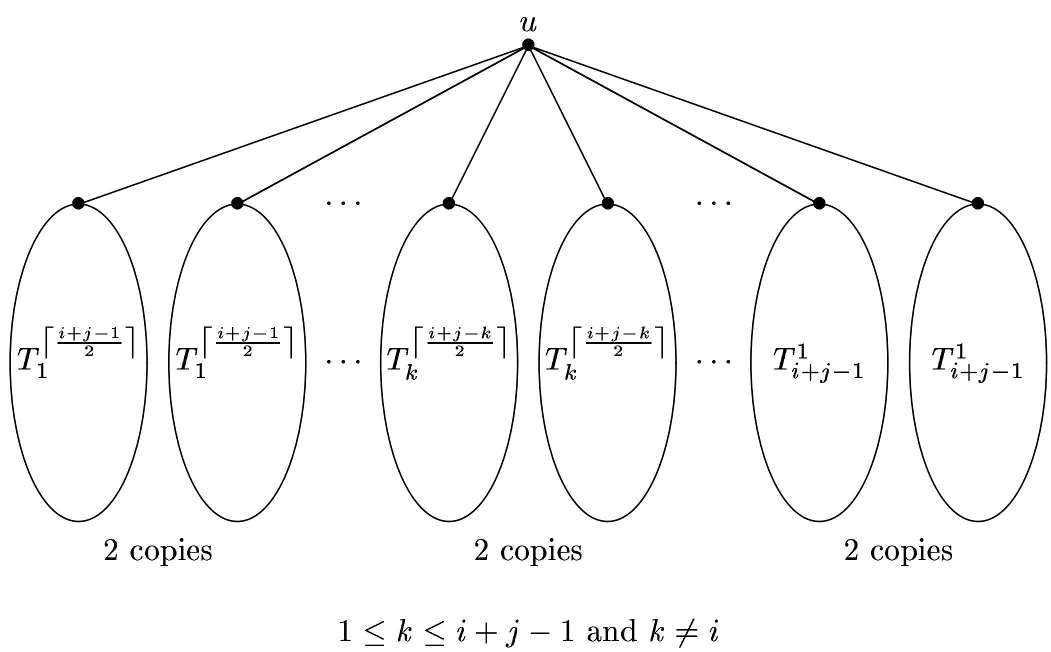

Figure 1. The construction of T_i^j

LEMMA For $(i,j) \in \mathbb{N} \times \mathbb{N}$, the construction of T_i^j is well-defined, and f_i^j is a proper coloring of T_i^j with color i at the root.

Proof: To show that T_i^j is well-defined, it suffices to show that when $(i,j) \neq (1,1)$, every tree used in the construction of T_i^j has been constructed previously. We use trees of the form T_k^m, where $k \in [i+j-1]-\{i\}$ and $m = \lceil (i+j-k)/2 \rceil$. It suffices to show that $k+m \le i+j$ and that $m < j$ when $k+m = i+j$.

For the first statement, we have $k+m \le \lceil (i+j+k)/2 \rceil \le i+j$, since $k \le i+j-1$. Equality requires $k = i+j-1$, which occurs only when $j \ge 2$ and yields $m = 1$. Thus $m < j$ when $k+m = i+j$. Since the trees whose indices sum to $i+j$ are generated in the order $T_{i+j-1}^1, \ldots, T_1^{i+j-1}$, the tree T_k^m exists when we need it.

Finally, f_i^j uses color i at the root of T_i^j, by construction. Since the subtrees used as descendants of the root have the form T_k^m with $k \neq i$, by induction the coloring f_i^j is proper. ∎

3. THE PROOF

The two-parameter construction enables us to prove a technically stronger statement. The additional properties of the construction facilitate the inductive proof. Recall that all colorings considered are labelings with positive integers.

THEOREM The construction of T_i^j and f_i^j has the following properties:

(1) If f' is a coloring of T_i^j different from f_i^j, then $\Sigma f' > \Sigma f_i^j$. Furthermore, if f' assigns a color different from i to the root of T_i^j, then $\Sigma f' - \Sigma f_i^j \ge j$;

(2) If $j = 1$, then $\Delta(T_i^j) = 2i-2$, achieved by the root of T_i^j. If $j \ge 2$, then $\Delta(T_i^j) = 2(i+j)-3$;

(3) The highest color used in f_i^j is $i+j-1$.

Proof: We use induction through the order in which the trees are constructed. As the basis step, T_1^1 is just a single vertex, and f_1^1 gives it color 1; conditions (1)-(3) are all satisfied.

Now consider $(i,j) \neq (1,1)$. For simplicity, we write T for T_i^j and f for f_i^j. To verify (1), let f' be a coloring of T different from f. We consider two cases.

Case 1. f' assigns i to the root u of T.

In this case, f' and f differ on $T-u$. Recall that $T-u$ is the union of $2(i+j-2)$ previously-constructed trees. The colorings f' and f differ on at least one of these trees. By the induction hypothesis, the total under f' is at least the total under f on each of these subtrees, and it is larger on at least one. Hence $\Sigma f' > \Sigma f$.

Case 2. f' assigns a color different from i to the root u.

In this case, we need to show that $\Sigma f' - \Sigma f \ge j$. Again the induction hypothesis gives f' as large a total as f on each component of $T-u$. If $f'(u) \ge i+j$, then the difference on u is large enough to yield $\Sigma f' - \Sigma f \ge j$.

Hence we may assume that $f'(u) = k$, where $1 \le k \le i+j-1$ and $k \neq i$. Since f' is a proper coloring, it assigns a label other than k to the roots v, v' of the two copies of T_k^m in $T-u$, where $m = \lceil (i+j-k)/2 \rceil$. Since f uses f_k^m on each copy of T_k^m, we have $f(v) = f(v') = k$. Since $f'(v)$ and $f'(v')$ differ from k, the induction hypothesis implies that on each copy of T_k^m the total of f' exceeds the total of f by at least m. Since the total is at least as large on all other components, we have

$$\Sigma f' - \Sigma f \ge k - i + 2m = k - i + 2 \left\lceil \frac{i+j-k}{2} \right\rceil \ge j.$$

Next we verify (2). In the construction of $T = T_i^j$, we place $2(i+j-2)$ subtrees under the root u. These have the form T_k^m for $1 \leq k \leq i-1$ and $i+1 \leq k \leq i+j-1$, and always $m = \lceil (i+j-k)/2 \rceil$. Note that $m = 1$ only when $k = i+j-1$ or $k = i+j-2$. The subtrees have maximum degree $2k-2$ (when $m = 1$) or $2(k+m)-3$ (when $m > 1$). Note that $2(k+m)-3 > 2k-2$ when $m \geq 1$. Thus

$$\Delta(T_k^m) \leq 2(k+m) - 3 = 2\left(k + \left\lceil \frac{i+j-k}{2} \right\rceil\right) - 3 = 2\left\lceil \frac{i+j+k}{2} \right\rceil - 3.$$

Also, we always have $k + m = \lceil (i+j+k)/2 \rceil$ for the subtree T_k^m.

When $j = 1$ we only have $k \leq i-1$, and thus $\Delta(T_k^m) \leq 2\lceil (i+1+k)/2 \rceil - 3 \leq 2i-3$. Hence each vertex in $T-u$ has degree at most $(2i-3)+1 = 2i-2$ in T. Since $d_T(u) = 2i-2$, we have $\Delta(T) = 2i-2$, achieved by the root.

When $j \geq 2$, the values of k for the subtrees are $1 \leq k \leq i-1$ and $i+1 \leq k \leq i+j-1$. By the induction hypothesis, the maximum degree of T_{i+j-1}^1 is $2(i+j-1)-2 = 2(i+j)-4$ and is achieved by its root. In T this vertex has degree $2(i+j)-3$, which exceeds $d_T(u)$. For $k \leq i+j-2$, we have $\Delta(T_k^m) \leq 2\lceil (i+j+k)/2 \rceil - 3 \leq 2(i+j)-5$. Hence $\Delta(T) = 2(i+j)-3$, achieved by the roots of the trees that are isomorphic to T_{i+j-1}^1.

It remains to verify (3): the maximum color used in f_i^j is $i+j-1$. By the induction hypothesis and the construction, the maximum color used by f_k^m on each T_k^m within f_i^j is $k+m-1 = \lceil (i+j+k)/2 \rceil - 1$. Since the largest k is $i+j-1$ when $j \geq 2$ and is $i-1$ when $j = 1$, this computation yields $i+j-1$ when $j \geq 2$ and $i-1$ when $j = 1$ as the maximum color on $T-u$. Since f assigns i to the root u, we obtain $i+j-1$ as the maximum color on T for both $j \geq 2$ and $j = 1$. ■

We have proved that f_i^j is the unique minimal coloring of T_i^j and that it uses $i+j-1$ colors. Hence $s(T_i^j) = i+j-1$. The maximum degree is $2i-2$ or $2(i+j)-3$, depending on whether $j = 1$ or $j \geq 2$. In particular, T_i^1 is a tree with strength i and maximum degree $2i-2$.

COROLLARY 1. There exists for each positive integer i a tree T_i with $s(T_i) = i$ and $\Delta(T_i) = 2i-2$. ■

COROLLARY 2. For every real number $\alpha \in (0, 1/2)$, there is a sequence of trees $T'_1, T'_2, \ldots$ such that $\lim_{n\to\infty} s(T'_n)/\Delta(T'_n) = \alpha$.

Proof: Let $t = \lfloor (\frac{1}{\alpha} - 2)i \rfloor + 2$. Consider the construction of T_i^1. Form T'_i by adding t additional copies of the subtree T_{i-1}^1 under the root u of T_i^1. The strength of T'_i is i, but $\Delta(T'_i) = 2i-2+t$. As $i \to \infty$, we have

$$\frac{s(T'_i)}{\Delta(T'_i)} = \frac{i}{2i+t-2} = \frac{i}{2i + \lfloor (\frac{1}{\alpha} - 2)i \rfloor} \to \alpha.$$ ■

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