Tree-thickness and caterpillar-thickness under girth constraints

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Abstract

We study extremal problems for decomposing a connected n-vertex graph G into trees or into caterpillars. The least size of such a decomposition is the *tree thickness* $\theta_{\mathbf{T}}(G)$ or *caterpillar thickness* $\theta_{\mathbf{C}}(G)$. If G has girth g with $g \geq 5$, then $\theta_{\mathbf{T}}(G) \leq \lfloor n/g \rfloor + 1$. This also holds for girth 4 when additional subgraphs are forbidden.

We study $\theta_{\mathbf{C}}(G)$ when G is outerplanar. If $g \geq 4$, then $\theta_{\mathbf{C}}(G) \leq \lceil 3n/8 \rceil$; sharp for $n \equiv 5 \mod 8$. If $g \geq 5$, then $\theta_{\mathbf{C}}(G) \leq \lceil 3n/10 \rceil$; sharp for $n \equiv 6 \mod 10$. If G is 2-connected, then the bounds are $\lfloor n/g \rfloor$ when $n \geq g^2/2$, $\left\lfloor \frac{n-g}{g-2} \right\rfloor$ when $3g-4 \leq n \leq g^2/2$, and 2 when $g \leq n \leq 3g-4$.

If $g \geq 6$ and $n > 6$, outerplanar or not, then $\theta_{\mathbf{C}}(G) \leq \lceil (n-2)/4 \rceil$. All these bounds for $\theta_{\mathbf{T}}(G)$ and $\theta_{\mathbf{C}}(G)$ are sharp for n as specified.

1 Introduction

A *decomposition* of a graph G is a set of pairwise edge-disjoint subgraphs with union G. We study decompositions of connected n-vertex graphs (general, planar, or outerplanar) into the fewest trees or the fewest caterpillars.

The complete graph K_n decomposes into $\lceil n/2 \rceil$ paths and no fewer. Gallai famously conjectured that every connected n-vertex graph decomposes into $\lceil n/2 \rceil$ paths. Chung [1]

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proved that $\lceil n/2 \rceil$ trees suffice. In fact, her proof decomposes every connected n-vertex graph into $\lceil n/2 \rceil$ caterpillars of diameter at most 4. A *caterpillar* is a tree having a single path (the *spine*) that contains at least one endpoint of every edge; equivalently, deleting the leaves yields a path. This class is intermediate between paths and trees. The connectedness condition is needed because $n/3$ disjoint triangles do not decompose into fewer than $2n/3$ trees. We study always connected graphs, and we use n for the number of vertices.

Given a class $\mathbf{F}$ of graphs, the $\mathbf{F}$-*decomposition number* or $\mathbf{F}$-*thickness* of a graph G, written $\theta_{\mathbf{F}}(G)$, is the minimum size of a decomposition of G into subgraphs that lie in $\mathbf{F}$. We seek the maximum of $\theta_{\mathbf{F}}$ over graphs in some class $\mathbf{G}$. We can refine such problems by seeking tighter bounds over classes smaller than $\mathbf{G}$ or by restricting the family $\mathbf{F}$. Let $\theta_{\mathbf{T}}$ and $\theta_{\mathbf{C}}$ denote the tree-thickness and caterpillar-thickness, respectively.

For connected graphs, the maximum tree-thickness $\lceil n/2 \rceil$ is attained by K_n. Forbidding triangles excludes this extremal graph. The *girth* of a graph is the length of a shortest cycle. For $g \geq 5$, we prove in Theorem 4 that $\theta_{\mathbf{T}}(G) \leq \lfloor n/g \rfloor + 1$ when G is connected and has girth g, and equality is achievable. The conclusion also holds when $g = 4$ among graphs containing no subdivision of $K_{2,3}$ with girth 4.

In a larger class of graphs with girth 4, we obtain a weaker upper bound that nevertheless is tighter than the bound for general graphs. We prove in Theorem 5 that $\theta_{\mathbf{T}}(G) \leq \lceil n/3 \rceil$ when G is a connected graph with girth 4 not containing the graph obtained from $K_{4,3}$ by deleting one edge. We know of no connected graph with girth 4 having tree-thickness more than $\lfloor n/4 \rfloor + 1$ and conjecture that $\lfloor n/4 \rfloor + 1$ is the maximum.

We next turn to caterpillar-thickness. Even on special families of planar graphs, $\theta_{\mathbf{C}}$ may be larger than the maximum of $\theta_{\mathbf{T}}$ on more general families. A graph is *outerplanar* if it has an embedding in the plane with all vertices lying on the unbounded face. A *cactus* is a connected graph in which every edge appears in at most one cycle; equivalently, every block is an edge or a cycle. Every cactus is outerplanar.

Always $\theta_{\mathbf{T}}(G) \leq \theta_{\mathbf{C}}(G) \leq \lceil n/2 \rceil$ (by Chung's proof), with equality throughout when $n \equiv 4 \mod 6$ for special cacti with triangles (Example 1). Forbidding triangles tightens the bound. We prove in Theorem 7 that $\theta_{\mathbf{C}}(G) \leq \lceil 3n/8 \rceil$ when G is outerplanar and triangle-free, sharp when $n \equiv 5 \mod 8$. With girth at least 5, a similar proof yields $\theta_{\mathbf{C}}(G) \leq \lceil 3n/10 \rceil$, sharp when $n \equiv 6 \mod 10$. Regardless of outerplanarity, girth at least 6 forces $thc(G) \leq \lceil (n-2)/4 \rceil$ when $n > 6$ (Theorem 9), achieved using trees (Example 2).

The extremal graphs presented in Example 1 are cacti. When we exclude cacti by considering only 2-connected outerplanar graphs, the bound on caterpillar-thickness tightens. We prove in Theorem 6 that if G is a 2-connected n-vertex outerplanar graph with girth g, then the maximum possible value of $\theta_{\mathbf{C}}(G)$ is $\lfloor n/g \rfloor$ if $n \geq g^2/2$, is $\left\lfloor \frac{n-g}{g-2} \right\rfloor$ if $3g-4 \leq n \leq g^2/2$, and is 2 if $g \leq n \leq 3g-4$.

It is natural to ask whether the bounds on caterpillar thickness of outerplanar graphs with fixed girth continue to hold when all planar graphs with that girth are allowed. These questions remain open. That is, is it true that $\theta_{\mathbf{C}}(G) \leq \lfloor n/g \rfloor$ whenever G is a 2-connected planar graph with girth at least g and that $\theta_{\mathbf{C}}(G) \leq \lceil 3n/8 \rceil$ whenever G is a connected planar graph with girth at least 4?

2 Lower Bound Constructions

In this section we present examples showing that the bounds in our later theorems are sharp.

Example 1 Let $H_{k,g}$ denote the cactus with $kg+1$ vertices formed from k disjoint g-cycles by adding one vertex having one neighbor in each cycle (see Figure 1). The cut-edges in $H_{k,g}$ imply that only one tree in a decomposition can extend out from each cycle. However, two trees must be used within each cycle. Hence at least k trees are confined to the cycles, and at least one more tree must be used. There is such a decomposition, so $\theta_{\mathbf{T}}(H_{k,g}) = k+1 = \lfloor n/g \rfloor + 1$.

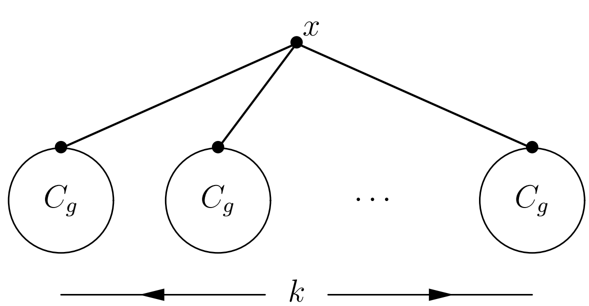

Figure 1: The graph $H_{k,g}$

This decomposition of $H_{k,g}$ is unavailable for caterpillar thickness, since the one large tree has an edge on each cycle and hence is not a caterpillar (when $k \geq 3$). A caterpillar in

$H_{k,g}$ has edges in at most two of the cycles, because a caterpillar cannot have three paths of length 2 with a common endpoint. Therefore, since only k paths can start along and depart from a cycle, the best we can do is save $\lfloor k/2 \rfloor$ by combining into pairs the paths that leave the cycles. Thus $\theta_{\mathbf{C}}(H_{k,g}) = 2k - \lfloor k/2 \rfloor = \lceil 3k/2 \rceil$.

More generally, let $n = kg + r$, where $1 \le r \le g$. Note that $k = \lfloor (n-1)/g \rfloor$. Let x be the central vertex of $H_{k,g}$. When k is even and $r = 3$, form $H'_{k,g}$ by appending a path of two edges to $H_{k,g}$ at x. We show that this increases the caterpillar thickness by 1 by using the same lower bound idea. There are $2k+1$ paths needed in the $k+1$ components of $H'_{k,g} - x$, only one of these paths can extend from each component, and they can at best combine in pairs, so $\theta_{\mathbf{C}}(H'_{k,g}) = 2k + 1 - \lfloor (k+1)/2 \rfloor = (3k/2) + 1$, since k is even.

For all other n with $r \ne 1$, we simply append a leaf to the construction for $n-1$, without increasing $\theta_{\mathbf{C}}$. We thus obtain an n-vertex cactus G with caterpillar-thickness as listed below.

$n = 2lg + r$	$\theta_{\mathbf{C}}(G)$
$1 \le r \le 2$	$3l$
$3 \le r \le g$	$3l + 1$
$g + 1 \le r \le 2g$	$3l + 2$

□

Note that $\theta_{\mathbf{C}}(H_{k,3}) = n/2$ when $n \equiv 4 \mod 6$. Thus for such n, the maximum value of caterpillar-thickness over n-vertex graphs is achieved not only by K_n, but also by a cactus.

Example 2 When $g > 6$, a special tree needs more caterpillars than the construction of Example 1. Form T_n by subdividing $\lfloor (n-1)/2 \rfloor$ edges in the star $K_{1,\lceil (n-1)/2 \rceil}$ (each subdivided once); this yields n vertices. At most two of the edges not containing the center can lie in a single caterpillar, so $\lceil \lfloor (n-1)/2 \rfloor /2 \rceil$ caterpillars are needed, and this many suffice. For $n > 2$, we obtain $\theta_{\mathbf{C}}(T_n) = \lceil (n-2)/4 \rceil$. For $g = 6$, this construction beats that of Example 1 in some congruence classes; for $g > 6$, it always wins.

The tree T_n is also the n-vertex graph whose total interval number is largest (see [2]). In a triangle-free graph, the total interval number equals the number of edges plus the minimum number of trails needed to touch at least one endpoint of every edge. In a tree, trails are paths; touching every edge with t paths means decomposing into t caterpillars. □

The family $H_{k,g}$ can be excluded by restricting to 2-connected graphs, but the tree-thickness can still be almost as large as for $H_{k,g}$. Again the graphs are outerplanar.

Example 3 For $k \geq g/2$, let $J_{k,g}$ denote the graph formed from the cycle C_n, where $n = kg$ and the vertices are $v_1, \ldots, v_n$ in order, by adding chords of the form $v_{gi-g+1}v_{gi}$ for $1 \leq i \leq k$ (see Figure 2). Note that $J_{k,g}$ has girth g.

Each chord forms a cycle, which requires two trees in the decomposition. Only one of those two trees can continue on to the next higher cycle in the direction of increasing indices, so a new tree must start within that cycle. In traversing the full outer cycle, at least k trees must be started. Hence $\theta_{\mathbf{T}}(J_{k,g}) \geq k = n/g$, and equality holds using n/g paths.

When n is not a multiple of g, we can start with a cycle of length n and insert $\lfloor n/g \rfloor$ chords in this way while maintaining girth g (if $n \geq g\lceil g/2 \rceil$), so for $n \geq g^2/2$ we obtain examples with tree-thickness (and caterpillar-thickness and path-thickness) $\lfloor n/g \rfloor$.

When $g \leq n < g^2/2$ (or $k < g/2$), the cycle on the "inside" is too short. Instead of inserting all k chords, insert only the first m. The cycle through these chords and the remaining higher-indexed vertices has length $2m + (n - mg)$. We require $2m + (n - mg) \geq g$ and set $m = \left\lfloor \frac{n-g}{g-2} \right\rfloor$. As above, decomposing G needs m trees, so $\theta_{\mathbf{T}}(G) = \theta_{\mathbf{C}}(G) = \left\lfloor \frac{n-g}{g-2} \right\rfloor$. When $g \leq n < 3n - 4$, the existence of one chord yields $\theta_{\mathbf{T}}(G) = \theta_{\mathbf{C}}(G) = 2$. □

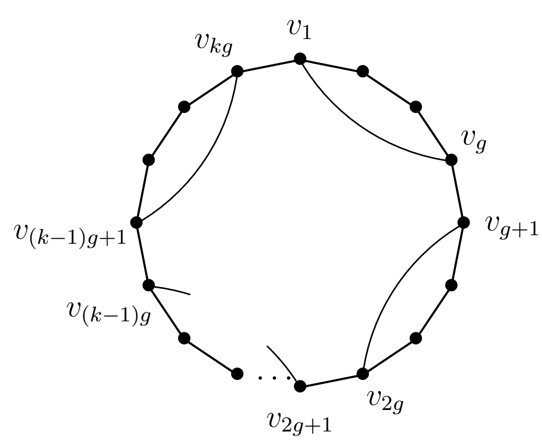

Figure 2: The graph $J_{k,g}$

3 Tree-thickness and Girth

We write $G[A]$ for the subgraph of G induced by a vertex set A. The tree-thickness arguments for connected graphs with girth at least 5 and for connected graphs with girth at least 4 that avoid subdivisions of $K_{2,3}$ are essentially the same, so we combine them.

Theorem 4 *Let G be an n-vertex connected graph. If* $\text{girth}(G) \geq g \geq 5$*, or if $g = 4$ and G contains no subdivision of $K_{2,3}$ with girth 4, then $\theta_{\mathbf{T}}(G) \leq \lfloor n/g \rfloor + 1$, and this is sharp.*

Proof. Sharpness is shown by the graph $H_{k,g}$ of Example 1, with pendant vertices added when $g \nmid n$. For the upper bound, we use induction on n. If $n < g$, then G has no cycle and is a tree itself. If $n = g$, then G is a cycle and decomposes into two trees. For the induction step, consider $n > g$. We may assume that G is not a tree, since then $\theta_{\mathbf{T}}(G) = 1$.

Let P be a longest path in G, with vertices $v_1, \ldots, v_m$ in order. Since $\text{girth}(G) \geq g$, we have $m \geq g$. Let $R = \{v_1, \ldots, v_g\}$. No two vertices in R have more than one common neighbor outside R, because this would create a subdivision of $K_{2,3}$ containing a 4-cycle (vertices in R with a common neighbor cannot be consecutive on R, since $\text{girth}(G) \geq 4$). The same observation holds for $R - \{v_g\}$.

Let T be a spanning tree of G that contains P. For $1 \leq i \leq m$, let S_i be the set of vertices outside P whose path to $V(P)$ in T arrives at v_i. Let $S = S_1 \cup \cdots \cup S_g$; note that $S_1 = \varnothing$. Among all the spanning trees that contain P, let T be one that minimizes $|S|$. With this choice of T, no vertex in S has a neighbor outside $S \cup R$.

Case 1: $S \neq \varnothing$.

Let $A = S \cup R - \{v_g\}$. Note that $G - A$ is connected. Also $|V(G - A)| \leq n - g$, since $S \neq \varnothing$. By the induction hypothesis, $\theta_{\mathbf{T}}(G - A) \leq \lfloor (n - g)/g \rfloor + 1 = \lfloor n/g \rfloor$. Call the trees in such a decomposition the "old" trees. We will incorporate the edges incident to A by adding some edges to old trees and creating one additional tree for the rest.

The key observation is that $G[A]$ is a forest. If there is a cycle C among the vertices of A, then it has at least g vertices. Combining a path around C with a shortest path from $V(C)$ to v_g in $G[A \cup \{v_g\}]$ contradicts the choice of P as a longest path in G.

Let $W_1, \ldots, W_t$ be the components of $G[A]$. One component contains all of $v_1, \ldots, v_{g-1}$ and $S_1, \ldots, S_{g-1}$, and the others form $G[S_g]$. By the choice of T, v_g is the only vertex outside A having neighbors in S, and v_g has a neighbor in each W_i. Use one such edge to each W_i

along with $G[A]$ to form a new tree T' for the decomposition. Add the other edges from v_g to S to the tree containing $v_g v_{g+1}$.

We have now assigned all edges of G to trees in the decomposition except those from $v_1, \dots, v_{g-1}$ to neighbors outside A. We can add these edges to T' unless two of them reach a common vertex outside A. Since G has girth at least g, the only vertices in $v_1, \dots, v_{g-1}$ that can have such a common neighbor are v_1 and v_{g-1}. We have observed that they can have only one such common neighbor; call it x. If $x = v_g$, then we have already put one of $\{v_1 x, v_{g-1} x\}$ into T' and the other into an old tree containing an edge incident to x. If $x \neq v_g$, then we can do the same, since x has no other neighbor in the component of T' containing v_{g-1}, and v_1 has no other neighbor in the old tree.

Case 2: $S = \varnothing$.

Let $A = \{v_1, \dots, v_g\}$. Note that $G - A$ is connected, since $S = \varnothing$; also, it has $n - g$ vertices. By the induction hypothesis, $\theta_{\mathbf{F}}(G - A) \leq \lfloor n/g \rfloor$. Call the trees in such a decomposition "old" trees. An additional tree T' will contain all other edges incident to the path $G[A]$, with a few possible exceptions.

Since girth$(G) \geq g$, the only pairs in A that can have common neighbors (and only one for each pair, as noted earlier) are $\{v_1, v_g\}, \{v_1, v_{g-1}\}, \{v_2, v_g\}$. Let x, y, z denote their possible common neighbors, respectively.

If the edge $v_1 v_g$ exists, then actually $y = v_g$ and $z = v_1$, and x does not exist. In this case we add $v_{g-1} v_g$ and $v_g v_{g+1}$ to an old tree and put all other edges incident to A in T'.

If $v_1 v_g \notin E(G)$, then we can add all of $\{x v_1, y v_{g-1}, z v_2\}$ that exist into old trees and put all other edges incident to A into T'. □

We have not proved that the bound of Theorem 4 holds for all connected graphs with girth 4. However, we can improve the upper bound from Chung's theorem for a class that allows subdivisions of $K_{2,3}$. It suffices to forbid $K'_{4,3}$ as a subgraph, where $K'_{4,3}$ is the graph obtained from $K_{4,3}$ by deleting one edge. In this proof our method follows that of Chung [1], inductively establishing a decomposition with special additional properties.

Theorem 5 *Let G be a connected n-vertex graph. If girth$(G) \geq 4$ and $K'_{4,3} \not\subseteq G$, then $\theta_{\mathbf{F}}(G) \leq \lceil n/3 \rceil$.*

Proof. Let $k = \lceil n/3 \rceil$. We use induction on n to prove the stronger statement that G has a tree decomposition $T_1, \ldots, T_k$ and a vertex partition $S_1, \ldots, S_k$ such that $S_i \subseteq V(T_i)$ for all i and S_i satisfies the following additional properties:

1) $|S_i| \le 4$, with equality only if $G[S_i]$ consists of two disjoint edges, and

2) If S_i is an independent set of size 3, then its vertices have a common neighbor in T_i.

For the basis step, observe that G is a tree if $n \le 3$, and we set $T_1 = G$ and $S_1 = V(G)$.

For the induction step, consider $n > 3$. A *pendant claw* in G is an induced copy of $K_{1,3}$ whose set S of leaves does not separate G. A *pendant cherry* in G is an induced copy of P_3 whose set S of vertices does not separate G. If G has a pendant claw or cherry whose leaves have no common neighbors outside this subgraph, then we let $S_k = S$ and let T_k consist of all edges incident to S_k. In the claw case, condition (2) holds. Since G has girth at least 4, T_k is a tree. Since T_k contains all edges incident to S_k, applying the induction hypothesis to the connected graph $G - S_k$ completes the decomposition. We call such pendant claws or cherries *deletable*.

In most cases, we find deletable claws or cherries. Let P be a longest path in G, with vertices $v_1, \ldots, v_m$ in order. For a spanning tree T of G that contains P, let A_i be the set of vertices outside P whose path to P in T arrives at v_i. Since P is a longest path, $A_1 = \varnothing$. Among all spanning trees that contain P, choose T to minimize $|A_2|$, and within this to minimize $|A_3|$. For this T, each vertex in A_2 has no neighbor outside $A_2 \cup \{v_2\}$, and each vertex in A_3 has no neighbor outside $A_3 \cup \{v_2, v_3\}$.

By the choice of P, $A_2 \cup \{v_1\}$ is an independent set of neighbors of v_2. Among $A_2 \cup \{v_1\}$, only v_1 can have neighbors other than v_2. If $|A_2| \ge 2$, then G has a deletable claw consisting of v_1, v_2, and two vertices of A_2. If $|A_2| = 1$, then G has a deletable cherry. Hence we may assume that $A_2 = \varnothing$.

If $A_3 \ne \varnothing$, then let R be the vertex set of a component of $G[A_3]$. Since P was chosen to be a longest path, $G[R]$ is a star, and the only edge leaving R is the edge to v_3 from a central vertex of the star. When $|R| \ge 4$, we thus have a deletable claw, and when $|R| = 3$ we have a deletable cherry.

When $|R| \le 2$, we let $S_k = \{v_1, v_2\} \cup R$ and let T_k consist all edges incident to S_k. Note that $|S_k| \le 4$, with equality only if $G[S_k]$ consists of two disjoint edges, as required. The only edges incident to S_k are those in the path induced by $S_k \cup \{v_3\}$ and edges incident to v_1 and v_2 (which have no common neighbor since they are adjacent). Thus T_k is a tree, and we can apply the induction hypothesis to $G - S_k$ to complete the decomposition.

The remaining case is $A_2 = A_3 = \varnothing$. Let $S_k = \{v_1, v_2, v_3\}$. Here we cannot simply let T_k consist of all edges incident to S_k, since v_1 and v_3 may have common neighbors. Let $G' = G - \{v_1, v_2, v_3\}$. By the induction hypothesis, G' has a tree decomposition $T'_1, \ldots, T'_{k-1}$ with corresponding special sets $S'_1, \ldots, S'_{k-1}$. For G, set $S_i = S'_i$ for $1 \le i \le k-1$.

Let $Z = N(v_1) \cap N(v_3)$. Each vertex of Z lies in S_i for some $i \le k-1$. We will modify T'_i to absorb half of the edges from S_i to $\{v_1, v_3\}$, becoming T_i. After this, the remaining edges incident to S_k will form a tree T_k to complete the decomposition.

Let $X = Z \cap S_i$ for some $i \in \{1, \ldots, k-1\}$. If $X = \{w\}$, then put wv_1 into T_i and wv_3 into T_k. If $X = \{w, y\}$, then put $\{wv_1, yv_3\}$ into T_i and $\{wv_3, yv_1\}$ into T_k.

If $|X| \in \{3, 4\}$, then the absence of triangles in G forces X to be an independent set. Hence $|S_i| = 3$ by the requirement that $G'[S_i]$ induces two edges when $|S_i| = 4$. When S_i is an independent set of size 3, the decomposition guaranteed by the induction hypothesis gives these vertices a common neighbor in T_i. The two additional common neighbors v_1 and v_3 complete $K_{3,3}$. Also, v_2 is adjacent to v_1 and v_3, completing $K'_{4,3}$, which is forbidden.

Thus always $|X| \le 2$, and T_k consists of the path induced by S_k and at most one edge from S_k to each vertex of $G - S_k$. This completes the desired decomposition. $\square$

4 Caterpillar Thickness

In this section, we study caterpillar-thickness of outerplanar graphs. We first consider the 2-connected case.

Theorem 6 *If G is a 2-connected n-vertex outplanar graph with girth at least g, then $\theta_{\mathbf{C}}(G)$ is bounded as given below, and all these bounds are sharp.*

$$\theta_{\mathbf{C}}(G) \le \begin{cases} 2 & \text{if } g \le n \le 3g-4, \\ \left\lfloor \frac{n-g}{g-2} \right\rfloor & \text{if } 3g-4 \le n \le g^2/2, \\ \lfloor n/g \rfloor & \text{if } n \ge g^2/2 \text{ (except } n=5 \text{ when } g=3). \end{cases}$$

Proof. In Example 3, we presented 2-connected outerplanar graphs with girth g having tree-thickness and caterpillar-thickness as specified above. Note that $g^2/2 < 3g - 4$ when $g = 3$; the middle "range" is empty.

For the upper bound, let C be the outer boundary in an outerplanar embedding of G. Since G is 2-connected, C is a cycle with vertices $v_1, \ldots, v_n$ in order, and G has no other vertices. A chord v_iv_j of C is *minimal* if one of the v_i, v_j-paths on C has no other endpoint of a chord as an internal vertex. Let m be the number of minimal chords.

If $m \le 1$, then G is a cycle with at most one chord, and two caterpillars (in fact, paths) suffice. Hence we may assume that $m \ge 2$.

Because G has girth at least g, the computation in Example 3 yields $m \le \left\lfloor \frac{n-g}{g-2} \right\rfloor$. Therefore, to complete the proof for the case $n \le g^2/2$ it suffices to show that $\theta_{\mathbf{C}}(G) \le m$ always. For $n > g^2/2$, we will prove (inductively) that $\theta_{\mathbf{C}}(G) \le n/g$.

Bound 1: *If $m \ge 2$, then $\theta_{\mathbf{C}}(G) \le m$.* Decompose C into m "boundary paths" $P_1, \ldots, P_m$ such that the endpoints of each path are internal to the paths generated by the chords. In particular, if v_rv_s is the ith minimal chord, then some internal vertex of the path from v_r to v_s along C is the end of P_i and the beginning of P_{i+1}. By the minimality of v_rv_s, no chord is incident to the common vertex of P_i and P_{i+1}. We use $P_1, \ldots, P_m$ as the spines of the caterpillars in the decomposition.

Each chord of C joins vertices from two boundary paths; we assign it to one of these two paths (we have observed that each end is incident to only one boundary path). Since every chord incident to P_i is incident at its other end to exactly one other boundary path, it suffices to show that the chords joining P_i and P_j can be distributed to those two paths in such a way that the chords assigned to each have distinct endpoints in the other.

Let H be a graph H consisting of paths $\langle u_1, \ldots, u_r \rangle$ and $\langle w_1, \ldots, w_s \rangle$ joined by noncrossing chords of the form u_iw_j. "Noncrossing" means that the chords obey a linear order L such that the indices of the vertices from each path are nondecreasing. The subgraph of G consisting of P_i and P_j and the chords joining them has this form.

Process the chords in H in the order L. If the next chord shares an endpoint with the current chord, assign it to the path containing the shared vertex; otherwise assign it to either path (this case covers the initial chord). If two vertices on one path have chords to a common neighbor on the other path, then the second chord among these two is assigned to the other path. Hence the chords assigned to each path have distinct endpoints on the other path.

Bound 2: *If $n \geq 2g$, then $\theta_{\mathbf{C}}(G) \leq n/g$.* We prove inductively that G decomposes into $\lfloor n/g \rfloor$ caterpillars whose spines cover $E(C)$. Two such caterpillars suffice when $m \leq 1$.

If no two minimal chords share an endpoint, then the minimal chords lie on m disjoint cycles, and $n \geq mg$. If $n \leq g^2/2$, then $m \leq (n-g)/(g-2) \leq n/g$. In either case, the construction for Bound 1 suffices, since the union of its spines is C. Hence we may assume that $n > g^2/2$ and that some two minimal chords share an endpoint.

If $m = 2$, then all chords have a common endpoint, and G is the edge-disjoint union of a star and a path, each of which is a caterpillar. Two edges of the star lie on C and form the spine of this caterpillar; the path is the remainder of C. Hence we may assume that $m > 2$.

Let v_iv_j and v_jv_k be two minimal chords with a common endpoint; we may assume that $i < j < k$. Let P be the v_i, v_k-path through v_j along C. Form a smaller 2-connected outerplanar graph G' as follows: If $g = 3$, then delete $V(P) - \{v_i, v_k\}$ from G and add the edge v_iv_k (if not already present); if $g \geq 4$, then delete $V(P) - \{v_i, v_j, v_k\}$. In the first case, we deleted $k - i - 1$ vertices; in the second, we deleted $k - i - 2$.

Since $k - i - 1 \geq 2g - 3 \geq g$ if $g = 3$ and $k - i - 2 \geq 2g - 4 \geq g$ if $g \geq 4$, there are at most $n - g$ vertices in G'. We can apply the induction hypothesis unless G' has fewer than $2g$ vertices. If so, then G' is a cycle. Since G has at least three minimal chords, this case arises only if $g = 3$ and G is the union of a spanning cycle and a triangle. Such a graph decomposes into two paths; all edges lie along the spines.

Now the induction hypothesis provides a decomposition of G' into at most $\lfloor n/g \rfloor - 1$ caterpillars whose spines cover the outer edges. When $g \geq 4$, it suffices to add P to this decomposition. When $g = 3$ and $v_iv_k \notin E(G)$, the edge v_iv_k lies on the outer face in G' and hence is on the spine of its caterpillar T in the decomposition of G'. Replacing v_iv_j with v_iv_j and v_jv_k in T yields again a caterpillar. The desired decomposition is now completed by adding the caterpillar consisting of P and all other edges incident to v_j except v_iv_j and v_kv_j.

Finally, suppose that $g = 3$ and $v_iv_k \in E(G)$. The edges $v_{i-1}v_i$, v_iv_k, and v_kv_{k+1} all lie on spines in the decomposition of G'. If they are not in the same caterpillar in the decomposition, then we add v_iv_j and v_jv_k to two different caterpillars and add P as a new caterpillar. If these three edges are in the same caterpillar T, then we break the spine of T at v_i; the piece containing $v_{i-1}v_i$ continues along P to v_j and then directly to v_k and v_{k-1}, while the piece containing $v_{k+1}v_k$ and v_kv_i continues directly to v_j and then along P to v_{k-1}. All the edges of G that are not in G' become spine edges in the resulting decomposition. □

The case of subdividing v_iv_k when $g = 3$ is the only part of the proof of Bound 2 that needs the stronger statement about containing C in the union of the spines.

Our next result is our most intricate. Essentially, we show that the graphs of girth 4 constructed in Example 1 are extremal for $\theta_{\mathbf{C}}$ among connected triangle-free outerplanar graphs wth n vertices. Theorems 8 and 9 discuss graphs with larger girth.

We believe the bound improves to $\lceil 3n/8 \rceil - 1$ when $n \not\equiv 5 \mod 8$ and $n > 8$. Gaining 1 in these congruence classes would require extensive case analysis, so we omit it. One source of difficulty is that the savings does not occur in most congruence classes until n exceeds 8. Another is that the optimal formula (without using $\lceil\ \rceil$) is not uniform across congruence classes. Hence we are content with a uniform formula that is optimal infinitely often.

Theorem 7 *If G is a connected triangle-free outerplanar graph with n vertices, then $\theta_{\mathbf{C}}(G) \le \lceil 3n/8 \rceil$, and this bound is sharp when $n \equiv 5 \mod 8$.*

Proof. In Example 1 we presented outerplanar graphs (in fact, cacti) with the specified caterpillar-thickness. When $k = 2l+1$, the graph $H_{k,4}$ has $8l+5$ vertices and has caterpillar-thickness $3l + 2$, which equals $\lceil 3(8l + 5)/8 \rceil$.

For the upper bound, we consider a counterexample G with fewest vertices, n. We will derive structural properties of G that eventually forbid its existence.

A subgraph H is *deletable* if it has a vertex subset S such that $G - E(H) - S$ is connected and $\theta_{\mathbf{C}}(H) \le \lfloor 3|S|/8 \rfloor$. With $a = 3|S|/8$, we have $\lceil 3(n - |S|)/8 \rceil + \theta_{\mathbf{C}}(H) \le \lceil 3n/8 - a \rceil + \lfloor a \rfloor \le \lceil 3n/8 \rceil$. Therefore, a minimal counterexample contains no deletable subgraph. When S is a set of at least three vertices, and the edges incident to S form a caterpillar, and $G - S$ is connected, we also say that S is deletable.

For 2-connected outerplanar graphs, Theorem 6 already provides an upper bound of $\lfloor n/4 \rfloor$, which is always at most $\lceil 3n/8 \rceil$. Thus G is not 2-connected and has at least 2 blocks. Let B be a leaf block of G (a block sharing only one vertex with the rest of G). Note that G is embedded with all vertices on the unbounded face.

Step 1: *Every leaf block is an edge or a 4-cycle.* Suppose that B has at least five vertices. Let v_1 be the cut-vertex of G in B, and let $v_1, \ldots, v_k$ be the vertices of B in order on the unbounded face. If $N(v_{i-1}) \cap N(v_{i+1}) = \{v_i\}$ for some i with $3 \le i \le k - 1$, then the girth condition implies that the edges incident to S form a caterpillar, where $S = \{v_{i-1}, v_i, v_{i+1}\}$. Also, $G - S$ is connected, so S is deletable.

If $\{v_2, v_3, v_4\}$ is not deletable, then there exists $v_j \in N(v_2) \cap N(v_4) - \{v_3\}$. If $j \neq 5$, then $\{v_3, v_4, v_5\}$ is deletable. If $j = 5$, then girth 4 forces $k > 5$, and now $\{v_4, v_5, v_6\}$ is deletable.

Step 2: *Every vertex in at most one non-leaf block lies in at most one leaf block.* Let v be a vertex in leaf blocks B and B'. If B is a 4-cycle, with vertices v, w, x, y in order, and z is a neighbor of v in B' (whether B' is an edge or a 4-cycle), then $\{x, y, z\}$ is deletable (the edges incident to $\{x, y, z\}$ form a path). If three leaf blocks that are edges share v, then their leaves form a deletable triple.

Thus at most two leaf blocks can contain v, and if so both are edges. This forbids all blocks being leaf blocks, since then $n = 3$ and G is a path. If leaf blocks B and B' are both edges containing v, and v is in at most one non-leaf block, then $V(B \cup B')$ is now deletable.

Step 3: *G has no "spear".* Define a *spear* to be a subgraph H consisting of two leaf blocks and a nontrivial path P connecting them, such that only the (possibly equal) vertices w and w' of P in the same penultimate block B^* have neighbors outside H, and $G - S$ is connected, where $S = V(H) - \{w'\}$.

If the leaf blocks B and B' are edges, then H is a path and $|S| \geq 3$, so H is deletable.

If B is a 4-cycle, then let the vertices be v, x, y, z in cyclic order, with v the cut-vertex of G. If B' is an edge, then H decomposes into the edge xv of B and a path. Thus H is deletable if $|S| \geq 6$, which fails only if P has length 1 and $w \neq w'$. In that case, delete only the path $H - xv$, with marked set S' consisting of $\{y, z\}$ and the leaf of B'.

If B and B' are 4-cycles, then H decomposes into the edge xv, one edge of B', and a path. Thus H is deletable if $|S| \geq 8$, which fails only if P has length 1 and $w \neq w'$. In that case, we may assume by symmetry that $w' = v$. Now delete only $H - xw'$ (a path plus an edge of B'), with marked 6-set S' consisting of $V(B') \cup \{y, z\}$.

Step 4: *Every penultimate block is an edge.* We observed that not all blocks are leaf blocks. A *penultimate block* is a leaf block in the graph obtained by deleting the non-cut-vertices of leaf blocks. Let B^* be a penultimate block. Since B^* is not a leaf block, it has a vertex v that lies in at least one leaf block and in no other non-leaf block. By Step 2, v belongs to exactly one leaf block; call it B.

Suppose that B^* is not an edge. Among the two neighbors of v along the unbounded face of B^*, we may choose x to avoid the only vertex that B^* can share with another non-leaf block. If x lies in a leaf block B', then $B \cup B' \cup xv$ is a spear. Otherwise, we find a deletable triple. It is $V(B) \cup \{x\}$ if B is an edge, and it is x together with two adjacent vertices of B other than v if B is a 4-cycle.

Step 5: *Two penultimate blocks cannot intersect.* Suppose that B_1^* and B_2^* are penultimate blocks with a common vertex w. By Step 3, each B_i^* is an edge; let v_i be the endpoint opposite w. By Step 2, each v_i lies in one leaf block B_i. Now $B_1 \cup B_2 \cup B_1^* \cup B_2^*$ is a spear.

Step 6: *A peripheral penultimate block intersects only one other non-leaf block.* A *chain* of blocks is a list of distinct blocks in which any two consecutive blocks intersect and the shared cut-vertices are all distinct. The leaf block and penultimate block at the beginning or end of a longest such chain are *peripheral* such blocks. Choose B and B^* to start a longest chain C, so B and B^* are a peripheral leaf block and peripheral penultimate block.

By Step 3, B^* is an edge; as usual, let w be the vertex it shares with a non-leaf block. If there are two such blocks, then one of them is not in the chain of blocks starting with B and B^*. By Step 5 it is not a penultimate block. It therefore has another vertex in a non-leaf block. Thus it yields a chain of at least three blocks that can replace B and B^* in C to form a longer chain, contradicting the choice of C.

Step 7: *There is no minimal counterexample.* Let B be a peripheral leaf block, sharing v with a penultimate block B^*. Let w be the other vertex of B^*. By Step 5, w lies in only one other non-leaf block, B_0.

If w lies in a leaf block, B', then $B \cup B^* \cup B'$ is a spear, since w lies in only one other block. Hence w lies only in B_0 and B^*.

Let x be a neighbor of w in B_0, along the unbounded face, and let $S = V(B) \cup \{w, x\}$. Note that either (1) $|S| = 6$ (if B is a 4-cycle) and the subgraph of edges incident to S decomposes into two caterpillars, or (2) $|S| = 4$ (if B is an edge) and the subgraph of edges incident to S decomposes into one caterpillar. If a leaf block or a penultimate block is attached at x, then we obtain a spear. If a longer chain is attached at x, then it contradicts B' being a leaf block. Hence nothing is attached at x. Now $G - S$ is connected, and the subgraph is deletable. □

When the girth is at least 5, a proof similar to that of Theorem 7 yields the following theorem. We omit the details of the proof and just mention a few of the differences.

Theorem 8 *If G is a n-vertex connected outerplanar graph with girth at least 5, then $\theta_{\mathbf{C}}(G) \le \lceil 3n/10 \rceil$, and this is sharp when $n \equiv 6 \mod 10$.*

Proof. (sketch) First, when $k = 2l + 1$, the graph $H_{k,5}$ has $10l + 6$ vertices and has caterpillar-thickness $3l + 2$, which equals $\lceil 3(10l + 6)/10 \rceil$.

A subgraph H is *deletable* if it has a vertex subset S such that $G - E(H) - S$ is connected and $\theta_{\mathbf{C}}(H) \le \lfloor 3|S|/10 \rfloor$. We follow the steps similar to those in the proof of Theorem 7.

Step 1: *Every leaf block is an edge or a 5-cycle.* Otherwise we can find a deletable set with 4 vertices.

Step 2: *Every vertex in at most one non-leaf block lies in at most one leaf block, unless the vertex lies in two leaf blocks, each of which is a single edge.*

Step 3: *G has no spear unless the path P in the spear has length 1 and the two leaf blocks are edges.*

Step 4 and 5 concern penultimate blocks in a longest chain of blocks. This differs from the proof of Theorem 7, where the conclusions were proved about all penultimate blocks.

Step 4: *Every peripheral penultimate block is an edge.*

Step 5: *Two peripheral penultimate blocks cannot intersect.*

Step 6: *A peripheral penultimate block intersects only one other non-leaf block.*

Step 7: *There is no minimal counterexample.* □

When the girth is at least 6, an argument like those of Section 3 suffices, and we are no longer confined to considering outerplanar graphs. The bound here is roughly $n/4$ rather than the $n/6$ of Theorem 4, because here we are restricted to using caterpillars.

Theorem 9 *If G is an n-vertex graph with girth at least 6, then $\theta_{\mathbf{C}}(G) \le \lceil (n-2)/4 \rceil$ for $n \ge 7$, and this is sharp.*

Proof. We observed in Example 2 that the bound is achieved by the tree obtained by subdividing $\lfloor (n-1)/2 \rfloor$ edges in the star with $\lceil (n-1)/2 \rceil$ edges.

For the upper bound, we use induction on n. Every graph with at most six vertices having girth at least 6 is a caterpillar except the 6-cycle. Also, every connected edge-disjoint union of a 6-cycle and a caterpillar decomposes into two caterpillars. Hence it suffices to show for $n \geq 7$ that $V(G)$ contains a set A of size at least 4 such that $G - A$ is connected and the set of edges incident to A forms a caterpillar.

Let P be a longest path in G, with vertices $v_1, \ldots, v_m$ in order. The girth requirement yields $m \geq 6$. Let $R = \{v_1, v_2, v_3\}$. No vertex has two neighbors in R.

Let T be a spanning tree of G that contains P. For $1 \leq i \leq m$, let S_i be the set of vertices outside P whose path to $V(P)$ in T arrives at v_i (note that $S_1 = \varnothing$). Let $S = S_2 \cup S_3 \cup S_4$. Among all the spanning trees that contain P, consider those that minimize $|S|$, and among these choose T to maximize $|S_2 \cup S_3|$.

With this choice of T, no vertex in S has a neighbor outside $S \cup R \cup \{v_4\}$. Furthermore, every component of $G[S_3]$ is a star whose center is adjacent to v_3, and S_2 is an independent set. If $|S_2 \cup S_3| \geq 2$, then let A consist of v_1, v_2, S_2, and the vertices in a largest component of $G[S_3]$, or the vertices in two components of $G[S_3]$ if S_3 is independent and $S_2 = \varnothing$. Except for the edges from S_3 to v_3, only v_1 and v_2 have neighbors outside A, and no two vertices of A have common neighbors. Thus A has the desired properties.

If $|S_2 \cup S_3| = 1$, then let $A = R \cup S_2 \cup S_3$. Again only vertices on the path formed by R have neighbors outside A, so A has the desired properties.

If $S_2 \cup S_3 = \varnothing$ and $S_4 \neq \varnothing$, consider a component H of $G[S_4]$; H is a tree whose vertices have distance at most 3 from v_4. If H contains a vertex with distance 3 from v_4, then H is a path, by the choice of T. Otherwise, H is a star with center adjacent to v_4. In either case, the choices of P and T prevent further edges from $V(H)$ to R. Let $A = V(H) \cup R$. Now $A \cup \{v_4\}$ induces a caterpillar, and the only edges leaving A are incident to R and reach distinct neighbors. Thus A has the desired properties. □

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