On the 3-Reconstructibility of Trees and Rooted Trees

Alexandr V. Kostochka*, Mina Nahvi†, Douglas B. West‡, Dara Zirlin§

June 28, 2021

Abstract

A graph is ℓ-reconstructible if it is determined by its multiset of subgraphs obtained by deleting ℓ verties. Using centroids and rooted trees, we prove that trees with at least 22 vertices are 3-reconstructible.

1 Introduction

A graph is ℓ-reconstructible if it is determined by its multiset of (unlabeled) subgraphs obtained by deleting ℓ vertices. The famous Reconstruction Conjecture of Kelly [4, 5] and Ulam [14] is that every graph with at least three vertices is 1-reconstructible. Kelly [5] made a stronger conjecture.

Conjecture 1.1 (Kelly [5]). *For all $\ell \in \mathbb{N}$ there exists M_ℓ such that every graph with at least M_ℓ vertices is ℓ-reconstructible (in particular, $M_1 = 3$).*

We will prove that trees with at least 22 vertices are 3-reconstructible. It is conjectured that this holds with at least seven vertices, which would be sharp.

When discussing ℓ-reconstructibility, the multiset of subgraphs of an n-vertex graph G obtained by deleting ℓ vertices is called the $(n-\ell)$-*deck* or simply the *deck* of G, denoted

*University of Illinois at Urbana–Champaign, Urbana IL, and Sobolev Institute of Mathematics, Novosibirsk, Russia: kostochk@math.uiuc.edu. Research supported by NSF grant DMS-1600592, NSF RTG grant DMS-1937241, and grant 19-01-00682 of the Russian Foundation for Basic Research.

†University of Illinois at Urbana–Champaign, Urbana IL: mnahvi2@illinois.edu. Research supported by Arnold O. Beckman Campus Research Board Award RB20003 of the University of Illinois at Urbana-Champaign.

‡Zhejiang Normal Univ., Jinhua, China and Univ. of Illinois at Urbana–Champaign, Urbana IL: dwest@illinois.edu. Research supported by National Natural Science Foundation of China grants NSFC 11871439, 11971439, and U20A2068.

§University of Illinois at Urbana–Champaign, Urbana IL 61801: zirlin2@illinois.edu. Research supported by NSF RTG grant DMS-1937241 and by Arnold O. Beckman Campus Research Board Award RB20003 of the University of Illinois at Urbana-Champaign.

$\mathcal{D}_{n-\ell}(G)$. We are given only the isomorphism class of each subgraph and do not know which vertices were deleted. The resulting $\binom{n}{\ell}$ subgraphs are called the *cards* of the deck. For an n-vertex graph G, being ℓ-reconstructible means that no other graph has the same $(n-\ell)$-deck as G, or $\mathcal{D}_{n-\ell}(G) = \mathcal{D}_{n-\ell}(H)$ implies $G \cong H$.

The elementary observation that motivates the extension from 1-reconstructibility to ℓ-reconstructibility is the following.

Observation 1.2. *For any graph G, the k-deck $\mathcal{D}_k(G)$ determines the $(k-1)$-deck $\mathcal{D}_{k-1}(G)$.*

Proof. Each card in $\mathcal{D}_{k-1}(G)$ appears in exactly $|V(G)| - k + 1$ cards in $\mathcal{D}_{k-1}(G)$. □

Thus every graph that is ℓ-reconstructible is also $(\ell-1)$-reconstructible, and the goal is to determine for each graph G the maximum ℓ such that G is ℓ-reconstructible, which we may call its *maximum reconstructibility.*

For the family of all graphs, Nýdl [11] proved that M_ℓ grows superlinearly in ℓ. For more restricted families, the threshold number of vertices to guarantee ℓ-reconstructibility may be smaller. Spinoza and West [13] proved that all graphs having maximum degree at most 2 and at least $2\ell+1$ vertices are ℓ-reconstructible. Furthermore, this is sharp, since the path $P_{2\ell}$ and the disjoint union $C_{\ell+1} + P_{\ell-1}$ of a cycle and a path have the same ℓ-deck. In fact, **??** determined the maximum reconstructibility for all graphs with maximum degree at most 2.

For trees, nearly the same threshold is conjectured. Nýdl [10] conjectured that trees with at least $2\ell+1$ vertices are *weakly ℓ-reconstructible*, meaning that no two such n-vertex trees have the same $(n-\ell)$-deck. Groenland et al. [3] found one counterexample by computer, consisting of two 13-vertex trees with the same 7-deck. In [7], The present authors proved that n-vertex acyclic graphs vertices form an *ℓ-recognizable* family when $n \ge 2\ell+1$ (and $(n,\ell) \ne (5,2)$), meaning that membership in the family is determined by the $(n-\ell)$-deck. The combination of ℓ-recognizability and weak ℓ-reconstructibility is ℓ-reconstructibility, so the modification of Nýdl's conjecture with the counterexample from [3] is now the following.

Conjecture 1.3. *For $n \ge 2\ell+1$ (except $(n,\ell) \in \{(5,2),(13,6)\}$), every n-vertex tree is ℓ-reconstructible, and this threshold is sharp.*

For sharpness, Nýdl provided two trees with $2k$ vertices having the same k-deck. A short proof of this was provided by Kostochka and West [8] using the tools developed by Spinoza and West [13] for maximum degree 2. A *spider* is a tree having one vertex with degree at least 3. Let $S_{a_1,\dots,a_d}$ denote the spider consisting of one vertex of degree d that is the endpoint of paths with lengths $a_1,\dots,a_d$; the tree has $1+\sum a_i$ vertices. Nýdl provided the spiders $S_{k-1,k-1,1}$ and $S_{k,k-2,1}$ with $2k$ vertices having the same k-deck.

According to Nýdl [12], Bondy and Hemminger [1] reported the existence of a preprint of Giles proving that sufficiently large trees are ℓ-reconstructible, but this was apparently never published. The result has now been proved in a recent paper by Groenland, Johnston, Scott, and Tan [3] that substantially generalizes and strengthen some of the earlier results. They proved that n-vertex trees are reconstructible from their k-decks when $k \geq \frac{8}{9}n+\frac{4}{9}\sqrt{8n+5}+1$. For $k = n-3$, this result implies that n-vertex trees are 3-reconstructible (reconstructible from the $(n-3)$-deck) when $n \geq 194$. We extend this threshold for $\ell = 3$ to $n \geq 22$.

Results on ℓ-reconstructibility are also known for degree lists, connectedness, random graphs, disconnected graphs, complete multipartite graphs, 3-regular graphs, and r-regular graphs that are not 2-connected. Kostochka and West [8] surveyed these results. In addition to the results on trees, the paper by Groenland et al. [3] also extends some of the other earlier results. They proved that the degree list of an n-vertex graph is reconstructible from the k-deck when $k \geq \sqrt{2n\log(2n)}$ and that connected n-vertex graphs are ℓ-recognizable when $n \geq 10\ell$. We will use our earlier result [6] that degree lists of graphs with at least seven vertices are 3-reconstructible; in particular, we know the number of leaves of a tree from its $(n-3)$-deck.

Our main tools are the ℓ-reconstructibility of rooted trees from rooted connected subtrees (for $\ell \leq 3$, with some exceptions), and the idea of "centroid" in a tree. The term "centroid" has been used in other ways in the literature, but this definition seems most common.

Definition 1.4. A *centroid* of a tree is a vertex whose deletion minimizes the maximum number of vertices in a single component of the remaining forest. The *cost* of a vertex v in a tree T is the maximum number of vertices in a component of $T-v$. The *cost* of T, which we write as $c(T)$, is the minimum cost among the vertices of T. Thus a centroid is a vertex with minimum cost.

It is well known that a tree has a unique centroid or has two adjacent centroids, yielding *unicentroidal* or *bicentroidal* trees, respectively. We need this fact in a stronger form. Myrvold [9] heavily used centroids and a more detailed version of this lemma in proving that trees with at least five vertices are 1-reconstructible from only three cards; that is, every such tree has "reconstruction number" 3.

Lemma 1.5. *Every n-vertex tree has either a unique centroid, with cost less than $n/2$, or two adjacent centroids, with cost exactly $n/2$.*

Proof. If a vertex v in a tree T has cost greater than $n/2$, then its neighbor in the large component of $T-v$ has smaller cost. Hence a vertex with smallest cost has cost at most $n/2$. For such a vertex v and any neighbor u, the forest $T-u$ has a component consisting of v plus all the other components of $T-v$, thereby yielding cost at least $n/2$. Furthermore, if v has cost less than $n/2$, then u has cost more than $n/2$. Moving further away from v increases the cost more.

Hence there are at most two centroids, and the cost is at most $n/2$. Furthermore, we have also shown that if the cost is less than $n/2$, then the centroid is unique. $\square$

Definition 1.6. The *pieces* of a unicentroidal tree T having centroid z are the components of $T - z$; when we know T and z, the neighbors of z in the pieces are the *roots* of the pieces. In a bicentroidal tree, the two subtrees obtained by deleting the edge joining the centroids are the *branches* of the tree, and the roots of the branches are the centroids. The *size* of a piece or branch is the number of vertices.

Our overall approach is to identify centroids in certain connected cards, in which case the cards will give us subtrees of rooted trees that enable us to apply reconstruction of rooted trees to obtain pieces of the original tree. For this we need an analogue for rooted trees of the notion of pieces in unrooted trees.

Definition 1.7. A *rooted tree* is a tree with one vertex distinguished as a *root*; all other vertices are undistinguished, and there is no specification of order among neighbors of a vertex. The *rooted pieces* or *r-pieces* of a rooted tree are the rooted subtrees that are the components obtained by deleting the root, with the original neighbors of the root designated as the roots in the r-pieces. The *size* of an r-piece is its number of vertices.

A *rooted connected card* or *rc1-card* of a rooted tree T with root z is a rooted tree T' with root z obtained by deleting a leaf of T other than z. More generally, the *rcℓ-cards* of a rooted tree with n vertices and root z are the rooted subtrees with $n - \ell$ vertices that have root z. The *rcℓ-deck* of a rooted tree is the multiset of its rcℓ-cards; the root vertex is known in each card, but otherwise the vertices are unlabeled.

A rooted tree is *weakly ℓ-reconstructible* if it is determined by its rcℓ-deck; that is, no other rooted tree has the same rcℓ-deck.

We use "weakly ℓ-reconstructible" because we are given that the full structure is a rooted tree, but this is the natural notion of ℓ-reconstructibility for rooted trees.

2 Reconstructibility of rooted trees

In proving that trees are 3-reconstructible, we will use ℓ-reconstructibility of rooted trees for $\ell \in \{1, 2, 3\}$. As we will discuss later, there will be some exceptions when $\ell \in \{2, 3\}$.

In the theory of reconstruction, proving reconstructibility for the graphs in a particular family $\mathcal{G}$ often is done in two steps. First we show that every graph having the same deck as a graph in $\mathcal{G}$ is also in $\mathcal{G}$; this is showing that the family is *recognizable*. We can then restrict our attention to reconstructions in $\mathcal{G}$ to show that only one graph (in $\mathcal{G}$) has this deck; this is showing that the family is *weakly reconstructible*. *Weakly ℓ-reconstructible* means doing this with the $(n - \ell)$-deck.

In the application of our result on rooted trees, we will know that we have the rcℓ-deck of a rooted tree. Thus we consider only rooted trees as reconstructions and ignore the problem of showing that the deck comes from a rooted tree. In order to be precise, we therefore describe these results as proving weak reconstructibility.

Theorem 2.1. *Rooted trees are weakly* 1*-reconstructible.*

Proof. We assume that we are given the rc1-cards of a rooted n-vertex tree T. We use induction on n. When $n \leq 2$, there is only one n-vertex rooted tree. When $n = 3$, there are two n-vertex rooted trees, and they are distinguished by the number of rc1-cards. For the induction step, suppose $n > 3$, and let z be the root of T.

Since $n \geq 4$ and z is given in each card, T has only one r-piece if and only if every rc1-card has only one r-piece. Hence we know whether T has one r-piece or more than one.

If T has only one r-piece, then let T' be the rooted tree obtained from $T - z$ by designating the neighbor of z in T as the root z'. The rc1-cards of T' are obtained from those of T by deleting z and designating its neighbor as the root z'. By the induction hypothesis, we can reconstruct T' from these, and we reconstruct T by adding z to T', adjacent to z'.

Hence we may assume that T has more than one r-piece. Over all the rc1-cards of T, the r-pieces include the r-pieces of T plus some rooted trees that are not r-pieces of T, obtained by deleting a leaf of an r-piece of T. In particular, all the largest r-pieces that arise are actual r-pieces of T, since they cannot arise from a larger r-piece by deleting a vertex.

If all r-pieces in all cards have only one vertex, then T is a star rooted at the center. We can recognize this, so in the remaining case some r-piece of T has more than one vertex.

Among all the largest r-pieces of the rc1-cards, let M be one that occurs most often. Cards with fewer than the maximum number of r-pieces isomorphic to M arise by deleting a leaf of an r-piece isomorphic to M. Let v be a leaf of M, and let $L = M - v$, with the same root. Among the rc1-cards of T with fewer than the maximum number of r-pieces isomorphic to M, find a card T' with the maximum number of r-pieces isomorphic to L. Reconstruct T by replacing an r-piece isomorphic to L in T' with an r-piece isomorphic to M. □

Before we discuss weak 2-reconstructibility of rooted trees, we note some exceptions.

Example 2.2. *Rooted trees with common rc2-decks.* For $n \geq 3$, two n-vertex rooted trees have the same rc2-deck with one rc2-card. Let $\hat{P}_n$ denote the path P_n as a rooted tree with an endpoint as the root. The only rc2-card of $\hat{P}_n$ is $\hat{P}_{n-2}$. Let $\hat{P}'_n$ denote the rooted tree consisting of $\hat{P}_{n-2}$ with two children added at its leaf. Again $\hat{P}_{n-2}$ is the only rc2-card.

For $n \geq 5$, two n-vertex rooted trees have the same rc2-deck with three rc2-cards. Let $\hat{Q}_n$ be the n-vertex rooted tree obtained from P_{n-4} by adding two copies of $\hat{P}_2$ under the leaf. Let $\hat{Q}'_n$ be the n-vertex rooted tree obtained from $\hat{P}_{n-4}$ by adding $\hat{P}_1$ and $\hat{P}'_3$ under the leaf. Both $\hat{Q}_n$ and $\hat{Q}'_n$ have rc2-deck consisting of two copies of $\hat{P}_{n-2}$ and one copy of $\hat{P}'_{n-2}$. These rooted trees are shown in Figure 1.

A rooted tree is *trivial* if it has only one vertex. The *root-degree* of a rooted tree is the degree of the root. We write a rooted tree with r-pieces $T_1, \dots, T_d$ as $U(T_1, \dots, T_d)$.

Theorem 2.3. *Except for $\{\hat{P}_n, \hat{P}'_n, \hat{Q}_n, \hat{Q}'_n\}$, rooted trees are weakly 2-reconstructible. Knowing the number of leaves distinguishes between the exceptions.*

Proof. We have the rc2-deck of an n-vertex rooted tree T, with root z specified in each rc2-card. If $n = 2$, then $T = \hat{P}_2$. If $n = 3$, then $T \in \{\hat{P}_3, \hat{P}'_3\}$.

If T has no vertices other than z and its children, then T has $\binom{n-1}{2}$ rc2-cards, all stars, and no other n-vertex rooted tree has this many rc2-cards. If T has exactly one vertex other than z and its children, then T has $\binom{n-2}{2} + 1$ rc2-cards; again T is determined. For $n = 4$, this leaves only $\hat{P}_4$ and $\hat{P}'_4$, which have the same rc2-deck but different numbers of leaves. This completes the proof for $n \le 4$; we proceed inductively with $n \ge 5$.

Let $d^* = d_T(z)$. In the remaining cases, T has at least two vertices that are not children of z. Now d^* is the maximum of the root-degrees in the rc2-cards, and $d^* \le n - 3$. The root-degree is the number of r-pieces of T, which we now know.

If $d^* = 1$, then T has only one r-piece; let z' be the child of z. Let T' be the r-piece of T (that is, $T' = T - z$ with z' as root). The rc2-cards of T' arise from those of T by deleting z and are rooted at z'. By the induction hypothesis, we can reconstruct T' from its rc2-deck unless $T' \in \{\hat{P}_{n-1}, \hat{P}'_{n-1}, \hat{Q}_{n-1}, \hat{Q}'_{n-1}\}$, which holds if and only if T is the corresponding member of $\{\hat{P}_n, \hat{P}'_n, \hat{Q}_n, \hat{Q}'_n\}$ (the cases $T \in \{\hat{Q}_n, \hat{Q}'_n\}$ do not arise when $d^* = 1$ until $n \ge 6$). We reconstruct T by adding z adjacent to z' in T'; in the exceptional cases the number of leaves distinguishes between the two members of a pair with the same rc2-deck.

Hence we may assume $2 \le d^* \le n - 3$. When $n = 5$, we thus consider only instances with $d^* = 2$; they are $\hat{Q}_5$ and $\hat{Q}'_5$ (from Example 2.2) and $U(\hat{P}_1, \hat{P}_3)$. Since $U(\hat{P}_1, \hat{P}_3)$ has only two rc2-cards while $\hat{Q}_5$ and $\hat{Q}'_5$ have three, $U(\hat{P}_1, \hat{P}_3)$ is reconstructible; $\hat{Q}_5$ and $\hat{Q}'_5$ have the same rc2-deck but have different numbers of leaves. Hence we may assume $n \ge 6$.

Note that T has at least two trivial r-pieces if and only if T has an rc2-card with root-degree $d^* - 2$. In this case, we reconstruct T by adding two trivial r-pieces to such a card.

Hence we may assume that all rc2-cards have root-degree at least $d^* - 1$, and T has at most one trivial r-piece. Define a "slim card"to be an rc2-card with root degree exactly $d^* - 1$. There are at least $d^* - 1$ slim cards when T has a trivial r-piece, with equality only when all other r-pieces are paths with at least three vertices. When T has no trivial r-piece, the number of slim cards is the number of r-pieces isomorphic to $\hat{P}_2$, which is at most d^*. Hence we know whether T has a trivial r-piece unless the deck has $d^* - 1$ or d^* slim cards.

Suppose first that T has exactly d^* slim cards and has reconstructions both with and without a trivial r-piece. The reconstruction without is $U(\hat{P}_2, \dots, \hat{P}_2)$, in which no rc2-card has an r-piece of size at least 3. This also requires $n = 2d^* + 1$, which yields $d^* \ge 3$ since $n > 5$. In a reconstruction having a trivial r-piece, $d^* - 2$ r-pieces are paths with size at least 3, so some rc2-card does have an r-piece with size at least 3. Hence when T has d^* slim cards we can recognize whether T has a trivial r-piece.

Now suppose that T has exactly $d^* - 1$ slim cards. If T has a trivial r-piece, then all other r-pieces are paths with size at least 3. Here rc2-cards with two trivial r-pieces arise only by deleting two vertices from an r-piece of size 3, so there are at most $d^* - 1$ of them. If T has no trivial r-piece, then every r-piece except one is $\hat{P}_2$, and the other r-piece has size more than 2. In this case there are exactly $\binom{d^*-1}{2}$ rc2-cards with two trivial r-pieces. Since $\binom{d^*-1}{2} > d^* - 1$ when $d^* \geq 5$, we may assume $d^* \leq 4$. If confusion remains, then the instances with one trivial r-piece or no trivial r-pieces are as below, where $\hat{T}_m$ denotes any rooted tree with m vertices (with $n \geq 9$ when $d^* = 3$ and $n \geq 5$ when $d^* = 2$). In each case, the two possibilities are distinguished by the number of rc2-cards.

	one trivial r-piece	#rc2-cards	no trivial r-piece	#rc2-cards
$d^* = 4$	$U(\hat{P}_1, \hat{P}_3, \hat{P}_3, \hat{P}_3)$	9	$U(\hat{P}_2, \hat{P}_2, \hat{P}_2, \hat{T}_4)$	≥ 10
$d^* = 3$	$U(\hat{P}_1, \hat{P}_3, \hat{P}_{n-5})$	5	$U(\hat{P}_2, \hat{P}_2, \hat{T}_{n-5})$	≥ 6
$d^* = 2$	$U(\hat{P}_1, \hat{P}_{n-2})$	2	$U(\hat{P}_2, \hat{T}_{n-3})$	≥ 3

Therefore, the rc2-deck determines whether T has a trivial r-piece. Now consider the case where the rc1-deck of T has a slim card. If either T has no trivial r-piece or T has both a trivial r-piece and a slim card with a trivial r-piece, reconstruct T from a slim card by adding $\hat{P}_2$ as an r-piece.

In the remaining case with a slim card, T has a trivial r-piece and the slim cards are the rc1-cards for the rooted tree T' obtained by deleting the trivial r-piece from T. By Theorem 2.1, we can reconstruct T' from these and obtain T by adding a trivial r-piece.

Hence we may assume that T has no slim cards. Thus every r-piece of T has size at least 3, and every rc2-card has d^* pieces. Since $d^* \geq 2$ and every r-piece has size at least 3, every actual r-piece of T appears as an r-piece in some rc2-card. As in Theorem 2.1, the largest r-pieces that appear in rc2-cards are actual r-pieces of T, and we see them. Let b be the maximum size of an r-piece of T.

Suppose first that every r-piece of T has size b. We recognize this case from rc2-cards whose r-pieces all have size b except for one with size $b-2$, which must have lost two vertices because the others could not have lost any. Call such rc2-cards "pure" cards. Among all the pure cards, let C be one in which the multiplicity of some r-piece R in the list of r-pieces of size b is as small as possible (possibly 0). Reconstruct T by replacing the small r-piece in C with a copy of R.

Hence we may assume that some r-piece of T has size less than b. Now an rc2-card C' having a smallest r-piece over all the rc2-cards arises by deleting two vertices from a smallest r-piece of T. In C' we see all r-pieces of T of size b, with their multiplicities.

Let M be a largest r-piece of T, and let d' be the number of r-pieces of T isomorphic to M. Let T' be obtained from T by deleting the d' r-pieces of T isomorphic to M. The rc2-cards of T' are obtained from the rc2-cards of T having d' r-pieces isomorphic to M by deleting the copies of M. If $d' < d^* - 1$, then T' has at least two r-pieces, each of size at least 3. Hence T' is not any of the exceptional rooted trees, and by the induction hypothesis we can reconstruct T' and replace the copies of M to obtain T.

In the final case, T consists of $d^* - 1$ pieces isomorphic to M and one smaller piece R', and we know this. Let L be a rooted tree obtained from M by deleting one leaf; let a be the number of leaves whose deletion from M yields L. In every rc2-card of T having $d^* - 2$ r-pieces isomorphic to M and one piece isomorphic to L, the remaining r-piece is an rc1-card of R' (smaller than L). Each rc1-card of R' arises this way on exactly $a(d^* - 1)$ cards. Hence we obtain the rc1-deck of R'. By Theorem 2.1, we can reconstruct R' and thus T. □

Definition 2.4. A *broom* is a tree obtained from a star by growing a path from the center of the star. A *rooted broom* is a broom with root chosen as the endpoint of the path grown from the center of the star. In particular, a rooted path and a star rooted at its center are both degenerate examples of rooted brooms.

Example 2.5. In the statement of the next lemma, when $\ell = 2$ the second class includes $\hat{P}_n$ and $\hat{P}'_n$, which are brooms, plus one rooted tree $\hat{P}''_n$ obtained by putting $\hat{P}_1$ and $\hat{P}_2$ below the leaf of $\hat{P}_{n-3}$. Note that $\hat{P}''_n$ has two leaves and two rc2-cards, while $\hat{P}_n$ and $\hat{P}'_n$ each have only one rc2-card. All rc2-cards for these three rooted trees are copies of $\hat{P}_{n-2}$, but the three examples are distinguished by knowing the number of leaves and the number of rc2-cards.

Lemma 2.6. *For $\ell \geq 2$ and $n \geq \ell + 2$, the rcℓ-cards of an n-vertex rooted tree T are pairwise isomorphic if and only if T is a rooted broom or T is formed by merging the leaf of $\hat{P}_{n-\ell-1}$ with the root of a rooted tree with $\ell + 2$ vertices.*

Proof. The rcℓ-cards of a rooted broom are a single rooted broom. In the second case desribed, every rcℓ-card is $\hat{P}_{n-\ell}$.

Let T be a rooted tree with pairwise isomorphic rcℓ-cards. We may assume $T \neq \hat{P}_n$. Let v be the branch vertex of T nearest to the root. If T is not in the class described, then at least $\ell + 2$ vertices lie below v in the tree. Also, the subtree rooted at v has at least two r-pieces, since v is a branch vertex.

It suffices to prove that a rooted tree $\widetilde{T}$ with at least two r-pieces and at least $\ell + 3$ vertices has distinct rcℓ-cards if it is not a rooted star. If there are at most ℓ vertices below the root outside the largest piece, then there is an rcℓ card with one r-piece and an rcℓ-card with more than one r-piece. If there are more than ℓ vertices below the root outside a largest r-piece, then deleting ℓ vertices from smallest r-pieces yields an rcℓ-card whose list of sizes of r-pieces differs from that of the rcℓ-card obtained by deleting vertices from largest r-pieces. □

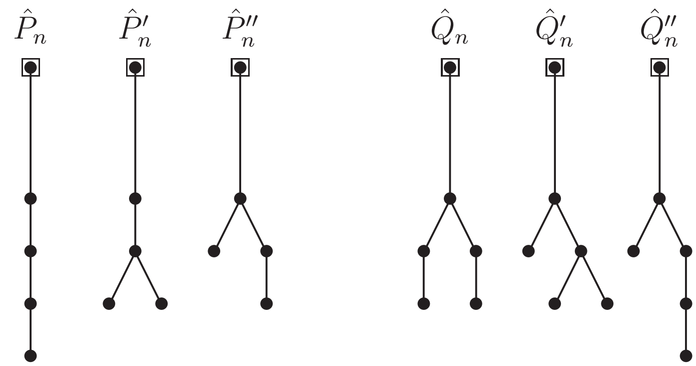

Figure 1: Special rooted trees.

When we want to reconstruct rooted trees from the rooted subtrees obtained by deleting three vertices, more exceptions arise.

Example 2.7. *Rooted trees with common rc3-decks.* We describe specific rooted trees by attaching a rooted forest below the leaf of a rooted path. Several appear in Figure 1, including $\hat{P}''_n$ as defined in Example 2.5. With $\hat{Q}_n$ and $\hat{Q}'_n$ defined as in Example 2.2, let $\hat{Q}''_n$ be the rooted tree obtained by putting $\hat{P}_1$ and $\hat{P}_3$ below the leaf of $\hat{P}_{n-4}$. Let $\hat{B}_{n,t}$ be the n-vertex rooted broom with t-leaves (thus $\hat{B}_{n,1} = \hat{P}_n$ and $\hat{B}_{n,2} = \hat{P}'_n$).

Every way of putting a rooted forest having ℓ vertices below the leaf of $\hat{P}_{n-\ell}$ yields an n-vertex rooted tree whose rcℓ-deck has a single card, $\hat{P}_{n-\ell}$. It is more helpful to describe this as putting a rooted tree with $\ell+1$ vertices under the leaf of $\hat{P}_{n-\ell-1}$. When $\ell = 2$, the resulting trees are $\hat{P}_n$ and $\hat{P}'_n$. When $\ell = 3$, the tree whose root is put under the leaf of $\hat{P}_{n-4}$ is one of $\{\hat{P}_4, \hat{P}'_4, \hat{P}''_4, \hat{B}_{4,3}\}$.

More generally, every way of adding $\ell+1$ vertices in d nonempty rooted trees under the leaf of $\hat{P}_{n-\ell-1}$ yields an n-vertex rooted tree whose rcℓ-deck consists of d copies of $\hat{P}_{n-\ell}$. For $(\ell, d) = (3, 2)$ and $n \geq 5$, we obtain three trees: $\hat{Q}_n$, $\hat{Q}'_n$, and $\hat{Q}''_n$, where $\hat{Q}''_n$ is obtained from $\hat{P}_{n-4}$ by adding $\hat{P}_1$ and $\hat{P}_3$ under the leaf.

For $n \geq 6$, several pairs of n-vertex rooted trees obtained from $\hat{P}_{n-6}$ by adding one of two 6-vertex rooted trees under the leaf have the same rc3-deck as listed below. Within each pair, the two rooted trees are distinguished by the number of leaves.

under leaf of $\hat{P}_{n-6}$	#rc3-cards	rc3-deck	#leaves
$U(\hat{P}_1, \hat{P}_4)$ or $U(\hat{P}_1, \hat{P}'_4)$	2	one $\hat{P}_{n-3}$, one $\hat{P}'_{n-3}$	2 or 3
$U(\hat{P}_2, \hat{P}_3)$ or $U(\hat{P}_1, \hat{P}''_4)$	3	two $\hat{P}_{n-3}$, one $\hat{P}'_{n-3}$	2 or 3
$U(\hat{P}_2, \hat{P}'_3)$ or $U(\hat{P}_1, \hat{B}_{4,3})$	4	three $\hat{P}_{n-3}$, one $\hat{P}'_{n-3}$	3 or 4

Finally, let $\hat{T}^+$ denote the rooted tree obtained from a rooted tree $\hat{T}$ by adding a trivial r-piece as an extra child of the root. If $\hat{T}$ and $\tilde{T}$ are rooted trees that have the same rcℓ-deck and also have the same rc$(\ell-1)$-deck, then $\hat{T}^+$ and $\tilde{T}^+$ also have the same rcℓ-deck. Their deck consists of the common rc$(\ell-1)$-deck of $\hat{T}$ and $\tilde{T}$ together with the cards of their common rcℓ-deck extended by adding a trivial r-piece. When $\ell = 3$, this occurs just when $\{\hat{T}, \tilde{T}\}$ is $\{\hat{P}_{n-1}, \hat{P}'_{n-1}\}$ (for $n \geq 4$) or $\{\hat{Q}_{n-1}, \hat{Q}'_{n-1}\}$ (for $n \geq 6$), since those are the only pairs of $(n-1)$-vertex trees having the same rc2-deck.

When $\ell = 3$, all these examples have root-degree at most 2, except for root-degree 3 when $n = 6$ in the pair obtained by adding a trivial r-piece to $\hat{Q}_5$ or $\hat{Q}'_5$, and when $n = 4$ for $\hat{B}_{4,3}$ in the set $\{\hat{P}_4, \hat{P}'_4, \hat{P}''_4, \hat{B}_{4,3}\}$. Besides $\hat{P}'_3$ in the pair $\{\hat{P}_3, \hat{P}'_3\}$ the examples with root-degree 2 are of those of the form $\{\hat{T}^+, \tilde{T}^+\}$ in the preceding paragraph where $\hat{T}$ and $\tilde{T}$ are exceptions with root-degree 1 that also have the same rc2-deck.

We mention two more pairs with seven vertices and root-degree 2: $\{U(\hat{P}_2, \hat{P}'_4), U(\hat{P}_3, \hat{P}_3)\}$ and $\{U(\hat{P}_2, \hat{P}''_4), U(\hat{P}_3, \hat{P}'_3)\}$. The first pair have the same rc3-deck consisting of two copies of $\hat{P}_4$ and two copies of $\hat{P}''_4$, and they are distinguished by the number of leaves. The second pair have the same rc3-deck consisting of one copy of $\hat{P}_4$, one copy of $\hat{P}'_4$, and three copies of $\hat{P}''_4$, but both of these rooted trees have three leaves.

Remark 2.8. *Rooted trees having the same rc2-deck also have the same rc3-deck.* We showed in Theorem 2.3 that a rooted tree is determined by its rc2-deck, except for $\{\hat{P}_n, \hat{P}'_n, \hat{Q}_n, \hat{Q}'_n\}$. If rooted trees with the same rc2-deck are the same, then they also have the same rc3-deck. Also, as noted in Example 2.7, $\hat{P}_n$ and $\hat{P}'_n$ have the same rc3-deck, as do $\hat{Q}_n$ and $\hat{Q}'_n$.

Theorem 2.9. *For $n \geq 4$, the n-vertex rooted trees not described in Example 2.7 are weakly 3-reconstructible. For the exceptions, it is sufficient to know also the number of leaves, except for the general pairs $\{\hat{P}'_n, \hat{P}''_n\}$ and $\{\hat{Q}_n, \hat{Q}''_n\}$ and the 7-vertex pairs $\{U(\hat{P}_2, \hat{P}'_4), U(\hat{P}_3, \hat{P}_3)\}$ and $\{U(\hat{P}_2, \hat{P}''_4), U(\hat{P}_3, \hat{P}'_3)\}$. Within each of these pairs, the two trees are distinguished by the number of copies of $S_{2,1,1}$ as an unrooted subtree.*

Proof. The behavior of the exceptions is verified by checking all the examples in Example 2.7. The examples having the same rc3-deck are distinguished by their number of leaves, except for the two pairs listed in the theorem statement. There is one more copy of $S_{2,1,1}$ in $\hat{P}''_k$ than in $\hat{P}'_k$, and there is one more copy of $S_{2,1,1}$ in $\hat{Q}_k$ than in $\hat{Q}''_k$.

Hence we may assume that we are given the rc3-deck of an n-vertex rooted tree T not in the list of exclusions. Let z be the root of T, specified in each rc3-card. Let $d^* = d_T(z)$. We use induction on n.

Step 1: $n \leq 6$ *or* $d^* \geq n-3$ *or* $d^* = 1$. For $n \leq 4$, all trees are exceptional.

Consider $n = 5$. The trees with $d^* = 1$ are in $\{\hat{P}_5, \hat{P}'_5, \hat{P}''_5, \hat{B}_{5,3}\}$, all exceptional. Those with $d^* = 2$ are in $\{\hat{Q}_5, \hat{Q}'_5, \hat{Q}''_5\}$, all exceptional. The only tree with $d^* = 3$ is $U(\hat{P}_1, \hat{P}_1, \hat{P}_2)$, whose rc3-deck is three copies of $\hat{P}_2$, and the only one with $d^* = 4$ is $U(\hat{P}_1, \hat{P}_1, \hat{P}_1, \hat{P}_1)$, whose rc3-deck is four copies of $\hat{P}_2$. These decks differ from those of the rooted trees in Example 2.7. Hence we may proceed inductively with $n \geq 6$.

Let s be the number of vertices of T other than z and its children. If $s \le 3$, then T has a rooted star with $n-4$ leaves as an rc3-card, while if $s \ge 4$ there is no such card. Furthermore, if $s = 3$, then there is exactly one such card, while if $s < 3$ there is more than one such card. Hence we can distinguish the cases $s < 3$, $s = 3$, and $s > 3$.

If $s = 0$, then $T = \hat{B}_{n,n-1}$ and T has $\binom{n-1}{3}$ rc3-cards (all stars). If $s = 1$, then T has $n-3$ trivial r-pieces and one 2-vertex r-piece, producing $\binom{n-2}{3} + (n-3)$ rc3-cards. If $s = 2$, then the r-pieces of T are trivial except for two copies of $\hat{P}_2$, or one copy of $\hat{P}_3$, or one copy of $\hat{P}'_3$. In these three cases the numbers of rc3-cards are $\binom{n-3}{3} + 2(n-4)$, or $\binom{n-3}{3} + (n-4) + 1$, or $\binom{n-2}{3} + 1$, respectively. These numbers and the count when $s = 1$ are distinct, except that $\binom{3}{3} + 4 = \binom{4}{3} + 1$ when $n = 6$, which occurs for the listed exceptions $\hat{Q}_5^+$ and $\hat{Q}_5'^+$. Hence the number of rc3-cards distinguishes all the nonexceptional rooted trees with $s \le 2$, which corresponds to $d^* \ge n-3$.

In the remaining cases, T has at least three vertices other than z and its children. Now d^* is the largest root-degree among the rc3-cards, and $d^* \le n-4$. We now know d^*, which is the number of r-pieces of T.

If $d^* = 1$, then T has only one r-piece; let z' be the child of z. Let T' be the r-piece of T, namely $T - z$ with z' as root. The rc3-cards of T' are obtained from those of T by deleting z and naming z' as the root. By the induction hypothesis, we can reconstruct T' from its rc3-deck unless T' is one of the exceptional trees on $n-1$ vertices, which holds if and only if T is one of the exceptional trees on n vertices. Outside the exceptional cases, we apply the induction hypothesis to T' and then reconstruct T by adding z above the root of T'.

Hence we may assume $2 \le d^* \le n-4$. When $n = 6$, this leaves only instances with $d^* = 2$. There are six such rooted trees, and they occur in three pairs of two rooted trees having the same rc3-deck, as listed in Example 2.7: $\{U(\hat{P}_1, \hat{P}_4), U(\hat{P}_1, \hat{P}'_4)\}$ (same as $\{\hat{P}_5^+, \hat{P}_5'^+\}$), $\{U(\hat{P}_2, \hat{P}_3), U(\hat{P}_1, \hat{P}''_4)\}$, and $\{U(\hat{P}_2, \hat{P}'_3), U(\hat{P}_1, \hat{B}_{4,3})\}$. Within each pair, the trees are distinguished by the number of leaves.

Hence in all other cases we have $n \ge 7$ and $2 \le d^* \le n-4$, and we know d^*.

Step 2: *At least three trivial r-pieces.* Note that T has at least three trivial r-pieces if and only if T has an rc3-card with root-degree $d^* - 3$. In this case, we reconstruct T by adding three trivial r-pieces to such a card. Otherwise, all rc3-cards have root-degree at least $d^* - 2$, and T has at most two trivial r-pieces.

Step 3: *Two trivial r-pieces.* A card in the rc3-deck of T is *j-slim* if it has root-degree $d^* - j$. A 2-slim card can arise by deleting two trivial r-pieces and a third vertex or by deleting one trivial r-piece and one r-piece of size 2; there are no 2-slim cards when T has no trivial r-pieces. There are at least $d^* - 2$ 2-slim cards when T has two trivial r-pieces, with equality only when all other pieces are paths of size at least 3. There are at most $d^* - 1$ 2-slim cards when T has exactly one trivial r-piece, with equality if and only if all other pieces are $\hat{P}_2$. Hence the number of trivial r-pieces is known unless the number of 2-slim cards is $d^* - 2$ or $d^* - 1$.

Suppose first that T has exactly $d^* - 1$ 2-slim cards. If T has only one trivial r-piece, then $T = U(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_2)$ and $n = 2d^*$. Since $n > 6$, we have $d^* \geq 4$. No rc3-card has an r-piece of size at least 3. On the other hand, in a reconstruction with two trivial r-pieces and $n = 2d^*$, some r-piece must have size at least 3, and with $d^* \geq 4$ we can see such a piece in some rc3-card. Hence in this case we either reconstruct T or know that every reconstruction has two trivial r-pieces.

Now suppose T has exactly $d^* - 2$ 2-slim cards. If T has only one trivial r-piece (Case 1), then $T = U(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_2, \hat{T}')$, where $\hat{T}'$ has at least three vertices. If T has exactly two trivial r-pieces (Case 2), then all the nontrivial r-pieces are paths of size at least 3.

In Case 1, each 2-slim card has $d^* - 3$ pieces that are $\hat{P}_2$, plus one piece that is $\hat{T}'$, and $\hat{T}'$ must be a path since all pieces in Case 2 are paths. To obtain such a 2-slim card in Case 2, we must delete the two trivial pieces and can reduce at most one piece to $\hat{P}_2$. Since there must be $d^* - 2$ such 2-slim cards, in Case 2 the nontrivial pieces must all be $\hat{P}_3$. Since also the 2-slim cards can only leave one piece with size 3, in Case 2 we have $T = U(\hat{P}_1, \hat{P}_1, \hat{P}_3, \hat{P}_3)$, and in Case 1 we have $T = U(\hat{P}_1, \hat{P}_2, \hat{P}_2, \hat{P}_3)$. All rc3-cards with three trivial r-pieces are $U(\hat{P}_1, \hat{P}_1, \hat{P}_1, \hat{P}_2)$; in Case 1 there are three of these, but in Case 2 there are only two. Hence these cases are distinguished.

Therefore, no matter what d^* is we can recognize whether T has exactly two trivial r-pieces. If so, then the 2-slim rc3-cards are the rc1-cards of the tree T' obtained by deleting the trivial r-pieces of T. By Theorem 2.1, we can reconstruct T', and we add two trivial r-pieces to obtain T.

Step 4: *One trivial r-piece.* In all remaining cases, T has at most one trivial piece. If T has a 2-slim rc3-card, then T has a trivial piece and a piece of size 2; reconstruct T from a 2-slim card by replacing these pieces.

When the rc3-deck has no 2-slim card, suppose that there exist both a reconstruction T with a trivial piece and a reconstruction T° with no trivial piece; we seek a contradiction.

Since there is no 2-slim card, r-pieces of T other than the trivial piece have size at least 3. The number of 1-slim cards with a trivial r-piece is then $2p$, where p is the number of r-pieces of T having size 3, since we can delete such a piece or delete two of its vertices (in only one way) and the trivial piece. In the rc3-deck of T°, the number of 1-slim cards with a trivial piece is $q(q-1)$, where q is the number of r-pieces of T° having size 2; delete both vertices from one such piece and one from another. Hence $2p = q(q-1)$.

Now consider 1-slim cards with no trivial piece. From T, we must delete the trivial piece, so such a card has at most two r-pieces of size 2. From T°, we must delete one r-piece of size 2 or 3 and not create a trivial r-piece, so any such card will have at least $q-1$ r-pieces of size 2. Thus $q - 1 \leq 2$.

These two computations reduce the possibilities to $(p, q) \in \{(1, 2), (3, 3)\}$. First consider

$(p, q) = (3, 3)$. Each of T and T° has six 1-slim cards with a trival piece. In each such card of T, the remaining pieces all have size at least 3. However, each such card of T° has a piece of size 2. Hence T and T° cannot both be reconstructions in this case.

Now consider $(p, q) = (1, 2)$. If $d^* = 2$, then $T \in \{U(\hat{P}_1, \hat{P}_3), U(\hat{P}_1, \hat{P}_3')\} = \{\hat{Q}_5'', \hat{Q}_5'\}$ and $T^\circ = U(\hat{P}_2, \hat{P}_2) = \hat{Q}_5$, listed as exceptions in Example 2.7 for $n = 5$. Hence we may assume $d^* \geq 3$. The two smallest r-pieces still may be $\{\hat{P}_1\hat{P}_3\}$, $\{\hat{P}_1, \hat{P}_3'\}$, or $\{\hat{P}_2, \hat{P}_2\}$, but now there is at least one more piece. The two 1-slim cards with a trivial r-piece arise by deleting three of the four vertices in the two smallest r-pieces, and they must be the same for any reconstruction, so in each case deleting the two smallest r-pieces yields the same rooted tree; call it Y. Let y_j be the number of rc$j$-cards of Y. In the three cases stated above, the number of rc3-cards of the resulting full tree is $2y_0 + 2y_1 + 2y_2 + y_3$, or $2y_0 + 3y_1 + 3y_2 + y_3$, or $2y_0 + 3y_1 + 2y_2 + y_3$, respectively, where the coefficient of y_j is the number of ways of deleting $3 - j$ vertices from the two smallest r-pieces in forming cards. Since Y has at least four vertices, y_1 and y_2 are positive, so the numbers of rc3-cards differ in the three cases.

Hence the rc3-deck determines whether T has exactly one trivial r-piece; suppose that it does. We already reconstructed T in the case that T has a 2-slim card, so we may assume that all other r-pieces of T have size at least 3. Let S be the set of 1-slim cards having a trivial r-piece. As discussed earlier, $|S| = 2p$, where p is the number of r-pieces of T with size 3. Let Y be the rooted tree obtained from T by deleting all the r-pieces of size 3. Any rc3-card in S shows Y and $p - 1$ of the 3-vertex pieces. Each such piece appears in $2p - 2$ members of S. Hence when $p \geq 2$ we can determine from S the number of pieces of size 3 that are $\hat{P}_3$ and the number that are $\hat{P}_3'$, thereby reconstructing all pieces of T. If $p = 1$, then since we know Y we know the rc$j$-deck of Y for each j. As earlier, since $U(\hat{P}_1, \hat{P}_3)$ has two rc1-cards and two rc2-cards, while $U(\hat{P}_1, \hat{P}_3')$ has three rc1-cards and three rc2-cards, we can tell from the size of the rc3-deck of T what the piece of size 3 is.

Hence we may assume that all r-pieces of T other than the one trivial piece have size at least 4. Let Y be the rooted tree obtained by deleting the trivial piece. Now the 1-slim cards of T are precisely the rc2-cards of Y. If Y is determined by its rc2-deck, then T is determined by adding a trivial piece to Y. If there is an alternative reconstruction $\tilde{Y}$ from the rc2-deck of Y, then Y and $\tilde{Y}$ also have the same rc3-deck, as observed in Remark 2.8. Now T and $\tilde{T}$ have the same rc3-deck, where $\tilde{T}$ arises from $\tilde{Y}$ by adding a trivial piece. The resulting T and $\tilde{T}$ are now Y^+ and $\tilde{Y}^+$, occuring as exceptions in Example 2.7. Otherwise Y is determined by its rc2-deck, which determines T.

Step 5: *T has no trivial r-piece but has an r-piece of size* 2. The only way to obtain a 1-slim rc3-card with a trivial r-piece when T has no trivial r-piece is to have two (or more) r-pieces of size 2 and delete both vertices from one of them and one from another. Hence in the remaining case if there is a 1-slim rc3-card with a trivial r-piece we reconstruct T from it by replacing the trivial r-piece with two copies of $\hat{P}_2$.

We next need to be able to recognize when T has a piece of size 2. Suppose that the rc3-deck also has a reconstruction T° with no r-piece of size at most 2. A 1-slim card of T° arises only by deleting an r-piece of size 3, and hence such a card has no r-piece of size 2. If T has an r-piece of size 3, then T has a 1-slim card with an r-piece of size 2. Hence in T every r-piece other than one of size 2 has size at least 4. Now a 1-slim card of T has at most one r-piece of size 3. This means that T° has at most two r-pieces of size 3 and therefore at most two 1-slim cards. Hence in T there are at most two leaves outside the r-piece of size 2.

If there is only one 1-slim card, then $T = U(\hat{P}_2, \hat{P}_{n-3})$, and there are only three rc3-cards. However, there are four ways to delete at most three vertices from the r-piece of size 3 in T° (whether it is $\hat{P}_3$ or $\hat{P}_3'$), all leading to rc3-cards. Hence we may assume that there are two 1-slim cards, meaning that T has exactly two leaves outside the smallest piece. Now T has at most nine rc3-cards. If $d^* = 3$, then two r-pieces of size 3 and an additional piece of size at least 4 give T° more than nine rc3-cards.

Finally, if $d^* = 2$, then T° (and hence also T) has exactly seven vertices. Here $T \in \{U(\hat{P}_2, \hat{P}_4'), U(\hat{P}_2, \hat{P}_4'')$ and $T^\circ \in \{U(\hat{P}_3, \hat{P}_3\}, U(\hat{P}_3, \hat{P}_3'), U(\hat{P}_3', \hat{P}_3')\}$. The possibilities for T have four or five rc3-cards, respectively, while those for T° have four, five, or six, respectively. This yields the pairs $\{U(\hat{P}_2, \hat{P}_4'), U(\hat{P}_3, \hat{P}_3)\}$ and $\{U(\hat{P}_2, \hat{P}_4''), U(\hat{P}_3, \hat{P}_3')\}$ which are exceptions having the same rc3-deck as described in Example 2.7. In each pair the trees are distinguished by the number of unrooted copies of $S_{2,1,1}$.

Hence we can recognize when T has one r-piece of size 2 (and no trivial r-piece). Say that a 1-slim card having a piece of size 2 is *defective*. If T has no defective card, then all pieces other than the one of size 2 have size at least 4, and the 1-slim rc3-cards of T are the rc1-cards of the rooted tree T' obtained by deleting the smallest r-piece from T. By Theorem 2.1, we can reconstruct T' and add an r-piece of size 2 to obtain T.

If T has a defective card, then T has a piece of size 3. A defective card contains as r-pieces all the r-pieces of T having size at least 4. It remains to determine the r-pieces of size 3; let there be p that are copies of $\hat{P}_3$ and q that are copies of $\hat{P}_3'$. Defective cards arise by deleting an r-piece of size 3 or by deleting the r-piece of size 2 and one leaf from an r-piece of size 3. Thus each copy of $\hat{P}_3$ is missing from two defective cards, and each copy of $\hat{P}_3'$ is missing from three defective cards. This yields $t = 2p + 3q$, where t is the total number of defective cards. On the other hand $s = p + q + 1$, where s is the number of pieces of size 3 in each defective cards. Since we see s and t from the rc3-deck, we compute p and q.

Step 6: *Every r-piece of T has at least three vertices.* Note the exceptions $U(\hat{P}_3, \hat{P}_3)$ and $U(\hat{P}_3, \hat{P}_3')$, which share rc3-decks with $U(\hat{P}_2, \hat{P}_4')$ and $U(\hat{P}_2, \hat{P}_4'')$, respectively, as discussed in Step 5. If the rc3-deck of T is not from one of these exceptions, then in the remaining case we have determined that all r-pieces of T have size at least 3.

Since $d^* \ge 2$ and every r-piece has size at least 3, every r-piece of T appears as an r-piece of some rc3-card of T. Let b be the maximum size of an r-piece in an rc3-card of T, and let

c be the minimum such size, setting $c = 0$ if T has a 1-slim card. The smallest r-piece(s) of T have size $c + 3$. Every card having an r-piece of size c (or 1-slim card if $c = 0$) has $d^* - 1$ r-pieces that are all the r-pieces of T except one of size $c + 3$. Call these *trim cards*. (Note that $b \geq c + 3$, with equality possible.)

For every trim card, there is one r-piece of T with size $c + 3$ that does not appear. If some trim card has an r-piece of size $c + 3$, then T has more than one r-piece of size $c + 3$, and over all the trim cards we see all the pieces, in particular all the r-pieces of size $c + 3$. Choose a trim card C in which an r-piece R of size $c + 3$ appears fewer than the maximum number of times, and reconstruct T from C by replacing the smallest r-piece with a copy of R (or adding R as an r-piece if $c = 0$).

In the remaining case, T has exactly one r-piece R of size $c + 3$. Each trim card gives us the other r-pieces, with their multiplicities (hence we also know b). The set consisting of the smallest r-piece in each trim card is the rc3-deck of R. Now we can reconstruct R by the induction hypothesis and hence reconstruct T unless R is in the set of exceptions.

Let M be an r-piece of T with b vertices, and let d' be the number of r-pieces of T isomorphic to M. Obtain T' from T by deleting the r-pieces isomorphic to M. The rc3-cards of T' are obtained from the rc3-cards of T with d' r-pieces isomorphic to M by deleting the copies of M. If $d' < d^* - 1$, then T' has at least two pieces, one of which is the exception R and the others of which have at least four vertices. No exceptions in Example 2.7 fit this description, so by the induction hypothesis T' is reconstructible from its rc3-deck, which we have obtained. After reconstructing T', we add the copies of M to obtain T.

Hence we may assume that T consists of $d^* - 1$ r-pieces isomorphic to M and one smaller piece R with s vertices (here $s = c + 3$). We know M, s, and b. First suppose that $s \leq b - 3$. Let L be an rc2-card of M, and let a be the number of copies of L as an rc2-card of M. In every rc3-card of T having $d^* - 2$ r-pieces isomorphic to M and one piece isomorphic to L, the remaining r-piece is an rc1-card of R (smaller than L). Each rc1-card of R arises this way on exactly $a(d^* - 1)$ cards. Hence we obtain the rc1-deck of R. By Theorem 2.1, we can reconstruct R and thus reconstruct T. If $s = b - 2$ and $d^* \geq 3$, then we can similarly use an rc1-card L' of M and obtain the rc1-deck of R from cards with $d^* - 3$ r-pieces isomorphic to M and two r-pieces isomorphic to L.

Next suppose $s = b - 2$ and $d^* = 2$. Let L be an rc1-card of M, and let a be the number of copies of L as an rc1-card of M. In every rc3-card of T having an r-piece isomorphic to L, the other r-piece is an rc2-card of R (smaller than L). Each rc2-card of R arises this way on exactly a cards. Hence we obtain the rc2-deck of R. By Theorem 2.3, we can reconstruct R and hence T unless we have the common rc2-deck of $\hat{P}_s$ and $\hat{P}'_s$ or of $\hat{Q}_s$ and $\hat{Q}'_s$. To distinguish these, note that an rc3-card of T having pieces of sizes $b - 3$ and $b - 2$ arises by deleting three vertices from M (in which case the other piece is R) or by deleting two vertices from M and one vertex from R. Hence the number of these cards is $ij + j'$, where i is the number of leaves of R, j is the number of rc2-cards of M, and j' is the number of rc3-cards of M. Since we know M, we know j and j' and can compute i. This distinguishes between $\hat{P}_s$ and $\hat{P}'_s$ and between $\hat{Q}_s$ and $\hat{Q}'_s$ for R.

In the final case, $s = b - 1$. Since we know M, we know the rc1-deck $L_1, \ldots, L_m$ of M, where m is the number of leaves of M. Let S be the multiset of rc3-cards of T in which one piece has $b-3$ vertices, one has $b-1$ vertices, and the rest are isomorphic to M. Such cards arise by deleting three vertices from one copy of M or by deleting one vertex from a copy of M and two vertices from R. If for some member of S the piece with $b - 1$ vertices is not in the rc1-deck of M, then that piece is R and we complete the reconstruction.

Otherwise, R is an rc1-card of M. Let p be the number of rc3-cards of M, and let q be the number of rc2-cards of R; at present q is unknown. Let k be the multiplicity of a given rc1-card L of M. If $L \neq R$, then the number of members of S in which the piece with $b - 1$ vertices is L is $(d^* - 1)kq$. If $L = R$, then that number is $(d^* - 1)(kq + p)$. Since we know $|S|$ from the deck, and the multiplicities of the various rc1-cards of M sum to m, we can compute $q = \frac{|S|-(d^*-1)p}{(d^*-1)m}$. Since we know the multiplicity of each rc1-card of M, we can now determine which one occurs too often as a piece of a member of S, and that is R. $\square$

In applying Theorem 2.9, it will be helpful to know the number of leaves of a tree. In the unrooted setting, we proved in [6] that when $n \geq 7$, the $(n - 3)$-deck of an n-vertex graph determines its degree list. We have not yet found a short argument that directly determines the number of leaves of a tree from its $(n - 3)$-deck, though there is a short argument for obtaining the number of leaves from the $(n - 2)$-deck. The number of copies of $S_{2,1,1}$ is also determined by the $(n - 3)$-deck when $n \geq 8$, by Observation 1.2.

3 Trees with Cost at Most $(n - 4)/2$

For trees with small cost, we reduce 3-reconstructibility to weak 3-reconstructibility of rooted trees. We begin by showing that we can recognize the $(n - 3)$-decks of trees.

Lemma 3.1. *If T is an n-vertex tree with $n \geq 7$, then every graph having the same $(n-3)$-deck as T is a tree.*

Proof. The $(n - 3)$-deck provides the 2-deck, so every reconstruction has $n - 1$ edges. All cards are acyclic, so reconstructions have no cycles of length at most $n - 3$. Therefore, a non-tree reconstruction G must be $C_{n-1} + P_1$ or $C_{n-2} + P_2$ or the graph C' consisting of C_{n-2} plus a pendant edge and an isolated vertex.

The numbers of copies of P_{n-3} and $K_{1,3}$ in these alternatives are $(n-1, 0)$ or $(n-2, 0)$ or $(n, 1)$, respectively; we know these values, since $n-3 \geq 4$. A path has only four copies of P_{n-3}, but $4 < n-2$. Hence we may assume that T has a branch vertex. A non-tree reconstruction allows only one copy of $K_{1,3}$, and then G must be C'. However, the maximum number of copies of P_{n-3} in a tree with only one copy of $K_{1,3}$ is 5, achieved by $S_{n-3,1,1}$, and $5 < n$. □

It sometimes is helpful to exclude "extreme" trees from consideration by recognizing the decks of such trees. Here we are deleting only two vertices.

Lemma 3.2. *The connected cards in the $(n-2)$-deck of an n-vertex tree T are pairwise isomorphic if and only if T is a star or a path or has at most five vertices.*

Proof. The claim holds for stars and paths, and for $n = 5$ because all 3-vertex trees equal P_3. For $n \geq 6$, suppose that T is not a star or a path but satisfies the condition.

A *leaf-neighbor* is a neighbor of a leaf. A tree is *leaf-regular* if all leaf-neighbors have the same degree. It is *leaf-uniform* if all leaf-neighbors are adjacent to the same number of leaves. If two leaf-neighbors x and y have different degrees or different numbers of leaves adjacent to them, then deleting one leaf adjacent to x or one leaf adjacent to y yields two nonisomorphic subtrees (in the first case the degree lists are different; in the second the leaves are grouped by theie adjacent leaf-neighbors into sets of different sizes). Hence $T - w$ must be both leaf-regular and leaf-uniform for each leaf w.

Since T is not a star, T has at least two leaf-neighbors. Let u and u' be distinct leaf-neighbors, and let v and v', respectively, be leaves adjacent to u and u'. Let b and b' be the numbers of leaves adjacent to u and u', respectively. If b and b' are at least 2, then because $T - v$ and $T - v'$ are leaf-uniform, and u and u' remain leaf-neighbors in both, we have $b' = b - 1$ and $b = b' - 1$, which is impossible.

Hence we may assume $b' = 1$. Suppose $b \geq 2$. Since the preceding paragraph applies to any two leaf-neighbors, the leaf-uniformity of $T - v'$ now implies that T has no third leaf-neighbor, making T a tree obtained from a star by replacing one edge with a path. Now deleting two vertices from the end of the path yields a different $(n-2)$-card from the tree obtained by deleting two leaves adjacent to u, since $n \geq 6$.

Hence we may assume that every leaf-neighbor is adjacent to exactly one leaf. The leaf-neighbors at both ends of a longest path must now have degree 2. Since deleting the leaf at either end yields a leaf-regular tree having a leaf-neighbor with degree 2, all leaf-neighbors in T have degree 2.

Given that T is not a path, consider the distances from leaves to nearest branch vertices. Deleting the leaf and its neighbor from a shortest such path either reduces the number of leaves or reduces the minimum distance from a leaf to a branch vertex by 2. Deleting two leaves does neither and hence produces a different subtree with $n-2$ vertices. □

In studying connected cards in the $(n-\ell)$-deck of an n-vertex tree, it is helpful to know which vertices of the original tree can be the centroid of the card. When ℓ is fixed, the centroid of a connected card cannot be very far from the original centroid. Our primary interest will be in the two largest pieces of a tree. The remaining vertices form appendages from the centroid, so we use a horticultural term describing appendages on trees.

Definition 3.3. A *j-burl* in a tree T is a vertex v such that there are exactly j vertices in $T - v$ outside the two largest components. The *burl* of the tree T is the set of these outside vertices when v is the centroid of T. Recall that $c(T)$ denotes the cost of T, which is the number of vertices in a largest component of the forest obtained by deleting a centroid of T.

Lemma 3.4. *Let z be a centroid of an n-vertex tree T. Let u be a centroid of a connected card C in the $(n - \ell)$-deck of T. If $n > 2\ell$, then $z \in V(C)$. If $c(T) \le (n - \ell + 1)/2$, then u is z or a neighbor of z. If $c(T) = (n - \ell + 2)/2$, then u can have distance 2 from z only if their common neighbor has degree 2 in T. If $c(T) = n/2$, then u can be outside the neighborhoods of the centroids of T only if the closer centroid is a j-burl with $j \le (\ell - 4)/2$.*

Proof. Omitting z from a connected card requires the remaining vertices to lie in one piece of T, which requires $(n - \ell) \le c(T) \le n/2$.

Every component of $T - z$ has at most $c(T)$ vertices. Therefore, if u is a vertex outside the closed neighborhood of z, in $T-u$ there is a component with at least $n-c(T)+1$ vertices. In order to make u a centroid of C, this component must be cut down to at most $(n - \ell)/2$ vertices. Hence $n - c(T) + 1 - \ell \le (n - \ell)/2$, which simplifies to $c(T) \ge (n - \ell + 2)/2$.

When this inequality is violated, there is no such centroid. When it holds with equality, the large component of $T-u$ must have exactly $n-c(T)+1$ vertices, so the common neighbor of u and z has degree 2.

Suppose that T is bicentroidal and the centroid u of C has distance 2 from the closer centroid z of T, and let z be a j-burl in T. To become a piece of $C-u$, the component of $T-u$ containing z must be trimmed to at most $(n-\ell)/2$ vertices. Hence $2+j+n/2-\ell \le (n-\ell)/2$, which simplifies to $j \le (\ell - 4)/2$. □

Corollary 3.5. *Let T be an n-vertex tree. If $c(T) \ne (n - 1)/2$, then every centroid in every connected $(n-3)$-card C of T is or has a neighbor that is a centroid of T. If $c(T) = (n-1)/2$ and C has a centroid u that is not a neighbor of the centroid z of T, then C is bicentroidal and the common neighbor of u and z has degree 2 in T and is the other centroid of C.*

Proof. Let z be a centroid of T, and set $\ell = 3$ in Lemma 3.4. Now $c(T) \le (n - 2)/2$ keeps the centroid of C within distance 1 of z, while $c(T) = (n - 1)/2$ allows it to move one step farther when the common neighbor has degree 2 (the card is then bicentroidal).

When T is bicentroidal, there is no j-burl when j is negative, so the centroid of C must be a neighbor of a centroid of T. □

Our arguments for reconstruction of a tree T from the $(n-3)$-deck are based on the value of the cost $c(T)$. Therefore, we need lemmas that enable us to recognize this value from the deck. For a deck $\mathcal{D}$, let $c(\mathcal{D})$ denote the maximum cost among connected cards in $\mathcal{D}$.

Lemma 3.6. *The $(n-\ell)$-deck $\mathcal{D}$ of an n-vertex tree T satisfies*

$$c(\mathcal{D}) = \begin{cases} c(T) & \text{if } c(T) \leq (n-\ell)/2, \\ \lfloor (n-\ell)/2 \rfloor & \text{if } c(T) > (n-\ell)/2. \end{cases}$$

Also, if $c(T) \leq (n-\ell)/2$, then the centroid of T is a centroid in every connected card.

Proof. Let z be a centroid of T, and let X be a largest piece of T. By Lemma 3.4, z appears in every connected card C in $\mathcal{D}$.

First suppose $c(T) \leq (n-\ell)/2$. Components of $C - z$ are contained in components of $T-z$ and hence have at most $c(T)$ vertices. Since $c(T) \leq (n-\ell)/2 = |V(C)|/2$, we conclude that z is a centroid of C, and $c(C) \leq (n-\ell)/2$. Furthermore, if C arises by deleting ℓ vertices outside X, then X is still a piece, so $c(C) = c(T)$. Hence $c(\mathcal{D}) = c(T)$.

Now suppose $c(T) > (n-\ell)/2$. Every connected card C satisfies $c(C) \leq (n-\ell)/2$, so it suffices to construct a card C with cost $\lfloor (n-\ell)/2 \rfloor$. Delete successive leaves of X until exactly $\lfloor (n-\ell)/2 \rfloor$ vertices of X remain. Complete the card by successively deleting other leaves outside X. The number of vertices remaining outside X is $\lceil (n-\ell)/2 \rceil$, since the card has $n-\ell$ vertices. Thus z is the unique centroid if $n-\ell$ is odd, while both z and its neighbor x in X are centroids if $n-\ell$ is even. Hence $c(C) = \lfloor (n-\ell)/2 \rfloor$, and $c(\mathcal{D}) = \lfloor (n-\ell)2 \rfloor$. $\square$

Theorem 3.7. *For $n \geq 7$, trees with n vertices and cost at most $(n-5)/2$ are 3-reconstructible.*

Proof. Let $\mathcal{D}$ be the $(n-3)$-deck of such a tree T. By Lemma 3.6, we recognize that T is in this family: every connected card has cost at most $(n-5)/2$. Lemma 3.6 also implies that every connected card has the centroid z of T as its unique centroid. With z distinguished in each connected card, the connected cards form the rc3-deck of T as a rooted tree with root z. In addition, $c(T) \leq (n-5)/2$ requires $d_T(z) \geq 3$. The rooted trees in Example 2.7 that have root-degree 3 and are not reconstructible from their rc3-decks have six vertices. Hence by Theorem 2.9 we can reconstruct T from the deck. $\square$

For general ℓ, Lemma 3.6 will lead to ℓ-reconstructibility of n-vertex trees having cost less than $\lfloor (n-\ell)/2 \rfloor$ if rooted trees are proved to be weakly ℓ-reconstructible, since the centroid of T is can be determined in every connected card and used as a root. This also needs a suitable threshold for n and avoiding the exceptions to reconstruction of rooted trees.

Recognition of trees with cost at most $(n-5)/2$ means that we can also recognize trees with cost at least $(n-4)/2$. We will proceed in this way, successively recognizing and reconstructing a subfamily of the remaining family of graphs. As the cost increases, this becomes more difficult. For cost $(n-4)/2$, the proof of reconstruction is very similar to Theorem 3.7 once we prove that the family is recognizable. We will begin to need not only the number of vertices in the largest piece, but also the number in the next largest piece.

Definition 3.8. For a unicentroidal tree T, let $c'(T)$ denote the size of the second largest piece in T, which may equal $c(T)$; we call $c'(T)$ the *subcost* of T. An $\left(\frac{n-a}{2}, \frac{n-b}{2}\right)$*-card* is a connected card such that for a centroid u the two largest components of $T-u$ have $(n-a)/2$ and $(n-b)/2$ vertices, with $a \le b$. For a deck $\mathcal{D}$, let $c'(\mathcal{D})$ denote the maximum subcost among the connected cards with maximum cost.

Henceforth always when discussing a unicentroidal tree T, we let z denote the centroid, with neighbor x in a largest piece X and neighbor x' in a second largest piece X'. Note that the burl of T is $T-R$, where $R = V(X) \cup V(X') \cup \{z\}$.

By Lemma 3.6, we know the cost of T from the cost of its $(n-\ell)$-deck when that cost is less than $\lfloor (n-\ell)/2 \rfloor$. Now set $\ell = 3$. When n is even, $n-3$ is odd, so Lemma 3.6 implies that $c(\mathcal{D})$ can be $(n-4)/2$ when $c(T) \in \{(n-4)/2, (n-2)/2, n/2\}$. Our next task is to determine from the deck whether $c(T)$ equals $(n-4)/2$.

Definition 3.9. A card in the $(n-\ell)$-deck of an n-vertex graph is *balanced* if it consists of two components having $\lceil (n-\ell)/2 \rceil$ and $\lfloor (n-\ell)/2 \rfloor$ vertices, respectively.

Lemma 3.10. *If $\mathcal{D}$ is the $(n-3)$-deck of an n-vertex tree T, where n is even and $n \ge 20$, then $c(T) = (n-4)/2$ if and only if*

(a) $c(\mathcal{D}) = (n-4)/2$,

(b) $\mathcal{D}$ has no balanced cards, and

(c1) some connected card has cost $(n-10)/2$, or

(c2) $c'(\mathcal{D}) = (n-2j)/2$ for some $j \in \{2,3,4\}$, and T has a $\left(\frac{n-2j}{2}, \frac{n-10}{2}\right)$-card. Also, when $j = 4$ there is a $\left(\frac{n-6}{2}, \frac{n-8}{2}\right)$-card, and when $j = 3$ there is a $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-card and a $\left(\frac{n-8}{2}, \frac{n-8}{2}\right)$-card.

Proof. By Lemma 3.6, we may assume $c(T) \in \{(n-4)/2, (n-2)/2, n/2\}$. In each case $c(\mathcal{D}) = (n-4)/2$, by Lemma 3.6.

Case 1: $c(T) = (n-4)/2$. Any balanced card has components with $(n-2)/2$ and $(n-4)/2$ vertices, since $n-3$ is odd. Thus deleting any single vertex from T leaves a component with at least $(n-2)/2$ vertices, contradicting $c(T) = (n-4)/2$. Hence (b) holds.

Define j by $c'(T) = (n-2j)/2$; note that $j \ge 2$. Every piece of T has at most $(n-4)/2$ vertices. By Lemma 3.6, the centroid of every connected card is z, the centroid of T. In any connected card obtained by deleting three vertices of X, there remain $(n-10)/2$ vertices of X and the entire second largest piece X'. If $j \ge 5$, then such a card has cost $(n-10)/2$.

Hence we may assume $j \in \{2,3,4\}$. The card described is a $\left(\frac{n-2j}{2}, \frac{n-10}{2}\right)$-card unless besides X and X' there is another piece in T with at least $(n-8)/2$ vertices. This requires $n \geq 1 + \frac{n-4}{2} + \frac{n-2j}{2} + \frac{n-8}{2}$, which simplifies to $2j \geq n-10$. Since $2j \leq 8$, we obtain the $\left(\frac{n-2j}{2}, \frac{n-10}{2}\right)$-card unless $n \leq 18$. (When $n = 18$, for example, the spider $S_{7,5,5}$ has cost $(n-4)/2$, no connected card with cost 4 for (c1), and no $\left(\frac{n-8}{2}, \frac{n-10}{2}\right)$-card for (c2).)

For $c'(\mathcal{D}) = (n-2j)/2$, we need a $\left(\frac{n-4}{2}, \frac{n-2j}{2}\right)$-card. We keep the two largest pieces to make such a card by deleting three vertices from the burl. These are available, since there are $1 + \frac{n-2j}{2} + \frac{n-4}{2}$ vertices outside the burl, leaving $j+1$ vertices in the burl. Also $c'(\mathcal{D}) \leq c'(T)$ when each connected card has centroid z, so $c'(\mathcal{D}) = c'(T) = (n-2j)/2$.

Finally, deleting one vertex from X and two from the burl yields a $\left(\frac{n-6}{2}, \frac{n-2j}{2}\right)$-card. When $j = 3$, deleting two from X and one from X' yields a $\left(\frac{n-8}{2}, \frac{n-8}{2}\right)$-card.

Case 2: $c(T) = (n-2)/2$. Assume (a), (b), and (c). If $c'(T) \geq (n-4)/2$, then T has a balanced card, contradicting (b). Hence $c'(T) \leq (n-6)/2$.

Deleting ℓ vertices reduces the cost by at most ℓ. Thus no card of T has cost at most $(n-10)/2$, and (c1) fails. Hence (c2) holds, so $c'(\mathcal{D}) = (n-2j)/2$ for some $j \in \{2,3,4\}$, and with (a) there is a $\left(\frac{n-4}{2}, \frac{n-2j}{2}\right)$-card C.

The centroid of C is in $\{x, z, x'\}$, by Corollary 3.5. It cannot be x', since the component of $T - x'$ containing z has at least $(n+6)/2$ vertices, which cannot be cut to $(n-4)/2$ by deleting three vertices. If the centroid is z, then the second largest piece in T has at least $(n-2j)/2$ vertices. Since the largest piece in T has $(n-2)/2$ vertices, forming a $\left(\frac{n-2j}{2}, \frac{n-10}{2}\right)$-card with centroid z requires deleting at least four vertices.

Hence the centroid of C is x. The component of $T-x$ containing z has $(n+2)/2$ vertices. Since C is a $\left(\frac{n-4}{2}, \frac{n-2j}{2}\right)$-card, C must arise by deleting three vertices from that component of $T-x$. Thus X contains a piece Y of C with $(n-2j)/2$ vertices.

Now consider the required $\left(\frac{n-6}{2}, \frac{n-2j}{2}\right)$-card C' if $j \in \{3,4\}$. The centroid of C' cannot be x, since the two biggest components of $T-x$ would together have to lose at least four vertices. Hence the centroid of C' is z. Now, as earlier when we studied C, there must be pieces as large as $(n-6)/2$ and $(n-2j)/2$ in T, so again $c'(T) \geq (n-2j)/2$. Again we must delete at least four vertices from the two largest pieces of T to obtain a $\left(\frac{n-2j}{2}, \frac{n-10}{2}\right)$-card.

Hence $j = 2$, and C is a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card. We have shown that the centroid of C is x, with a piece Y contained in X. Since Y uses all of X except the one vertex x, we have $d_T(x) = 2$. Now deleting x and two vertices from X' yields a balanced card, contradicting (b).

Case 3: $c(T) = n/2$. Here T is bicentroidal, with centroids z and z'. Let X be a second largest component of $T - z$, with x its neighbor of z; similarly define X' and x' from $T - z'$. By (b), T has no balanced cards, which requires $d_T(z), d_T(z') \geq 3$ and $|V(X)|, |V(X')| \leq (n-6)/2$, so $c'(T) \leq (n-6)/2$. Since $c(T) = n/2$, every connected card has cost at least $(n-6)/2$, so again (c1) cannot hold. Also, there is no $\left(\frac{n-8}{2}, \frac{n-10}{2}\right)$-card and no $\left(\frac{n-8}{2}, \frac{n-8}{2}\right)$-card, which are required when j in (c2) is 4 or 3, respectively. Hence $j = 2$

and $c'(\mathcal{D}) = (n-4)/2$.

Thus we have a $(\frac{n-4}{2}, \frac{n-4}{2})$-card C. Since $c'(T) \le (n-6)/2$, moving away from z or z' to a vertex w allows only the component of $T - w$ containing z and z' to have more than $(n-6)/2$ vertices. Hence C cannot exist. □

Theorem 3.11. *For even n with $n \ge 20$, trees with n vertices and cost $(n-4)/2$ are 3-reconstructible.*

Proof. Let $\mathcal{D}$ be the $(n-3)$-deck of such a tree T. By Lemmas 3.6 and 3.10, we recognize from $\mathcal{D}$ that T is in this family. Lemma 3.6 also implies that every connected card has the centroid z of T as its unique centroid ($n-3$ is odd). Hence again the connected cards form the rc3-deck of T with root z, and again $d_T(z) \ge 3$. Hence by the same proof as in Theorem 3.7, we can reconstruct T from the deck. □

For trees with higher cost, there may be connected cards whose centroids are not the centroid of T. Nevertheless, in many cases we will find a subset of the connected cards whose centroid can be identified as a particular vertex of T. In that case we may be able to apply the same reconstruction argument.

Lemma 3.12. *Let T be an n-vertex tree, and let $\mathcal{D}$ be the $(n-3)$-deck of T. If a vertex v in T and a subset $\mathcal{D}'$ of $\mathcal{D}$ can be identified such that $\mathcal{D}'$ is the multiset of connected cards arising by deleting three vertices from one component H of $T - v$, and in each card of $\mathcal{D}'$ we know v and which neighbor of v is in H, then T is 3-reconstructible.*

Proof. For each card in $\mathcal{D}'$, deleting the remaining vertices of H yields $T - V(H)$. Over all cards in $\mathcal{D}'$, the vertices that belong to H provide the rc3-deck of H rooted at its vertex neighboring v.

Since we know the number of leaves of T and the number of leaves of T outside H, we know the number of leaves of T in H. Hence Theorem 2.9 allows us to reconstruct H unless $H \in \{\hat{P}'_k, \hat{P}''_k, \hat{Q}_k, \hat{Q}''_k\}$, where $k = |V(H)|$. By Observation 1.2, we know the number of copies of $S_{2,1,1}$ in T, and from the cards in $\mathcal{D}'$ we know the number of copies of $S_{2,1,1}$ using at least one vertex outside H. We therefore know the number of copies of $S_{2,1,1}$ contained in H. As noted in Theorem 2.9, we thus can reconstruct H and T. □

4 Trees with Cost $(n-3)/2$

For odd n, Lemma 3.6 allowed us to recognize when the cost is at most $(n-5)/2$, and in Theorem 3.7 we reconstructed such trees. The next two lemmas enable us to distinguish whether reconstructions from the deck have cost $(n-3)/2$ or $(n-1)/2$.

Lemma 4.1. *If $\mathcal{D}$ is the $(n-3)$-deck of an n-vertex tree T, where n is odd and $n \geq 9$, then $c(T) = c'(T) = (n-3)/2$ if and only if $\mathcal{D}$ has exactly one balanced card.*

Proof. Necessity. If $c(T) = c'(T) = (n-3)/2$ and T has centroid z, then $T - z$ has two components with $(n-3)/2$ vertices and two leftover vertices. Deleting those two vertices and z yields a balanced card. When $n \geq 9$, this is the only balanced card deleting z. A balanced card keeping z must delete a vertex v in one of the large components of $T - z$, but then $T - v$ cannot have two components with $(n-3)/2$ vertices.

Sufficiency. A balanced card C must lack a vertex from the path in T connecting the two components of C. Thus T has a vertex z such that $T - z$ has two components with at least $(n-3)/2$ vertices, and only two since $n \geq 9$. If either such component has more vertices, then T has more than one balanced card, since any leaf of such a component can be deleted. Hence z has a neighbor in each large component of $T - z$, and the other two vertices are adjacent to z or form P_2 adjacent to z. Now $c(T) = c'(T) = (n-3)/2$. □

Lemma 4.2. *If $\mathcal{D}$ is the $(n-3)$-deck of an n-vertex tree T with cost at least $(n-3)/2$, where n is odd and $n \geq 13$, then $c(T) = (n-3)/2$ and $c'(T) \leq (n-5)/2$ if and only if $\mathcal{D}$ has no balanced cards and at least one of the following happens:*

(a) $\mathcal{D}$ has a card with cost at most $(n-9)/2$, or

(b) $\mathcal{D}$ has a $\left(\frac{n-7}{2}, \frac{n-7}{2}\right)$-card and a $\left(\frac{n-7}{2}, \frac{n-9}{2}\right)$-card, or

(c) $\mathcal{D}$ has a $\left(\frac{n-7}{2}, \frac{n-7}{2}\right)$-card and among its bicentroidal cards has exactly one with a centroid of degree 2 or has one with both centroids of degree 2.

Proof. As usual, let z be the centroid of T, let X be a largest piece of T (rooted at x), and let X' be a next largest piece of T.

Case 1: $c(T) = (n-3)/2$. By Lemma 4.1, $c'(T) = (n-3)/2$ guarantees a balanced card, so we need only show that the conditions are necessary when $c'(T) \leq (n-5)/2$. Since the components of $T - z$ have at most $(n-3)/2$ vertices and the cards have $n-3$ vertices, z is a centroid in every connected card.

Since $n-3$ is even, a balanced card has two components with $(n-3)/2$ vertices. When the path connecting them has more than one internal vertex, deleting any single vertex of T leaves a component with at least $(n-1)/2$ vertices, contradicting $c(T) = (n-3)/2$. When the path has a single internal vertex, it is the centroid, contradicting $c'(T) \leq (n-5)/2$. Hence T has no balanced cards.

A connected card C obtained by deleting three vertices from X has centroid z and a piece with $(n-9)/2$ vertices. The card C has another piece with $c'(T)$ vertices. If $c'(T) \leq (n-9)/2$, then $c(C) = (n-9)/2$, and (a) holds. If $c'(T) = (n-7)/2$, then C is a $\left(\frac{n-7}{2}, \frac{n-9}{2}\right)$-card, and deleting two vertices from X and one from the burl yields a $\left(\frac{n-7}{2}, \frac{n-7}{2}\right)$-card, confirming (b).

If neither (a) nor (b) holds, then $c'(T) = (n-5)/2$, and z is a 3-burl. Deleting two vertices from X and one from X' yields a $\left(\frac{n-7}{2}, \frac{n-7}{2}\right)$-card. Since X' has only $(n-5)/2$

vertices, all bicentroidal cards have centroids x and z and arise by deleting three vertices outside X. When $n \geq 13$, we have $|V(X')| \geq 4$. Hence when $d_T(x) > 2$, the only way to create a bicentroidal card with a centroid of degree 2 is to delete the three vertices of the burl. If $d_T(x) = 2$, then there may be many bicentroidal cards with one centroid having degree 2, but we can also make one with both centroids having degree 2 by deleting the three vertices of the burl.

Case 2: $c(T) = (n-1)/2$. We show that (a), (b), and (c) all fail when T has no balanced cards. If $c'(T) = (n-1)/2$, then a balanced card arises by deleting z and leaves of X and X'. If $c'(T) = (n-3)/2$, then a balanced card arises by deleting z, the leaf neighbor of z, and a leaf of X. Hence $c'(T) \leq (n-5)/2$. If $d_T(x) = 2$, then deleting x and two vertices of X' yields a balanced card, so we must have $d_T(x) \geq 3$.

Deleting ℓ vertices reduces the cost by at most ℓ. Hence no card of T has cost at most $(n-9)/2$, and (a) fails.

Deleting three vertices outside X yields cards with x a centroid. Since $d_T(x) > 2$, in such cards x is the unique centroid and the cost is $(n-5)/2$. Deleting one vertex of X and two outside X yields a bicentroidal card with centroids x and z. All other cards have centroid z.

Both (b) and (c) require a $\left(\frac{n-7}{2}, \frac{n-7}{2}\right)$-card, which must have centroid z. Thus $T - z$ has two components with at least $(n-7)/2$ vertices, so $c'(T) \geq (n-7)/2$. With $c'(T) \geq (n-7)/2$, we must delete at least four vertices to obtain a $\left(\frac{n-7}{2}, \frac{n-9}{2}\right)$-card. Hence (b) fails.

Finally, suppose that (c) holds. When $c'(T) = (n-5)/2$, a $\left(\frac{n-7}{2}, \frac{n-7}{2}\right)$-card with centroid z requires deleting four vertices from $X \cup X'$, which is not allowed. Hence we may assume $c'(T) = (n-7)/2$. With T being a $\left(\frac{n-1}{2}, \frac{n-7}{2}\right)$-tree, z is a 3-burl in T. Using $d_T(x) \geq 3$, we have observed that all bicentroidal cards have centroids x and z. Since z is a 3-burl in T and $d_T(x) \geq 3$, no such card has 2-vertices as both centroids.

Since bicentroidal cards arise only by deleting one vertex of X and two outside X, no bicentroidal card has z as a centroid of degree 2. Hence there must be exactly one bicentroidal tree with x being a centroid of degree 2. Such cards arise only when $d_T(x) = 3$ and the deletions are the leaf neighbor of x and two vertices outside X. However, there is more than one way to delete two vertices outside X, so (c) cannot hold. $\square$

Theorem 4.3. *For $n \geq 13$ with n odd, n-vertex trees with cost $(n-3)/2$ are reconstructible.*

Proof. Lemma 4.1 allows us to recognize $\left(\frac{n-3}{2}, \frac{n-3}{2}\right)$-trees. Lemma 4.2 allows us to recognize when $c(T) = (n-3)/2$ and $c'(T) \leq (n-5)/2$. Consider at tree T in these cases.

Case 1: $c'(T) \leq (n-5)/2$. All bicentroidal cards have centroids x and z and are obtained by deleting three vertices outside X. Hence X appears as one branch in all such cards, and the other branch provides the rc3-deck of $T - V(X)$, rooted at z; let $Y' = T - V(X)$. If there is only one branch that appears in all bicentroidal cards, then it is X, and Lemma 3.12 applies to reconstruct Y' and T.

If all the bicentroidal cards are the same, then the rc3-cards of Y' are all the same. Hence those cards must be brooms or paths, by Lemma 2.6. Since $d_Y(z) \geq 2$, we conclude that Y is a star with centroid z, and actually $c'(T) = 1$. Now we can reconstruct T from any bicentroidal card by adding three leaves to the branch that is a star.

Case 2: $c'(T) = (n-3)/2$. Here T is a $\left(\frac{n-3}{2}, \frac{n-3}{2}\right)$-tree and z is a 2-burl. Since $(n-7)/2 \geq 3$ when $n \geq 13$, any $\left(\frac{n-5}{2}, \frac{n-7}{2}\right)$-card shows whether the two vertices of the burl are adjacent to each other or only to z. A connectee card of T is bicentroidal if and only if it arises by deleting three vertices outside X (having centroids z and x) or three vertices outside X' (having centroids z and x').

Let h be the number of vertices in $\{x, x'\}$ with degree 2 in T. We have $h = 2$ if and only if every bicentroidal card has a centroid of degree 2. In the bicentroidal cards having a centroid of degree 3, that centroid is then z. The branch not containing z is X or X' with its root located. Over all such cards, we obtain the rooted X and X' (identical or not).

We claim $h = 0$ if and only if no bicentroidal card has degree 2 at both centroids. Reducing z to degree 2 requires deleting bot vertices of the burl, but the third deleted vertex cannot be a neighbor of the other resulting centroid. In bicentroidal cards with one centroid of degree 2, that centroid now is z. These cards form the rc1-deck of T (minus the burl) as a tree rooted at z. By Theorem 2.1, we can reconstruct this subtree and hence T.

In the remaining case, some but not all bicentroidal cards have two centroids of degree 2, which implies $h = 1$. By symmetry, let $d_T(x) = 2$. In the bicentroidal cards with two centroids of degree 2, one branch is X, and after deleting z the other is an rc1-card of X' with root x'. If only one branch is common to all these cards, then it is X, and by Theorem 2.1 we can reconstruct X'. Otherwise, the bicentroidal cards are all the same. When both branches appear in all these cards and are distinct unrooted trees, then we can see which is X in the unique balanced card. If they are the same unrooted tree, with different roots, then the one that arises from X' must be the rooted broom, by Lemma 2.6. □

5 Trees with Cost $(n-2)/2$ and Subcost at most $(n-4)/2$

In light of Lemma 3.10 and Theorem 3.11, when n is even we henceforth restrict to $n \geq 20$ so that in reconstruction arguments we may assume that every reconstruction from $\mathcal{D}$ has cost at least $(n-2)/2$. By Lemma 3.10, we can recognize from the $(n-3)$-deck of T that $c(T) \in \{(n-2)/2, n/2\}$. To distinguish these two cases, we consider the subcost.

Definition 5.1. When T is an n-vertex tree with $c(T) = (n-a)/2$ and $c'(T) = (n-b)/2$, we say that T is an $\left(\frac{n-a}{2}, \frac{n-b}{2}\right)$*-tree.* An $\left(\frac{n-a}{2}, \frac{n-b}{2}\right)$*-vertex* in a tree T is a vertex v such that $T-v$ has largest component with $(n-a)/2$ vertices and next largest with $(n-b)/2$ (possibly $a = b$). In particular, an $\left(\frac{n-a}{2}, \frac{n-b}{2}\right)$-vertex in a $\left(\frac{n-a}{2}, \frac{n-b}{2}\right)$-tree is a centroid.

Lemma 5.2. *For n even, an n-vertex tree T is a $\left(\frac{n-2}{2}, \frac{n-b}{2}\right)$-tree with $b \geq 8$ if and only if $c(T) \geq (n-2)/2$ and the $(n-3)$-deck of T has a card with cost $(n-8)/2$.*

Proof. If $c(T) = (n-2)/2$ and $c'(T) \leq (n-8)/2$, then deleting three vertices from the largest piece of T yields the desired card. If $c(T) = n/2$, or if $c(T) = (n-2)/2$ and $c'(T) \geq (n-6)/2$, then $c(C) \geq (n-6)/2$ for each connected card C. □

Theorem 5.3. *For $n \geq 20$ with n even, n-vertex trees with cost $(n-2)/2$ and subcost at most $(n-8)/2$ are 3-reconstructible.*

Proof. Lemmas 3.6 and 3.10 enable us to recognize the case $c(T) \geq (n-2)/2$, and from Lemma 5.2 we recognize that $c(T) = (n-2)/2$ and $c'(T) \leq (n-8)/2$.

For $1 \leq j \leq 3$, a connected card C obtained by deleting j vertices of X and $3-j$ vertices outside X has centroid z and cost $(n-2-2j)/2$, since the components of $C-z$ contained in X and X' have $(n-2-2j)/2$ and at most $(n-8)/2$ vertices, respectively. Connected cards deleting three vertices outside X have centroid x and cost $(n-4)/2$, because the piece rooted at z has $(n-4)/2$ vertices. Thus the set $\mathcal{D}'$ of connected cards with cost $(n-6)/2$ consists precisely of those obtained by deleting two vertices from X and one outside X.

If T has a $\left(\frac{n-8}{2}, \frac{n-8}{2}\right)$-card C, then X' has $(n-8)/2$ vertices and C arises by deleting three vertices from X. In this case the full burl at z has four vertices, is present in C, and is distinguishable from what remains of X and X' since $n \geq 20$. Cards obtained by deleting three vertices from the burl are $\left(\frac{n-2}{2}, \frac{n-8}{2}\right)$-cards and show X and X' completely. Hence we obtain X and X' and the burl at z to reconstruct T.

If T has no $\left(\frac{n-8}{2}, \frac{n-8}{2}\right)$-card, then $c'(T) \leq (n-10)/2$. Now cards with cost $(n-8)/2$ are obtained by deleting three vertices of X, and what remains of X is the unique largest piece. By Lemma 3.12, T is 3-reconstructible. □

By Lemmas 3.6, 3.10, and 5.2, the $(n-3)$-deck of an n-vertex tree T allows us to recognize when $c(T) \geq (n-2)/2$ and $c'(T) \geq (n-6)/2$, for even n.

Lemma 5.4. *For $n \geq 14$ with n even, an n-vertex tree T satisfying $c(T) \geq (n-2)/2$ and $c'(T) \geq (n-6)/2$ is a $\left(\frac{n-2}{2}, \frac{n-6}{2}\right)$-tree if and only if*

(a) the $(n-3)$-deck $\mathcal{D}$ has a $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-card and a $\left(\frac{n-6}{2}, \frac{n-8}{2}\right)$-card, and

(b) T has a balanced card or the number of disconnected cards with components having $n/2$ and $(n-6)/2$ vertices is not 1.

Proof. To allow the second-largest piece to have $(n-8)/2$ vertices, the condition of having a $\left(\frac{n-6}{2}, \frac{n-8}{2}\right)$-card in the $(n-3)$-deck may require $(n-8)/2 \geq 3$, which simplifies to $n \geq 14$.

Case 1. *T is a $\left(\frac{n-2}{2}, \frac{n-6}{2}\right)$-tree.* Deleting two vertices of X and one from X' or from the burl yields the two cards required by (a). If $d_T(x) = 2$, then deleting x and two vertices of X' yields a balanced card. Hence if T has no balanced card, then $d_T(x) \geq 3$.

There are three vertices in the burl, so a card having components with $n/2$ and $(n-6)/2$ vertices cannot be obtained by deleting z. Also the component of $T - x'$ containing z has $(n+6)/2$ vertices, so $T - x'$ cannot contain such a card. Hence with z remaining and $d_T(x) \geq 3$, obtaining such a card requires $d_T(x) = 3$ and a leaf neighbor of x, and we must delete x, its leaf neighbor, and a leaf of the component of $T - x$ containing z. Hence when $d_T(x) = 3$ and x has a leaf neighbor there is more than one such card, while if $d_T(x) \neq 3$ or x has no leaf neighbor then there is no such card.

Case 2. *$c(T) = (n-2)/2$ and $c'(T) \geq (n-4)/2$.* First suppose $c'(T) = (n-4)/2$. The centroid z is a 2-burl in T. Let C be a $\left(\frac{n-6}{2}, \frac{n-8}{2}\right)$-card of T. The centroid u of C is a 3-burl in C; hence u must be a k-burl in T with $k \geq 3$. Thus $u \neq z$. Now the component of $T - u$ containing z has at least $(n+2)/2$ vertices and cannot reach $(n-6)/2$ by deleting three.

When $c'(T) = (n-2)/2$, the analogous argument uses the required $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-card C. Here z is a 1-burl, u is a 2-burl in C, and again the component of $T - u$ containing z has at least $(n+2)/2$ vertices.

Case 3. *$c(T) = n/2$.* Let z and z' be the centroids of T. Let Y and Y' be the components of $T - zz'$, with $z \in V(Y)$ and $z' \in V(Y')$. If T has a $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-card C, then the centroid of C must be z or z' (a centroid w anywhere else would leave too big a component of $T - w$ to cut down to $(n-6)/2$ vertices). By symmetry, we may let z be the centroid. Cutting the big piece of $T - z$ down to $(n-6)/2$ vertices requires C to be obtained by deleting three vertices of Y'. Now the second-largest component of $T - z$ must be contained in Y and have $(n-6)/2$ vertices, so z is a 2-burl.

In this setting, the centroid of a $\left(\frac{n-6}{2}, \frac{n-8}{2}\right)$-card cannot be in Y; the piece containing z' would be too big. Hence the centroid of such a card C' must be z', and deleting three vertices from Y yields a piece with $(n-6)/2$ vertices. No more vertices can be deleted, so $Y' - z'$ must have a component with $(n-8)/2$ vertices, and z' is a 3-burl.

By the argument above, every subtree with at least $(n-4)/2$ vertices contains z or z', so T has no balanced cards. In order to obtain a card with components having $n/2$ and $(n-6)/2$ vertices, it is necessary to delete one centroid and delete no vertices from the resulting component with $n/2$ vertices. Since also a component with $(n-6)/2$ vertices is needed, the only way to do this is to delete z and the two vertices of the burl at z. Hence there is only one such card, and (b) fails. □

Theorem 5.5. *For $n \geq 20$ with n even, $\left(\frac{n-2}{2}, \frac{n-6}{2}\right)$-trees are 3-reconstructible.*

Proof. By Lemma 5.4 and earlier cases, we recognize that T is a $\left(\frac{n-2}{2}, \frac{n-6}{2}\right)$-tree.

Label z, X, x, X', x' as usual. Since there are $(n+2)/2$ vertices outside X, cards obtained by deleting three vertices outside X have centroid x and cost $(n-4)/2$. Connected cards obtained by deleting one vertex of X and two outside X also have cost $(n-4)/2$, with centroid z, but they have subcost at most $(n-6)/2$. Connected cards obtained by deleting at least two vertices of X have cost at most $(n-6)/2$. Thus T has $\left(\frac{n-4}{2},\frac{n-4}{2}\right)$-cards if and only if $d_T(x)=2$, and x is the centroid of such cards.

Let $\mathcal{D}'$ be the set of $\left(\frac{n-4}{2},\frac{n-4}{2}\right)$-cards. If $\mathcal{D}'\neq\varnothing$, then $d_T(x)=2$. In cards in $\mathcal{D}'$, one piece is always $X-x$. The other piece yields the rc3-deck of $T-V(X)$, rooted at z. If only one piece appears in all cards of $\mathcal{D}'$, then it is X. By Lemma 3.12, we can reconstruct T.

If each $\left(\frac{n-4}{2},\frac{n-4}{2}\right)$-card has the same two pieces, then by Lemma 2.6 $T-V(X)$ is a rooted broom or is formed by merging the leaf of a rooted path with the root of a rooted tree with five vertices. When $(n-6)/2\ge 2$, this is impossible with z being a 3-burl in T.

Hence we may assume $d_T(x)\ge 3$ and $\mathcal{D}'=\varnothing$. Now consider cards with cost $(n-4)/2$. They arise with centroid x by deleting three vertices outside X or with centroid z by deleting one vertex of X and two outside X; in both cases the largest piece is unique. The root of the largest piece has degree 2 in the card if in the first case we deleted the three vertices of the burl or in the second case x is a 1-burl and we deleted it leaf neighbor.

Therefore, we recognize that x is not a 1-burl by there being exactly one card C with cost $(n-4)/2$ such that the neighbor of the centroid in the largest piece has degree 2. When this happens, the unique largest piece is rooted at z, and deleting z from it gives us X'. Deleting the largest piece from C gives us X, rooted at x. To find the configuration of the burl we examine any $\left(\frac{n-6}{2},\frac{n-8}{2}\right)$-card. Such cards arise only by deleting three vertices of $X\cup X'$; we distinguish the burl within it because $n\ge 16$.

In the remaining case, when there is more than one such card C, we know that x is a 1-burl in T. Consider the $\left(\frac{n-4}{2},\frac{n-10}{2}\right)$-cards in which the centroid is a 3-burl whose neighbor in the largest piece has degree 2. Such cards arise only by deleting the leaf neighbor of X and two vertices of X'. Since $n\ge 18$, we see the burl. Reconstruct X from the largest piece by giving the root a leaf neighbor, so we know the number of leaves of X'. The second largest pieces in these cards form the rc2-deck of X'; reconstruct X' using Theorem 2.3. $\square$

Lemma 5.6. *For $n\ge 16$ with n even, an n-vertex tree T satisfying $c(T)\ge(n-2)/2$ and $c'(T)\ge(n-4)/2$ is a $\left(\frac{n-2}{2},\frac{n-4}{2}\right)$-tree if and only if*

(a) the $(n-3)$-deck $\mathcal{D}$ has a $\left(\frac{n-6}{2},\frac{n-6}{2}\right)$-card and a $\left(\frac{n-4}{2},\frac{n-4}{2}\right)$-card, and

(b1) T has exactly one balanced card, or (b2) and at least one of $\{$(c1),(c2)$\}$ hold:

(b2) $\mathcal{D}$ has at least $2+q$ balanced cards, where all large components of balanced cards have at least q leaves, and their small components (except maybe one) are isomorphic.

(c1) No connected card has a vertex y whose deletion leaves largest components with $n/2$ and $(n-12)/2$ vertices and has a 2-neighbor in the component with $n/2$ vertices, or

(c2) T has a $\left(\frac{n-4}{2},\frac{n-8}{2}\right)$-card in which the piece of order $(n-8)/2$ has a 2-burl or 3-burl at distance at most 3 from the centroid of the card.

Proof. **Case 1:** *T is a $\left(\frac{n-2}{2},\frac{n-4}{2}\right)$-tree.* With $|V(X)|=(n-2)/2$ and $|V(X')|=(n-4)/2$, the centroid z is a 2-burl. Deleting two vertices from X and one from X' yields a $\left(\frac{n-6}{2},\frac{n-6}{2}\right)$-card; deleting the burl and one leaf from X yields a $\left(\frac{n-4}{2},\frac{n-4}{2}\right)$-card. Hence (a) holds.

Deleting z and the burl yields a balanced card, and this is the only balanced card omitting z. A non-leaf vertex whose deletion produces a component with at least $(n+4)/2$ vertices cannot be deleted in forming a balanced card. Hence a balanced card containing z must also contain x' and delete x. Since $T-x$ has components with $(n+2)/2$ and at most $(n-4)/2$ vertices, forming a balanced card also requires deleting two vertices from the large component of $T-x$ and requires $X-x$ to be connected, meaning $d_T(x)=2$. Hence if $d_T(x)\geq 3$, then T has exactly one balanced card, in which case (b1) holds.

Henceforth in this case we may assume that (b1) fails and $d_T(x)=2$. The components of order $(n-4)/2$ in balanced cards are X' when z and the burl are deleted, or $X-x$ when the deleted vertices are x and two vertices outside X. Hence these components are pairwise isomorphic except that when $X'\not\cong X-x$ the one that is X' is different from the others.

The number of balanced cards is 1 plus the number of subtrees of $T-V(X)$ obtained by deleting two vertices. Let q' be the number of leaves of $T-V(X)$ (note that $T-V(X)$ is the large component in one balanced card, so q' is at least the value q defined in the statement of (b2)). If $q\geq 4$, then we have more than q ways to delete two leaves from $T-V(X)$. If $q=2$, then $T-V(X)$ is a path and we have three ways to delete two vertices. If $q=3$, then $T-V(X)=S_{1,1,(n-4)/2}$, and we have three ways to delete two leaves plus one way to delete two vertices from the long leg, since $(n-4)/2\geq 2$. Hence when (b1) fails, the number of balanced cards is at least $2+q$, and (b2) holds.

If T has no connected card having a vertex y whose deletion leaves largest components with $n/2$ and $(n-12)/2$ vertices, then (c1) holds. Hence we may assume that y is such a vertex in such a card C. Note that y is a 2-burl in C. Also, y cannot be z, since $T-z$ has no component with at least $n/2$ vertices.

The largest component of $T-y$ contains z, the two vertices in the burl of T, the large components of $T-z$ not containing y, and the s internal vertices of the z,y-path in T. This amounts to $(n+4)/2+s$ vertices if $y\in X'$, and $(n+2)/2+s$ if $y\in X$. To reduce to $n/2$ in forming C, exactly $s+2$ of these vertices must be deleted if $y\in X'$, or $s+1$ if $y\in X$. Therefore $s\leq 2$, and y has distance at most 3 from z.

The $s+1$ or $s+2$ vertices that were deleted from T to form the largest component of $C-y$ were not in the burl of y, which in C has only two vertices. Hence the burl of y can have more than three vertices only if $s=0$ and $y\in X$. That is, y is the neighbor of z in X. In this case we can forom the desired $\left(\frac{n-4}{2},\frac{n-8}{2}\right)$-card by deleting three vertices of X, including one or two from the burl at y. In all other cases the burl in T at y has two or three vertices, and we obtain the desired $\left(\frac{n-4}{2},\frac{n-8}{2}\right)$-card by deleting the appropriate number of vertices from X and X' without disturbing the burl at y.

Case 2: *T is a $\left(\frac{n-2}{2},\frac{n-2}{2}\right)$-tree.* Since the centroid z of T is a 1-burl while the centroid u of any $\left(\frac{n-6}{2},\frac{n-6}{2}\right)$-card C is a 2-burl in C, vertex u cannot be z. For any $v\in V(T)$ with $v\neq z$, the largest component of $T-v$ has at least $(n+2)/2$ vertices. Deleting three vertices cannot reduce that component below $(n-4)/2$ vertices. Hence T cannot have a $nt66$-card, and (a) fails.

Case 3: *T is bicentroidal.* Let z and z' be the centroids of T, with Y the branch containing z and Y' the branch containing z'.

If $v \in V(T) - \{z, z'\}$, then $T - v$ has a component with at least $(n+2)/2$ vertices, which cannot be reduced to $(n-6)/2$ by deleting three. Hence the centroid of a $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-card C must be z or z'; by symmetry, we may assume it is z. Since the component Y' of $T - z$ containing z' has $n/2$ vertices, C must arise by deleting three vertices of Y'. Hence $T - z$ has exactly $(n-6)/2$ vertices in its second-largest component X, so z is a 2-burl in T. Now a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card C' must have centroid z' and arise by deleting one vertex from Y' and two from Y. To form the piece of C' with $(n-4)/2$ vertices that is contained in Y', the second-largest component X' of $T - z'$ must have $(n-4)/2$ or $(n-2)/2$ vertices, in which case z' is a 1-burl or a 2-vertex in T, respectively.

Because z is a 2-burl in T and $|V(X)| = (n-6)/2$, there is no balanced card omitting z. One way to create a balanced card is to delete z', a leaf of Y, and either the leaf neighbor of z' (if X' has $(n-4)/2$ vertices) or a leaf of X' (if X' has $(n-2)/2$ vertices). Since Y is a nontrivial tree, there are at least two such cards, and hence (b1) cannot hold.

If $|V(X')| = (n-4)/2$, then deleting z' and its leaf neighbor and a leaf of Y is the only way to form a balanced card. The number of balanced cards is then the number of leaves of Y. The component of order $(n-2)/2$ in any balanced card is obtained from Y by deleting one leaf. If the minimum number of leaves in such a component is q, then Y has at most $q+1$ leaves. Hence T has at most $q+1$ balanced cards, so (b2) cannot hold in this case.

In the remaining case, $|V(X')| = (n-2)/2$ and $d_T(z') = 2$. Still z is a 2-burl. There are now two ways to form balanced cards. Type 1 is to delete z' and two vertices from Y; the component of order $(n-4)/2$ is then contained in Y. Type 2 is to delete z' and one leaf each from the trees Y and X'; the component of order $(n-4)/2$ is then contained in X'.

The components of order $(n-4)/2$ in the balanced cards are the subtrees of Y obtained by deleting two vertices and the subtrees of X' obtained by deleting one leaf. These must be isomorphic, except possibly for one. If all subtrees obtained by deleting two vertices from Y are isomorphic, then Y is a path or a star, by Lemma 3.2. The star is forbidden, since z is a 2-burl in T and Y has $n/2$ vertices, with $n \ge 10$. However, Y may be a path. Now every subtree obtained from X' by deleting a leaf must also be a path, except possibly for one.

If X' is a path (with $(n-2)/2$ vertices), then attaching z' to it makes T constructed from $P_{n/2} + P_{n/2-1}$ by adding z' with one neighbor in each component, where the neighbor z in the component with $n/2$ vertices has distance 2 from a leaf; we return to this case later.

If X' is not a path, then deleting any leaf except one yields a path, so $X' = S_{1,1,n/2-4}$. Now there are two balanced cards in which the component having $(n-4)/2$ vertices is not a path, obtained by deleting the end of the long leg of X' and one of the two leaves of Y.

The other case is that all subtrees of X' obtained by deleting a leaf are isomorphic; let Z be that subtree. Now all leaves of T in X' have the same distance from z', and all vertices at a given distance from z' have the same degree. Furthermore, since Y has at least three subtrees obtained by deleting two vertices, at least two such subtrees of Y are also isomorphic to Z. In at least one of those copies of Z in Y, we delete one vertex from the burl of T, leaving z with degree 2 and a leaf neighbor in Z. In another copy, we delete two vertices of X, leaving z with degree 3 or a leaf at distance 2. (If there is only one subtree deleting two vertices of X, then Y is a broom with three leaves or a path; the broom fails.) The resulting trees cannot be isomorphic and have the properties of Z unless Z is a path. Now Y and Y' are both paths, with z having distance 2 from one end. This yields $T = S_{2,(n-6)/2,n/2}$, a special case of the construction above where z' is made adjacent to an endpoint of $P_{n/2-1}$.

In these two cases we are left with the spider $S_{2,(n-6)/2,n/2}$ or a tree with two (nonadjacent) vertices of degree 3 and the rest of degree at most 2, in which the legs at one branch vertex (which we have called z) have length 2 and $(n-6)/2$. Now consider conditions (c1) and (c2). Deleting z from the connected card obtained by deleting three vertices from the leg of length $(n-6)/2$ at z leaves components with $(n-12)/2$ and $n/2$ vertices, and the neighbor z' of z in the component with $n/2$ vertices has degree 2. Hence (c1) fails.

Hence T must satisfy (c2). The centroid of a $\left(\frac{n-4}{2}, \frac{n-8}{2}\right)$-card must be a 2-burl in the card. Hence the only choices for the centroid of such a care are z and the neighbor x' of z' in Y', which has distance 2 from z. To have x' as the centroid, note that the component of $T - x'$ containing z has $(n+2)/2$ vertices. We must delete three vertices there to get down to a piece with $(n-4)/2$ vertices. However, the piece with $(n-8)/2$ vertices is then a path without the required 2-burl or 3-burl.

To have z as the centroid, we must delete one vertex from the leg of length $(n-6)/2$ at z, since we cannot find a piece with $(n-4)/2$ vertices there. However, again the piece with $(n-8)/2$ vertices is then a path and cannot contain the required 2-burl or 3-burl. □

Theorem 5.7. *For $n \geq 20$ with n even, $\left(\frac{n-2}{2}, \frac{n-4}{2}\right)$-trees are reconstructible.*

Proof. Among the remaining trees, Lemma 5.6 allows us to recognize that T is a $\left(\frac{n-2}{2}, \frac{n-4}{2}\right)$-tree from its $(n-3)$-deck $\mathcal{D}$ (since $n \geq 20$). Let z, X, x, X', x' be as usual.

In a $\left(\frac{n-2}{2}, \frac{n-4}{2}\right)$-tree, the centroid z is a 2-burl. A $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-card has centroid z, is obtained by deleting two vertices from X and one from X', and determines whether the vertices of the burl are adjacent. Hence it suffices to reconstruct X and X' with their roots x and x' located.

As discussed in Lemma 5.6, here $d_T(x) \geq 3$ if and only if T has exactly one balanced card, obtained by deleting z and the burl. Consider this case. For $v \in V(T) - \{x, z\}$, always $T - v$ has a component with at least $(n+4)/2$ vertices. Hence the centroid of a connected card can only be z or x. Furthermore, since $T - x$ has a component with $(n+2)/2$ vertices, the centroid is x only when three vertices outside X are deleted, which means that when the centroid is x the centroid has degree at least 3 in the card.

Cards in which the centroid has degree 2 therefore have centroid z; they are obtained by deleting the burl and one leaf of X. All such cards are $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards in which one piece is X' and the other piece is an rc1-card of X. If the two pieces in these cards are not always the same, then we know X' and reconstruct X by Theorem 2.1. If the two pieces are always the same, then X has the property that all vertices at a given distance from the root x have the same number of children, so we can locate x within X as the larger component of the balanced card. We then also have X'.

Hence we may assume $d_T(x) = 2$. In every $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card, the centroid is a 1-burl, so the centroid must be z and not x. Such cards are obtained by deleting one vertex from the burl and either one leaf each from X and X' or two vertices from X. In the $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-cards where the centroid has a neighbor with degree at least 3, that neighbor is x'. When x' is in the piece with $(n-4)/2$ vertices, that piece is X'. For the cards with x' in the piece with $(n-6)/2$ vertices, the pieces with $(n-4)/2$ vertices form the rc1-deck of X rooted at x, and we reconstruct X by Theorem 2.1. Thus we have both X' and X to reconstruct T.

Finally, suppose that in every $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card both neighbors of the centroid have degree 2. When $(n-4)/2 \geq 3$, this requires $d_T(x') = 2$. Let y be the neighbor of x in X. In the $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards where the centroid has a neighbor with degree at least 3, the centroid is x; these cards arise by deleting three vertices outside X. If in some such card the neighbors of the centroid both have degree at least 3, then $d_T(y) \geq 3$. Now the $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards in which the centroid has exactly one neighbor of degree 2 arise by deleting the two vertices of the burl and one leaf of X'. The piece whose root has degree at least 3 in these cards is $X - x$, and the other piece gives the rc1-deck of X' with z prepended at the root. By Theorem 2.1, we reconstruct X'.

Otherwise, $d_T(y) = 2$, and the connected cards in which the centroid has one neighbor of degree 2 and one of degree 3 are obtained by deleting one vertex of the burl and two vertices from X'. The piece whose root has degree 2 is always $X - x$, rooted at y. The other piece is obtained from an rc2-card of X' by prepending z and a vertex of the burl at the root. This gives us the rc2-deck of X' (two copies of it when the vertices of the burl are not adjacent, which we can determine from the $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-cards). Since we know the total number of leaves in T and the number of leaves of T in X and in the burl, by Theorem 2.3 we can reconstruct X' with its root. □

6 High-Cost Trees with Special Structure

In this section we prove 3-reconstructibility of two special classes of trees, each of which is a family whose n-vertex members have cost at least $(n-2)/2$. With these results in hand, further recognition arguments can assume that the deck does not come from a tree in these families.

Definition 6.1. A *j-vertex* or *j^+-vertex* is a vertex with degree j or at least j, respectively. Similarly, a *j-neighbor* or *j^+-neighbor* is a neighbor that is a j-vertex or j^+-vertex, respectively. A *full vertex* is a 3^+-vertex that is not a 1-burl. A *caterpillar* is a tree having a single path incident with all edges; equivalently, it is a tree not containing $S_{2,2,2}$ as a subgraph.

Theorem 6.2. *For $n \geq 10$, every n-vertex caterpillar having maximum degree at most 3 is 3-reconstructible.*

Proof. Let $\mathcal{D}$ be the $(n-3)$-deck of such an n-vertex tree. When $n \geq 2\ell + 1$, acyclic graphs are ℓ-recognizable ([7]), and the number of edges is known from the 2-deck, so all reconstructions are trees. Also we know the degree list, since $n - \ell \geq 5$ and these trees have no 4^+-vertex. Finally, since $n - \ell \geq 7$ we see that a tree with the given deck has no copy of $S_{2,2,2}$, and hence it is a caterpillar. Thus this family of n-vertex graphs is ℓ-recognizable.

Let T be a reconstruction of $\mathcal{D}$. We know the number s of 3-vertices in T and hence the number of vertices in a longest path in T; it is just $n - s$. Let the *end-distance* of a 3-vertex be its minimum distance from an endpoint of a longest path.

Case 1: $s \leq 1$. If $s = 0$, then T is a path, which we know from the maximum degree. If $s = 1$, then the end-distance of the 3-vertex is the least j such that T has only one copy of $S_{1,1,j+1}$. By Observation 1.2, we can count the copies of $S_{1,1,j+1}$ in T if $j + 4 \leq n - \ell$, and $j \leq (n-2)/2$ suffices. We locate the 3-vertex on the $(n-1)$-vertex path to reconstruct T.

Case 2: $s = 2$. We seek the distance d between the two 3-vertices and the end-distance of one of them. Let H_i be the $(i+5)$-vertex subtree consisting of a path with $i+1$ vertices plus two leaves appended at each end. The value d is the only i such that $H_i \subseteq T$. By Observation 1.2, we find H_d in T if $d + 5 \leq n - 3$.

In this case, let r be smallest among the two end-distances. We know the number t_j of copies of $S_{1,1,j}$ in T if $j + 3 \leq n - \ell$. If $j \leq r$, then $t_j = 4$, except that $t_j = 6$ if $j = d + 1 \leq r$. If $j = r + 1$, then $t_j = 3$ unless $d = n - 3 - 2r$ (here both end-distances equal r, and $t_j = 2$) or $d = 4$ (here $t_j = 5$). Thus, if we find some j with $t_j = 5$, then $r = j - 1$. Otherwise, r is the least $j - 1$ such that $t_j = 3$, or if t_j is never odd for $j \leq n - 6$, then $r = (n - d - 3)/2$.

If $d \geq n - 7 \geq 3$ and we do not find H_d, then the two end-distances sum to at most 4 and are measured from opposite ends of the path. They are the same and equal r if $t_{r+1} = t_r - 2 = 2$, and otherwise they are the two values of r such that $t_{r+1} = t_r - 1$.

Case 3: $s = 3$. The approach is similar to Case 2. The distances d and d' from the middle 3-vertex to the other two are the least i and j such that we find H_i and H_j as subgraphs (possibly $i = j$), unless one of those distances is at least $n - 7$. In that case, since $s = 3$, the tree T arises from a path with $n - 3$ vertices by adding leaf-neighbors to vertices with end-distances 1, 1, and 2.

Otherwise, we know d and d'. We need to know the end-distance for a 3-vertex having a given distance (d or d') from the middle 3-vertex. Let $H'_{i,j}$ consist of a path P with $i + 2 + j$

vertices plus a leaf neighbor of the vertices at distances 1 and j from the opposite endpoints of P. If $i+j \le n-7$, then we can count the copies of $H'_{i,j}$ in T. Consider $H'_{d,j}$ and $H'_{d',j}$ for various j. Since the middle 3-vertex has distance at most $(n-4)/2$ from some endpoint of a longest path in T, for d or d' we find at least two (at least four if $d = d'$) copies of $H'_{d,j}$ for all j from 1 up to some value r, with only one copy of $H'_{d,r+1}$ existing (two or three copies if $d = d'$ and r is or is not $(n-4-2d)/2$). This value r is the end-distance for the 3-vertex at distance d from the middle 3-vertex, and this determines T.

Case 4: $s = 4$. All connected cards having a path P with $n-4$ vertices arise by deleting leaf neighbors of three 3-vertices, leaving one whose distance from an endpoint of P is its end-distance. Let r be the least such end-distance.

In T there are six subgraphs that are copies of H_i for various i, corresponding to the six pairs of 3-vertices. Since we can delete the two 3-vertices not in the pair, we see all these distances, except the largest in the case $r = r' = 1$, where the distance is $n-5$. Hence we know all six pairwise distances.

From these values, we determine the order of the distances between 3-vertices along the path. Let the positions be a, b, c, d in order, up to translation. The largest among the distances gives us $d-a$. The next largest is $c-a$ or $d-b$; up to reversal, we may let it be $c-a$. Now we know $d-c$; it is $(d-a)-(c-a)$. This leaves $\{b-a, c-b, d-b\}$. Among these three values, $b-a$ and $c-b$ are the two with sum $c-a$, and then which of the two is $c-b$ is the one whos sum with $d-c$ is $d-b$.

We now know a, b, c, d in order, except for translation and reversal. The value r is the end-distance for the 3-vertex as position a or d. To distinguish between these, we use the counts for the subgraphs $H'_{i,j}$ of Case 3. We associate position a with the least end-distance if T has fewer copies of $H'_{b-a,r+1}$ than $H'_{b-a,r}$, or position d if T has fewer copies of $H'_{d-c,r+1}$ than $H'_{d-c,r}$. However, more care is needed in the special case $c-b = r$. In that case, we can examine subgraphs with three 3-vertices to distinguish the two possibilities, and they are isomorphic if $d-c = b-a$.

Case 5: $s \ge 5$. Let $\mathcal{D}'$ be the set of connected cards having a path with $n-s$ vertices; such cards arise by deleting leaf neighbors of three distinct 3-vertices. Let v and w be the 3-vertices closest to the two ends of a longest path in T. The cards in $\mathcal{D}'$ having 3-vertices farthest apart have v and w as 3-vertices, telling us their end-distances. If they have distinct end-distances, then the various cards in $\mathcal{D}'$ having a 3-vertex closest to an endpoint of the path P with $n-s$ vertices give us the positions of all the other 3-vertices, reconstructing T.

If v and w have the same end-distance r, then consider the cards in $\mathcal{D}'$ where all 3-vertices have end-distance more than r. These arise by deleting the leaf neighbors of v, w, and one other 3-vertex; still at least two 3-vertices remain. Among these, the cards having 3-vertices farthest apart fix the end-distances of the second 3-vertex from each end.

Let q be the minimum end-distance among these two 3-vertices. If they both have end-distance q, then look at one card in $\mathcal{D}'$ having one 3-vertex with end-distance q and no 3-vertex with end-distance r. This shows us the remaining 3-vertices, and we know exactly where to add the three missing leaves. If only one of these two has end-distance q, then consider the cards in $\mathcal{D}'$ where the least end-distance of 3-vertices is q. Such cards are missing the leaf neighbors of v, w, and one other 3-vertex not having end-distance q. Since $s \geq 5$, over all such cards we obtain the positions of the other 3-vertices, since the 3-vertex with end-distance q distinguishes the two ends of P. $\square$

Note that an n-vertex tree having no full vertex as a centroid has cost at least $(n-2)/2$.

Theorem 6.3. *Let $\mathcal{D}$ be the $(n-3)$-deck of an n-vertex tree T. If T is unicentroidal with no unicentroidal card having a full vertex as centroid, or if T is bicentroidal with both centroids having degree 2, then T is 3-reconstructible, without knowing in advance the cost or subcost.*

Proof. Let T be a tree in this family, with $(n-3$-deck $\mathcal{D}$. If T has no full vertex at all, then T is a caterpillar with maximum degree at most 3, which by Theorem 6.2 is reconstructible from $\mathcal{D}$. Hence we may assume that T has a full vertex.

Any centroid z of T is a centroid with degree $d_T(z)$ in some connected card. Hence z must be a 1-burl or a 2-vertex. Thus T is a $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-tree, a $\left(\frac{n-1}{2}, \frac{n-3}{2}\right)$-tree, a $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$-tree, or a bicentroidal tree.

We consider first which vertices can be centroids of cards of T. When T is unicentroidal, label z, x, X, x', X' as usual. In a $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-tree, deleting one leaf from each nontrivial piece and then any leaf from the tree that remains yields a $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card or $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card with z as centroid. Deleting three vertices outside one of the nontrivial pieces yields a card with the root of that piece (x or x') as the centroid.

In a $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$-tree, deleting two vertices from one nontrivial piece and one from the other yields a bicentroidal card in which z is one of the centroids. Deleting three vertices outside X yields a unicentroidal card with x as centroid or, if x is a 2-vertex, a bicentroidal tree with centroids x and its neighbor in X (similarly for three vertices outside X'). The same discussion holds for a $\left(\frac{n-1}{2}, \frac{n-3}{2}\right)$-tree, except that deleting one vertex from X' and two from X yields a $\left(\frac{n-5}{2}, \frac{n-5}{2}\right)$-tree with centroid z, while deleting three vertices outside X' yields a bicentroidal tree with centroids z and x'.

Consider a bicentroidal tree with centroids z and z' as the roots of branches Y and Y', and $\langle x, z, z', x' \rangle$ being a path of nonleaf vertices. By symmetry, we describe only the cards where a majority of the deleted vertices are outside Y, since we are assuming $d_T(z) = d_T(z') = 2$. Deleting three vertices outside Y yields a card with centroid x, while deleting one vertex of Y and two outside Y yields a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card with centroid z.

Let v be a full vertex in T. In a card having v as a full vertex, we compute the distance from v to the centroid(s), taking the average of the two distances when the card is bicentroidal. In all types of trees above, this distance is minimized when all three deleted vertices are outside the piece or branch containing v. Otherwise, this distance is larger.

Let an *optimal card* be a connected card minimizing r, where in a unicentroidal tree r is the distance from the centroid to a closest full vertex, and in a bicentroidal card r is the minimum average distance from the two centroids to a single full vertex.

If no piece (or branch) occurs in all optimal cards, then each nontrivial piece of T has a full vertex at the same distance from the centroid in T. If these are v and v', then some optimal cards have v as the closest full vertex (distance r) from their centroid, while other cards have v' as this vertex (no card has both). Each optimal card contains one of the two resulting pieces, so over all optimal cards we obtain both. In each optimal card we see the centroid(s) of T and whether they have leaf neighbors, and in all cases we can determine T.

If all the optimal cards have the same piece containing a full vertex closest to the centroid, then either that piece arises from the same vertex in T or it arises from two different full vertices in T. In the latter case, the non-constant pieces of the optimal cards provide two copies of the rc3-deck of the constant piece. In the former case, they give the rc3-deck of the piece of T not containing the constant piece, and we reconstruct by Lemma 3.12.

In particular, when the neighbor x of a centroid z is the centroid in the optimal cards, the rc3-deck of the non-constant piece tells us whether z is a 1-burl or a 2-vertex, which distinguishes between $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-trees and bicentroidal trees in this family. The same notion distinguishes between $\left(\frac{n-1}{2}, \frac{n-3}{2}\right)$-trees and $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$-trees in this family. □

In this result we do not include the case of bicentroidal trees having a centroid z that is a 1-burl, because z becomes the unique centroid in a connected card both when three vertices are deleted from the other branch (leaving a constant piece with $(n-4)/2$ vertices) and when two vertices are deleted from the other branch and one from the branch containing z (leaving a $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-tree with neither piece constant).

Remark 6.4. Forbidding unicentroidal cards with centroids that are full vertices was used to ensure that in the unicentroidal case the centroid of T is a 2-vertex or 1-burl. It also forbids the neighbors of z from being full vertices. If some other argument determines that z is not a full vertex, then the remainder of the argument in Theorem 6.3 reconstructs T in the unicentroidal case even if one or both of the neighbors of z is a full vertex.

7 $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-Trees

At this point we know that the remaining trees have cost at least $(n-2)/2$, and those with cost $(n-2)/2$ are $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-trees. Among these, we have reconstructed all except those where a neighbor of the centroid z is a full vertex. Among the bicentroidal trees, we have reconstructed those where both centroids are 2-vertices, leaving those having a centroid that is a 3^+-vertex. Our lemma here will distinguish between these two classes.

Recall that every reconstruction from the $(n-3)$-deck of a tree has the same number of leaves, as remarked before Lemma 3.1. For unicentroidal trees, label z, x, X, x', X' as usual.

Lemma 7.1. *Let $\mathcal{D}$ be the $(n-3)$-deck of some tree that is a $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-tree having a full vertex adjacent to the centroid or is bicentroidal having a centroid that is a 3^+-vertex. Let t be the number of leaves in every tree whose $(n-3)$-deck is $\mathcal{D}$.*

(A) All reconstructions from $\mathcal{D}$ are $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-trees whose centroid has no 2-neighbor if and only if $\mathcal{D}$ has exactly $t-1$ balanced cards.

(B) All reconstructions from $\mathcal{D}$ are $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-trees whose centroid has exactly one 2-neighbor if and only if $\mathcal{D}$ has at least $t+2$ balanced cards and $\mathcal{D}$ has a connected card with cost $(n-4)/2$ whose centroid is a full vertex whose neighbor in the largest piece is a 1-burl.

Proof. Recall that when n is even, balanced cards have components with $(n-2)/2$ and $(n-4)/2$ vertices. Also, the centroid of a $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-tree is a 1-burl.

Case 1: *T is a $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-tree such that the centroid z has no 2-neighbors.* Since z has no 2-neighbors, deleting one of its non-leaf neighbors yields at least three components. Even if one component has only one vertex, the large component has $(n+2)/2$ vertices and cannot be trimmed small enough to make a balanced card. Hence every balanced card must delete z and its leaf neighbor and keep the other two neighbors of z. The remaining vertex deleted must be another leaf of T, and there are $t-1$ choices for it.

Case 2: *T is a $\left(\frac{n-2}{2}, \frac{n-2}{2}\right)$-tree whose centroid z has exactly one 2-neighbor.* By symmetry, we may assume that x is a 2-vertex and x' is a full vertex. Besides the $t-1$ balanced cards described in Case 1, T has balanced cards obtained by deleting x and two vertices from the component of $T-x$ containing z. Since this component has another leaf outside the leaf neighbor of z, we obtain at least three additional balanced cards, as needed for (B).

Consider a card obtained by deleting three vertices from X. The centroid is the full vertex x'. The largest piece has $(n-4)/2$ vertices, rooted at z, which is a 1-burl.

Case 3: *T is bicentroidal, with centroids z and z'.* If z (or z') is a full vertex, then it must lie in every balanced card, since deleting it leaves at least two components with at most $(n-6)/2$ vertices. Thus when both are full vertices there will be no balanced cards. When z or z' is a 1-burl, we obtain balanced cards by deleting it, its leaf neighbor, and one leaf from the larger remaining component. When both centroids are 1-burls, this generates exactly t balanced cards, and there are no others. When one centroid is a 1-burl and the other is a full vertex, the number of balanced cards is the number of leaves in the branch not containing the 1-burl centroid, which is at most $t-2$. Hence a bicentroidal card with no centroid of degree 2 cannot satisfy the conditions on balanced trees specified in (A) or (B).

Hence by symmetry we may assume $d_T(z') = 2$ and $d_T(z) \geq 3$. We obtain balanced cards by deleting z' and deleting one leaf from each resulting component, or two vertices from the larger component. In all cases, this yields at least $t+2$ balanced cards. Hence the condition in (A) cannot hold, but we have enough balanced cards to consider (B).

Let C be a card with cost $(n-4)/2$ whose centroid is a full vertex adjacent to a 1-burl in the largest piece. With $d_T(z')=2$ and $d_T(z)\ge 3$, every card has centroid in $\{x',z',z\}$. If C has centroid x' and x' is a full vertex, then the piece with $(n-4)/2$ vertices is rooted at z, which is not a 1-burl. Similarly, if z is the centroid and is a full vertex, then cost $(n-4)/2$ can be obtained only by deleting two vertices from the branch containing z', but again z' is its root and is not a 1-burl. Hence (B) cannot hold. □

Corollary 7.2. *For $n\ge 10$ with n even, $\left(\frac{n-2}{2},\frac{n-2}{2}\right)$-trees are 3-reconstructible.*

Proof. In earlier sections, we reconstructed all n-vertex trees with cost less than $(n-2)/2$ and $\left(\frac{n-2}{2},\frac{n-b}{2}\right)$-trees with $b\ge 4$. In Theorems 6.2 and 6.3, we reconstructed $\left(\frac{n-2}{2},\frac{n-2}{2}\right)$-trees having no full vertex adjacent to the centroid. By Lemma 7.1, we recognize that every reconstruction from the $(n-3)$-deck of a $\left(\frac{n-2}{2},\frac{n-2}{2}\right)$-tree having a full vertex adjacent to the centroid is in that family. By Remark 6.4, the argument of Theorem 6.4 now applies to reconstruct such a tree T. □

8 Trees with Cost $(n-1)/2$

When n is odd, the remaining trees all have cost $(n-1)/2$, but we need to recognize the subcost to facilitate reconstruction. Those with small subcost are easy to reconstruct.

Theorem 8.1. *For $n\ge 15$ with n odd, n-vertex trees with cost $(n-1)/2$ and subcost at most $(n-7)/2$ are reconstructible.*

Proof. By Lemmas 3.6, 4.1, and 4.2, we recognize this family. Define z,X,x,X' as usual in such a tree T.

Case 1: $c'(T)\le(n-9)/2$. We recognize this case by the existence of a $\left(\frac{n-7}{2},\frac{n-b}{2}\right)$-card with $b\ge 9$. Such cards have centroid z and arise by deleting three vertices from X. The unique largest piece is an rc3-card of X. Deleting it always leaves $T-V(X)$, rooted at z, and we know which piece is the rc3-card of X. By Lemma 3.12, we reconstruct X and T.

Case 2: $c'(T)=(n-7)/2$. This case occurs when there is a $\left(\frac{n-7}{2},\frac{n-7}{2}\right)$-card but no $\left(\frac{n-7}{2},\frac{n-b}{2}\right)$-card with $b\ge 9$. Again these cards arise by deleting three vertices from X. The centroid is z, which is a 3-burl both in T and in these cards. Thus when $(n-7)/2\ge 4$ there are exactly two pieces with $(n-7)/2$ vertices. In these cards, one large piece is the same and is X'. If another does not always appear, then we have determined X' and have the rc3-deck of X to reconstruct T as in Case 1.

If all the $(\frac{n-7}{2}, \frac{n-7}{2})$-cards are the same, then the rc3-cards of X are pairwise isomorphic. By Lemma 2.6, they are a rooted broom or a path. If X' is not a rooted broom or a path, then we know which piece is which and reconstruct as in Case 1.

If X' and the common rc3-card of X are identical, then it doesn't matter to which we apply Lemma 3.12. If they are not identical, then at least one is a broom. Since we can reconstruct the degree list of T, we know when the broom with more than one leaf is the rc3-card of X and can reconstruct X by adding three sibling leaves to that broom. Otherwise we reconstruct as in Case 1. $\square$

Recognition of $(\frac{n-1}{2}, \frac{n-5}{2})$-trees uses cards with restricted structure.

Definition 8.2. A *thin* card is a connected card having a largest piece rooted at a 2-vertex whose neighbor in the piece is a 2-burl.

Lemma 8.3. *For $n \geq 15$, an n-vertex tree T with $c(T) = (n-1)/2$ and $c'(T) \geq (n-5)/2$ is an $(\frac{n-1}{2}, \frac{n-5}{2})$-tree if and only if T has no balanced cards or all of the following hold:*

(a) T has a $(\frac{n-5}{2}, \frac{n-5}{2})$-card and has a $(\frac{n-5}{2}, \frac{n-7}{2})$-card in which the root of the piece with $(n-7)/2$ vertices has degree 2, and

(b) Every bicentroidal card has a centroid with degree 2, and

(c) T either has no thin $(\frac{n-5}{2}, \frac{n-5}{2})$-card or does have a thin $(\frac{n-5}{2}, \frac{n-7}{2})$-card.

Proof. Define z, X, x, X', x' as usual; X and X' have the same size when $c'(T) = (n-1)/2$.

Case 1: $c'(T) = (n-5)/2$. If T has no balanced cards, then there is nothing to prove, so assume that T has a balanced card, with two components of order $(n-3)/2$. Since X' has only $(n-5)/2$ vertices, a balanced card must contain z and lack both vertices of the burl. Hence it also must lack x, and then x must be a 2-vertex so $X - x$ is connected.

We now obtain a $(\frac{n-5}{2}, \frac{n-5}{2})$-card with centroid z by deleting two vertices from X and one leaf from the burl. We obtain a $(\frac{n-5}{2}, \frac{n-7}{2})$-tree with centroid z by deleting three vertices from X, and the neighbor of z in the piece with $(n-7)/2$ vertices is the 2-vertex x. This requires $(n-7)/2 \geq 2$, or $n \geq 11$. Thus (a) holds.

Since $T - V(X')$ has $(n+5)/2$ vertices, it cannot be reduced to $(n-3)/2$ by deleting three vertices. Hence every bicentroidal card has x and z as centroids. Since the balanced card requires $d_T(x) = 2$, every bicentroidal card has a centroid with degree 2, and (b) holds.

For (c), we obtain a thin $(\frac{n-5}{2}, \frac{n-7}{2})$-card when T has a thin $(\frac{n-5}{2}, \frac{n-5}{2})$-card C. Let u be the centroid of C, and let w be a 2-burl at distance 2 from u such that their common neighbor v is a 2-vertex. Because it is a 3^+-vertex whose deletion leaves two components of order at least $(n-5)/2$ and T is a $(\frac{n-5}{2}, \frac{n-5}{2})$-tree, the vertex u must be z in T. Since z is a 2-burl in T and a 1-burl in C, the card C arises from T by deleting two vertices of X and a leaf from the burl of T.

Now $v \in \{x, x'\}$. If $v = x$, then $w \in V(X)$ and we obtain a thin $\left(\frac{n-5}{2}, \frac{n-7}{2}\right)$-card by deleting the same two vertices of X and one leaf from X'. If $v = x'$, then $w \in V(X')$ and nothing was deleted from X' to form C; obtain a thin $\left(\frac{n-5}{2}, \frac{n-7}{2}\right)$-card by deleting three vertices of X. Thus (c) holds.

Case 2: $c'(T) = (n-3)/2$. Now z is a 1-burl. Deleting z, its leaf neighbor, and a leaf of X yields a balanced card. Hence we may assume that (a), (b), and (c) all hold.

From (a), let C be a $\left(\frac{n-5}{2}, \frac{n-7}{2}\right)$-card with centroid u whose neighbor in the piece with $(n-7)/2$ vertices has degree 2. The only choice for u is x, since otherwise too many vertices must be deleted to get the pieces down to the desired sizes. In a $\left(\frac{n-5}{2}, \frac{n-7}{2}\right)$-card, the centroid is a 2-burl, so x is a 2-burl. Now deleting one vertex from X and two vertices from X' other than the leaf neighbor of z yields a bicentroidal tree with centroids z and x, both having degree at least 3. Hence (b) fails.

Case 3: $c'(T) = (n-1)/2$. Now $d_T(z) = 2$. Deleting z and one leaf from each of X and X' yields a balanced card. Hence we may assume that (a), (b), and (c) all hold.

As in Case 2, (a) provides a $\left(\frac{n-5}{2}, \frac{n-7}{2}\right)$-card C with centroid u whose neighbor in the piece with $(n-7)/2$ vertices has degree 2. Since u is a 2-burl in C, vertex u must be x or x', not z; by symmetry let it be x. The piece of C with $(n-7)/2$ vertices is contained in X, since x is a 2-burl in C, and C arises by deleting three vertices outside X.

Hence the part of X outside the burl of x is too small to allow x to be the centroid of a $\left(\frac{n-5}{2}, \frac{n-5}{2}\right)$-card C' provided by (a). Also z cannot be the centroid of C', since the centroid has degree 3 in a $\left(\frac{n-5}{2}, \frac{n-5}{2}\right)$-card. The remaining possibility is x'. Since X' has $(n-1)/2$ vertices and the centroid of C' is a 1-burl, C' must arise by deleting three vertices in X, and x' is a 1-burl in T. With $(n-1)/2 - 3 \geq 4$, we can delete three vertices of X while leaving x as a 2-burl, and then C' is a thin $\left(\frac{n-5}{2}, \frac{n-5}{2}\right)$-card.

We have shown that z and x' are 1-burls in T, while x is a 2-burl. Now (c) requires T to have a thin $\left(\frac{n-5}{2}, \frac{n-7}{2}\right)$-card. The centroid of this card is a 2-burl, so it cannot be x' or z and must be x. Since $X - x$ has only $(n-7)/2$ vertices outside the burl of x, the neighbor of x in the large piece must be z. However, since x' is a 1-burl in T, we cannot complete a thin $\left(\frac{n-5}{2}, \frac{n-7}{2}\right)$-card. □

Theorem 8.4. *For $n \geq 19$ with n odd, n-vertex $\left(\frac{n-1}{2}, \frac{n-5}{2}\right)$-trees are reconstructible.*

Proof. By Lemma 8.3 and earlier results, we can recognize from the deck that all reconstructions are $\left(\frac{n-1}{2}, \frac{n-5}{2}\right)$-trees. We determine the reconstruction T uniquely. Label z, x, X, x', X' as usual. The centroid z is a 2-burl. Every $\left(\frac{n-5}{2}, \frac{n-7}{2}\right)$-card arises by deleting three vertices of X, leaving z as centroid; this tells us whether the vertices of the burl in T are adjacent. There are balanced cards if and only if $d_T(x) = 2$, so we can tell whether x is a 3^+-vertex.

Case 1: $d_T(x) \geq 3$. In this case, x and z are the centroids of all bicentroidal cards, which are obtained by deleting one vertex of X and two vertices outside X.

There are no bicentroidal cards with two centroids of degree 2 if and only if x is a full vertex. The bicentroidal cards having one centroid of degree 2 are then obtained by deleting the burl of T and one vertex of X; the centroid of degree 2 is z. Over all such cards, the branch containing the centroid of degree at least 3 gives us the rc1-deck of X, and we can reconstruct X by Lemma 2.1. The other branch is X' with z prepended at the root.

If $d_T(x) \geq 3$ and x is not a full vertex, then x is a 1-burl. We recognize this by having a bicentroidal card C with two centroids of degree 2, obtained by deleting the leaf neighbor $\hat{x}$ of x and the burl of T, but this card does not determine which branch is which. Now a bicentroidal card in which one centroid has degree 2 and the other is a 2-burl is obtained only by deleting the leaf neighbor of x and two vertices of X'. We obtain X from the branch whose root (which is x) has degree 2 by adding a leaf neighbor to that root. We then return to C and know which branch in C is $X - \hat{x}$, so we know which centroid receives one leaf neighbor and which receives the 2-burl.

Case 2: $d_T(x) = 2$. Deleting at least two vertices from X yields unicentroidal cards with centroid z, while deleting at at least two vertices outside X yields bicentroidal cards. All $\left(\frac{n-5}{2}, \frac{n-5}{2}\right)$-cards are obtained by deleting two vertices from X and one from the burl, yielding centroid z as a 1-burl. In every such card, one of the large pieces is X'. If exactly one piece with $(n-5)/2$ vertices is always present, then that piece is X'. We then also know the numbers of leaves in X', T, and X (since we already know the burl). The large pieces in these cards after removing X' form the rc2-deck of X (or two copies of it if the burl has two leaves), and by Theorem 2.3 we can reconstruct X.

Hence we may assume that all $\left(\frac{n-5}{2}, \frac{n-5}{2}\right)$-cards have the same two pieces with $(n-5)/2$ vertices. This requires that the rc2-cards of X are pairwise isomorphic. Hence X is a rooted broom or $\hat{P}''_{(n-1)/2}$. Since the other piece is always X', we know X' if it is neither a rooted broom nor $\hat{P}''_{(n-1)/2}$. Suppose otherwise.

Now consider the balanced cards of T. Since every balanced card consists of two components with $(n-3)/2$ vertices, the balanced cards are formed by deleting x and two vertices outside X. There are multiple such cards, but every one has $X - \{x\}$ has a component. This is a broom or is $S(1, 2, (n-11)/2)$ (when the rooted X is $\hat{P}''_{(n-1)/2}$), and this common component is distinguishable from the other components. Hence we now know X and its rc2-cards, so we can return to the $\left(\frac{n-5}{2}, \frac{n-5}{2}\right)$-cards and determine X'. □

Corollary 8.5. *For $n \geq 19$ with n odd, n-vertex trees are 3-reconstructible.*

Proof. In earlier sections, we reconstructed all n-vertex trees with cost less than $(n-1)/2$, and in this section we have reconstructed all $\left(\frac{n-1}{2}, \frac{n-b}{2}\right)$-trees with $b \geq 5$. Hence the remaining trees with n odd are $\left(\frac{n-1}{2}, \frac{n-3}{2}\right)$-trees and $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$-trees. In particular, we know that in every reconstruction from the deck of such a tree, the centroid z is not a full vertex. By Theorems 6.2 and 6.3, we reconstructed $\left(\frac{n-1}{2}, \frac{n-3}{2}\right)$-trees and $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$-trees having no full vertex adjacent to the centroid. Since we know that the centroid is not a full vertex in the remaining trees with n odd, by Remark 6.4 the argument of Theorem 6.4 applies to reconstruct such a tree T. □

9 Bicentroidal Trees

It remains to consider bicentroidal trees, where the number n of vertices is even, the trees have cost $n/2$, and all cards in the $(n-3)$-deck are unicentroidal. Since we have shown that all unicentroidal trees with at least 19 vertices are 3-rconstructible, we can recognize bicentroidal trees from their $(n-3)$-decks. In order to reconstruct such a tree from the deck, we will consider cases according to the status of the two centroids.

Remark 9.1. As usual for a bicentroidal tree T, in this section let z and z' be the centroids, and let Y and Y' be the two branches of T, with $z \in V(Y)$ and $z' \in V(Y')$. Let x be the neighbor of z in a largest component of $Y - z$, and let x' be the neighbor of z' in a largest component of $Y' - z'$. When v is a 1-burl, let $\hat{v}$ denote the leaf neighbor of v.

For all $v \in V(T)$, there is a component with at least $n/2$ vertices in $T - v$; hence every connected $(n-3)$-card of T has cost $(n-6)/2$ or $(n-4)/2$. Furthermore, every card with cost $(n-6)/2$ has centroid z or z'. Other vertices can be centroids only for cards with cost $(n-4)/2$: possibly x when $d_T(z) = 2$, and possibly x' when $d_T(z') = 2$. No other vertex of T can be a centroid of a connected card.

We begin by showing 3-reconstructibility in an easy case.

Theorem 9.2. *For $n \geq 20$, an n-vertex bicentroidal tree is reconstructible if at least one centroid is a full vertex.*

Proof. Let $\mathcal{D}$ be the $(n-3)$-deck of such a tree T. Using prior results, we recognize from the deck that T is bicentroidal. Let v be a centroid of T.

If v is not full, then deleting one vertex from its branch ($\hat{v}$ if v is a 1-burl) and two vertices from the other branch yields a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card whose centroid has degree 2 (it is v). If v is a full vertex, then v cannot become a centroid of degree 2 in a card in $\mathcal{D}$, because deleting the burl at v to reach degree 2 (and possibly deleting one other vertex) would leave a piece with at least $(n-2)/2$ vertices. Also a vertex that is not a centroid in T can be a centroid in a connected card only if it is adjacent to a centroid in T having degree 2. Hence both centroids of T are full vertices if and only if no connected card has a centroid of degree 2.

If we delete three vertices from the branch containing v, then we obtain a connected card with cost $(n-6)/2$ whose centroid is the centroid of T other than v, retaining all its neighbors from T. As noted in Remark 9.1, only centroids of T can be centroids of connected cards with cost $(n-6)/2$. Hence exactly one centroid of T is full if and only if some connected card has a centroid of degree 2 and another with cost $(n-6)/2$ has a full vertex as centroid.

Having recognized these two cases, we proceed to reconstruction.

Case 1: *Both centroids are full vertices.* By Remark 9.1, every card has centroid in $\{z, z'\}$. Cards with cost $(n-6)/2$ arise when and only when three vertices are deleted from one branch. The remaining $(n-6)/2$ vertices in that branch form a largest piece of the card. The centroid is the root of the other branch of T. If the resulting card is a $\left(\frac{n-6}{2}, \frac{n-b}{2}\right)$-card, then $(n-6)/2 + (c+1) + (n-b)/2 = n-3$, where the other centroid is a c-burl in T; this reduces to $c = b/2 - 1$.

Letting z be a $(b-2)/2$-burl and z' be a $(b'-2)/2$-burl, we thus have $\left(\frac{n-6}{2}, \frac{n-b}{2}\right)$-cards and $\left(\frac{n-6}{2}, \frac{n-b'}{2}\right)$-cards in $\mathcal{D}$, with $b, b' \geq 6$. If $b, b' \geq 8$, then the largest pieces in these cards are unique, and deleting them yields Y' or Y, respectively. Over all cards with cost $(n-6)/2$, we thus obtain both Y' and Y, whether $Y = Y'$ or not.

If (by symmetry) $b \geq 8$ and $b' = 6$, then we still obtain Y by deleting the largest piece in any $\left(\frac{n-6}{2}, \frac{n-b}{2}\right)$-card, in which the next largest piece is an rc3-card of Y'. Since the pieces are distinguished by their size, by Lemma 3.12 we reconstruct Y' and T.

In the remaining case, every connected card with cost $(n-6)/2$ is a $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-card, and both z and z' are 2-burls. Consider the $\left(\frac{n-4}{2}, \frac{n-8}{2}\right)$-cards in which the root of the biggest piece has degree 2. Such cards arise by deleting both vertices of the burl at z or z' and a leaf of the opposite branch not in the burl of the other centroid. For each such card, in the largest piece we see one branch (without its burl), and in the next-largest piece (when $(n-8)/2 > 2$) we see the configuration of the burl from the other branch (two leaves or P_2). Hence using all such cards we reconstruct T.

Case 2: *One centroid is a full vertex and one is not.* By symmetry, assume that z is full and z' is not. As in Case 1, every $\left(\frac{n-6}{2}, \frac{n-b}{2}\right)$-card with $b \geq 6$ arises by deleting three vertices of Y' and has centroid z as a $(b-2)/2$-burl. When $b \geq 8$, the largest piece (with $(n-6)/2$ vertices) is unique and is an rc3-card of Y'; deleting it yields the branch Y. Over all such cards we obtain the rc3-deck of Y'. By Lemma 3.12 we reconstruct Y' and T.

In the remaining case, $b = 6$ and z is a 2-burl. We still obtain Y and an rc3-card of Y' from any $\left(\frac{n-6}{2}, \frac{n-b}{2}\right)$-card, but now both objects have $(n-6)/2$ vertices. We can be confused which subtree is Y only if the rc3-cards of Y' are all the same, making the $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-cards identical. Otherwise, we know Y and can reconstruct Y' by Lemma 3.12.

If $b = 6$ and the $\left(\frac{n-6}{2}, \frac{n-6}{2}\right)$-cards are identical, then by Lemma 2.6 the common rc3-card of Y' is a rooted broom or path. Since z' is not a full vertex, this requires $d_T(z') = 2$. Since Y is rooted at a 2-burl, we can tell which piece is Y. If the piece contained in Y' is not a path, then we reconstruct T from such a card by adding three more sibling leaves at the end of the piece. If the piece is a path, then $Y' \in \{\hat{P}_{n/2}, \hat{P}'_{n/2}, \hat{P}''_{n/2}, \hat{Q}_{n/2}, \hat{Q}'_{n/2}, \hat{Q}''_{n/2}\}$, and the reconstruction of Y' is determined by the number of such cards, the number of leaves in T, and the number of copies of $S_{1,1,2}$. □

Lemma 9.3. *When $\mathcal{D}$ is the $(n-3)$-deck of a bicentroidal tree in which neither centroid is a full vertex, in every reconstruction the set of centroids satisfies the same case among the following: two 1-burls, one 1-burl and one 2-vertex, or two 2-vertices.*

Proof. Let T be a reconstruction from $\mathcal{D}$, with centroids z and z'. Since neither centroid is full, we have the path $\langle x, z, z', x' \rangle$ and branches Y and Y' as usual. Let $\rho(\mathcal{D})$ be the minimum, over all $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-cards, of the minimum distance from the centroid to a 3^+-vertex in the largest piece. A $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card where this minimum distance is j is *j-distant.*

Case 1: $\rho(\mathcal{D}) = 1$. The centroid of any $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card is a 1-burl. If it is x or x', then by Remark 9.1 its neighbor in the piece with $(n-4)/2$ vertices has degree 2, so the card is not 1-distant. Hence the centroid of a 1-distant card must be z or z', making it a 1-burl.

Now that it must have centroid z or z', a 1-distant card exists with the centroid having two 3^+-neighbors if and only if (0) both z and z' are 1-burls and at least one of $\{x, x'\}$ is a 3^+-vertex. Otherwise, in every 1-distant card the centroid has a 2-neighbor. In this case, either (1) the degrees along $\langle x, z, z', x' \rangle$ are $(2, 3, 3, 2)$, or (2) $\{z, z'\}$ has exactly one 2-vertex and one 1-burl, and the 1-burl has a 3^+-neighbor. In (2), deleting one vertex from the branch whose root has degree 2 and two vertices from the other branch yields a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card in which one piece is rooted at a 1-burl with a 1-burl neighbor. In (1), there is no such card; no $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card has centroid x or x' since only one piece of $T - x$ or $T - x'$ has at least $(n-4)/2$ vertices, and deleting the leaf $\hat{z}$ or $\hat{z}'$ to make a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card with centroid z or z' yields a 2-vertex within distance 1 of the root in each piece.

Hence in all situations having a 1-distant card we can distinguish among (0), (1), and (2), determining whether both centroids or exactly one centroid of T are 1-burls.

Case 2: $\rho(\mathcal{D}) = 2$. If z and z' are both 1-burls, then deleting two vertices from $Y - \{z, \hat{z}\}$ and one from $Y' - \{z', \hat{z}'\}$ yields a 1-distant card, which is forbidden. If z and z' are both 2-vertices, then the centroid of a 2-distant card must be x or x', and the vertex at distance 2 in the piece with $(n-4)/2$ vertices is z' or z, whose degree is too small. Hence if $\rho(\mathcal{D}) = 2$, then exactly one of $\{z, z'\}$ is a 1-burl.

Case 3: $\rho(\mathcal{D}) \geq 3$. Here the two vertices nearest the centroid in the largest piece of a $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card are 2-vertices (also in the full tree, since $(n-4)/2 \geq 4$). Neither centroid of the original tree is full, and already $\rho(\mathcal{D}) > 1$ implies that they cannot both be 1-burls. Hence we must decide whether they are two 2-vertices or one 2-vertex and one 1-burl.

Suppose that $\mathcal{D}$ has reconstructions of both types. Let T be a reconstruction with centroids z and z' such that $d_T(z) = d_T(z') = 2$. Let $\widetilde{T}$ be a reconstruction with centroids $\tilde{z}$ and $\tilde{z}'$ of unequal degree; by symmetry, we may assume that $\tilde{z}'$ is a 1-burl and $\tilde{z}$ is a 2-vertex. Since $\tilde{z}'$ is a 1-burl in $\widetilde{T}$, the deck has $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-cards. Since T also has deck $\mathcal{D}$, having

$\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-cards makes x or x' a 1-burl; by symmetry, we may let x' be a 1-burl in T. With $\mathcal{D}$ having no 1-distant card, $d_{\widetilde{T}}(\tilde{x}') = 2$. With $\mathcal{D}$ having no 2-distant card, $d_{\widetilde{T}}(\tilde{x}) = d_{\widetilde{T}}(\tilde{y}') = 2$, where $\tilde{y}'$ is the nonleaf neighbor of $\tilde{x}'$ other than $\tilde{z}'$. We now split this case.

Subcase 3a: $\rho(\mathcal{D}) = 3$. When x' is the centroid of a card in T, the largest piece contains z and z'. Hence a 3-distant card requires $d_T(x) \geq 3$. If x is a full vertex in T, then deleting three vertices from Y' in T yields a $\left(\frac{n-4}{2}, \frac{n-b}{2}\right)$-card with $b \geq 8$. Since $\tilde{x}$ and $\tilde{x}'$ are 2-vertices in $\widetilde{T}$, the deck $\mathcal{D}$ has no such card. Hence x is a 1-burl in T.

Now x and x' are 1-burls in T. Hence T has $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards with 1-burls (and no larger burl) at distance 1 and 2 in opposite directions from the centroid. Such cards in the deck of $\widetilde{T}$ require the nonleaf neighbor $\tilde{y}$ of $\tilde{x}$ other than $\tilde{z}$ to be a 1-burl in $\widetilde{T}$. Since $n/2 \geq 8$, deleting three vertices from $\widetilde{Y}$ in $\widetilde{T}$ (not the leaf neighbor of $\tilde{y}$) now yields a $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card whose piece with $(n-6)/2$ vertices has a 1-burl at distance 3 from the centroid.

A $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card arises from T by deleting three vertices from one branch, and its centroid is x' or x. The piece with $(n-6)/2$ vertices does not contain the centroids of T. Therefore, to obtain a card like we just found from $\widetilde{T}$, in T there must be a 1-burl at distance 3 from x or x' in the direction away from z and z' (the intervening vertices are 2-vertices). Thus we obtain $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-cards from T such that in both pieces the nearest 3^+-vertices to the centroid are 1-burls at distance 3 from the centroid.

Such cards must also occur in the deck of $\widetilde{T}$, so $\widetilde{T}$ has 1-burls at distance 3 in both directions from the 1-burl $\tilde{z}'$, since in $\widetilde{T}$ the only possible centroid of a $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card is $\tilde{z}'$.

We now have in T a 1-burl at x' or x with nearest 1-burls at distance 3 in both directions; by symmetry, let it be x'. In $\widetilde{T}$ we have a 1-burl at $\tilde{z}'$ and nearest 1-burls at distance 3 in both directions. By deleting the leaf neighbor of $\tilde{z}'$ and two vertices of $\widetilde{Y}$ (other than the leaf neighbor of $\tilde{y}$) from $\widetilde{T}$, we obtain a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card such that in each piece the nearest 3^+-vertex is distance 3 from the centroid. However, T has no such card; all $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards of T have centroid at z or z', and the distance in such a card from the centroid to the nearest 3^+-vertex in Y' is 1, 2, 4, 5, or larger.

Subcase 3b: $\rho(\mathcal{D}) > 3$. With no 3-distant card, $d_T(x) = 2$. Every $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card of T arises by deleting three vertices of the branch in T containing x and z (yielding the large piece). The piece with $(n-6)/2$ vertices in these cards is the same, rooted at the neighbor y' of x'; call it W.

On the other hand, $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-cards arise from $\widetilde{T}$ by deleting three vertices of $\widetilde{Y}$ or by deleting two vertices from $\widetilde{Y}$ and one from $\widetilde{Y}'$. In one case the piece with $(n-6)/2$ vertices is contain in $\widetilde{Y}$; in the other it is in $\widetilde{Y}'$. That piece must always be W due to the cards from T, so in $\widetilde{T}$ the rc3-cards of the branch $\widetilde{Y}$ and the rc1-cards of the subtree obtained from $\widetilde{Y}'$ by deleting $\tilde{z}'$ and its leaf neighbor all equal W. By Lemma 2.6, W is a path or a broom.

The tree $\widetilde{T}$ now consists of the 1-burl $\tilde{z}'$, its leaf neighbor, and two copies of W rooted at the neighbors of $\tilde{z}'$ in $\widetilde{Y}$ and $\widetilde{Y}'$, with $\widetilde{Y}'$ completed by adding one vertex beyond the copy of W and $\widetilde{Y}$ completed by adding three vertices beyond W. Let C be a connected card obtained from $\widetilde{T}$ by deleting the leaf neighbor of $\tilde{z}'$ and two vertices of $\widetilde{Y}$, leaving the copy of W rooted at $\tilde{z}$ in $\widetilde{Y}$. Note that C consists of $\tilde{z}$, two copies of W rooted at its neighbors, and one vertex beyond W in each direction.

In order to obtain C as a card from T, we must delete the leaf neighbor of x' and have x' become the root of a copy of W. This means that deleting a leaf from W and prepending a vertex at the root can yield another copy of W. Hence W must be a path.

Deleting three vertices from Y' in T yields a connected card C' with a vertex v such that $C' - v$ has components of orders $(n-12)/2$, $(n+2)/2$, and 1. Viewing C as a card of T, the role of v is played by the 1-burl x'. To obtain C as a card of $\widetilde{T}$, it cannot be that $\tilde{z}'$ plays the role of v, because the component of $\widetilde{T} - \tilde{z}'$ with $(n-4)/2$ vertices cannot be cut down to $(n-12)/2$ by deleting three vertices. To find another 3^+-vertex in $\widetilde{T}$ to serve as v, we must move to within one vertex of the end of W, but the sizes of the components when we delete a vertex out there (even after deleting three vertices to form the card) are too unbalanced.

Hence we cannot have both T and $\widetilde{T}$ as reconstructions, and in all reconstructions the set of centroids has the same type: both 2-vertices or just one 2-vertex. □

Theorem 9.4. *Bicentroidal trees with at least* 20 *vertices are* 3*-reconstructible.*

Proof. Let $\mathcal{D}$ be the deck of a bicentroidal tree T with n vertices. From earlier sections, every reconstruction is bicentroidal. By Theorem 9.2 and earlier results, we may assume that all reconstruction have neither centroid a full vertex, so we have the usual path $\langle x, z, z', x'\rangle$ and branches Y and Y'. By Lemma 9.3, all reconstructions have $\{z, z'\}$ in the same status: two 1-burls, two 2-vertices, or one of each. If no card has a full vertex, then Theorem 6.2 reconstructs T. Hence we may assume T has a full vertex v.

When both centroids are 2-vertices, Theorem 6.3 applies to reconstruct T. When both are 1-burls, a similar argument works. In this case, every connected card is a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card or $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-card with centroid z or z'. The $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards arise by deleting the leaf neighbor of a centroid and two vertices from the other branch.

Let an *optimal card* be a $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-card having a full vertex closest to the centroid, say at distance r. As in Theorem 6.3, if no piece occurs in all optimal cards, then each branch of T has a full vertex at distance r from the root of the branch. If these are v and v', then some optimal cards have v as the full vertex at distance r from their centroid, while others have v'; none has both. Over the optimal cards we see both $Y - z$ and $Y' - z'$ and reconstruct T.

If all optimal cards have the same piece containing a full vertex closest to the centroid, then either that piece arises from one branch in T or the two branches of T are isomorphic. In the latter case, we have T. In the former case, we know one branch and have the rc2-deck of the other. We also know the number of leaves, so we can reconstruct T.

Hence we may assume that z' is a 1-burl and z is a 2-vertex in T. Every connected card has centroid in $\{x, z, z'\}$, where x is the nonleaf neighbor of z other than z'.

Case 1: *x is a full vertex.* Connected cards whose centroid is a full vertex have centroid x and arise by deleting three vertices of Y'. They are $\left(\frac{n-4}{2}, \frac{n-b}{2}\right)$-cards with $b \geq 8$, so the largest piece is unique. The largest piece is an rc3-card of Y' plus z prepended at the root. The next largest piece combines with z, x, and the burl to form Y. Hence we obtain Y and the rc3-deck of Y', and we know which is which. By Lemma 3.12, we reconstruct Y' and T.

Case 2: *x is a 2-vertex.* If v is a full vertex in Y, then v is closest to the centroid only in $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards where three vertices are deleted from Y', with centroid x. Since $n/2 - 3 \geq 3$, we can keep z' as a 1-burl in such cards, meaning that the piece not containing v is rooted at a 2-vertex with a 1-burl neighbor. If v' is a full vertex in Y', then v' is closest to the centroid in $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-cards where three vertices are deleted from Y, with centroid z', in which cards v' is in the big piece. There are also $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-cards where v' is the same distance from centroid z' by deleting two vertices from Y and one from Y', but in these v' is not in the big piece. There are also $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards where v' is the same distance from centroid z' by deleting the leaf neighbor of z' and two vertices from Y, but in these the two closest vertices in the piece not containing v' are 2-vertices.

Among the connected cards minimizing the distance from a full vertex to the centroid, consider the $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$-cards where the full vertex is in the largest piece and the $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards in which the piece not containing the full vertex is rooted at a 2-vertex with a 1-burl neighbor. If both types arise, then since the pieces are distinguished in both types we obtain Y and Y'. If only one type arises, then we know which type it is and obtain Y or Y' and the rc3-deck of the other, from which which can reconstruct T.

Case 3: *x is a 1-burl.* Connected cards in which a full vertex in Y is closest to the centroid have centroid x, all of which are $\left(\frac{n-4}{2}, \frac{n-6}{2}\right)$ cards that arise by deleting three vertices of Y' and have full vertices of Y only in the piece with $(n-6)/2$ vertices. Connected cards in which a full vertex in Y' is closest to the centroid have centroid z'; one way they arise is as $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards obtained by deleting the leaf neighbor of z' and two vertices of Y.

Let us reuse the term *optimal* for connected cards *of these two types* with minimum distance r from the centroid to a full vertex. If there are any of the second type, the $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards, then they all give us Y' from the piece containing the closest full vertex and the rc2-deck of Y from the other piece. Knowing the number of leaves, we can reconstruct Y and T.

If there are no optimal $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$-cards, then Y' does not have a full vertex at distance r from z'. Now all optimal cards have the first type, and their constant piece gives us Y, while the other pieces gives the rc3-deck of Y', and we reconstruct. Because Y' has a 3-vertex z' at its root, its rc3-cards cannot all be rooted brooms, so the exception cannot occur and we really have only one piece that is constant over these cards. □

References

[1] J. A. Bondy and R. L. Hemminger, Graph reconstruction—a survey, *J. Graph Theory* 1 (1977), 227–268.

[2] W. B. Giles, Reconstructing trees from two-point deleted subtrees, *Discrete Math.* 15 (1976), 325–332.

[3] C. Groenland, T. Johnston, A. Scott, and J. Tan, Reconstructing trees from small cards, arXiv:2103.13359.

[4] P. J. Kelly, On isometric transformations, PhD Thesis, University of Wisconsin-Madison, 1942.

[5] P. J. Kelly, A congruence theorem for trees, *Pacific J. Math.* 7 (1957), 961–968.

[6] A. V. Kostochka, M. Nahvi, D. B. West, and D. Zirlin, Degree lists and connectedness are 3-reconstructible for graphs with at least seven vertices, *Graphs and Combinatorics* 36 (2020), 491–501.

[7] A. V. Kostochka, M. Nahvi, D. B. West, and D. Zirlin, Acyclic graphs with at least $2\ell+1$ vertices are ℓ-recognizable, (submitted).

[8] A. V. Kostochka and D. B. West, On reconstruction of graphs from the multiset of subgraphs obtained by deleting ℓ vertices, *IEEE Transactions on Information Theory*, (published on-line March, 2020, DOI: 10.1109/TIT.2020.2983678).

[9] W. J. Myrvold, The ally-reconstruction number of a tree with five or more vertices is three, *J. Graph Theory* 14 (1990), 149–166.

[10] V. Nýdl, A note on reconstructing of finite trees from small subtrees, *Acta Univ. Carol. Math. Phys.* 31 (1990), 71–74.

[11] V. Nýdl, Finite undirected graphs which are not reconstructible from their large cardinality subgraphs, *Discrete Math.* 108 (1992), 373–377.

[12] V. Nýdl, Graph reconstruction from subgraphs, *Discrete Math.* 235 (2001), 335–341.

[13] H. Spinoza and D. B. West, Reconstruction from the deck of k-vertex induced subgraphs, *J. Graph Theory* 90 (2019), 497–522.

[14] S. M. Ulam, A collection of mathematical problems, *Interscience Tracts in Pure and Applied Mathematics* 8 (Interscience Publishers, 1960).