EVERY OUTERPLANAR GRAPH IS THE UNION OF TWO INTERVAL GRAPHS

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An *intersection representation* f of a graph G is an assignment of sets to the vertices so that vertices are adjacent if and only if the corresponding sets intersect. The *interval number* $i(G)$ is the minimum t such that G has an intersection representation in which each set is a union of at most t intervals on the real line. The graphs with interval number 1 are the *interval graphs*. Scheinerman and West [4] proved that planar graphs have interval number at most 3. Their proof includes the statement that outerplanar graphs have interval number at most 2.

A more restrictive intersection model is obtained by using sets that consist of an interval from each of t parallel lines. In [3], Kumar and Deo proposed the study of intersection graphs of such sets. Gyárfás and West [2] proposed calling these sets *t-track intervals*. Such an representation of G is a t-track representation, and the *track number* of G is the minimum t such that G has a t-track representation. Equivalently, the track number (written $t(G)$) is the minimum number of interval graphs whose union is G. In [2], it was shown that recognition of graphs with track number 2 is NP-hard.

Since the tracks could be placed in disjoint portions of a single line, the track number of a graph is at least its interval number. In general, one may ask by how much $t(G)$ can exceed $i(G)$. Trotter and Harary [5] proved that $i(K_{m,n}) = \lceil (mn+1)/(m+n) \rceil$. Beineke [1] and Gyárfás and West [2] proved that $t(K_{m,n}) = \lceil mn/(m+n-1) \rceil$, by decomposing $K_{m,n}$ into this many caterpillars.

In this note, we continue the comparison of $t(G)$ with $i(G)$ by showing that $t(G) \le 2$ when G is outerplanar. It remains open whether always $t(G) \le 3$ when G is planar. One might expect that every outerplanar graph decomposes into two forests of caterpillars, yielding a stronger result analogous to the optimal decomposition of bicliques, but we provide a counterexample to this. By modifying our argument for 2-track representations, we obtain that every outerplanar graph decomposes into three caterpillar forests.

A *displayed portion* of the set assigned to a vertex v [or an edge uv] is an interval in $f(v)$ [or $f(u) \cap f(v)$] that does not intersect the image of any other vertex.

It is convenient to view each set $f(v)$ as a list of disjoint intervals, with the edge uv arising if and only if some interval assigned to u intersects some interval assigned to v. A *small interval* in a representation is an interval assigned to a vertex that does not contain an endpoint of an interval assigned to any other vertex.

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THEOREM 1. Every outerplanar graph is a union of two interval graphs.
Proof: Let G be an outerplane graph; we produce a 2-track representation.

Choose a root vertex r. Let T be a spanning tree grown from r by breadth-first search. Let V_i be the set of vertices at distance i from r (level i of T). Index the vertices of V_i in clockwise order according to the embedding of G, starting from the unbounded face. The breadth-first search order will generate the elements in this order if we always append new vertices to the search queue in clockwise order in the embedding, starting after the parent edge.

We claim that the subgraph induced by V_i is a disjoint union of paths, with edges joining only consecutive vertices in the ordering. If two nonconsecutive vertices at level i are adjacent, then the edge between them and the paths from these vertices to r enclose the vertices between them on level i, which contradicts the outerplanarity of the embedding.

Let $S = \bigcup_k V_{2k}$, and let $S' = \bigcup_k V_{2k-1}$; note that $S \cap S' = \emptyset$. Since no edges skip levels, S and S' induce disjoint unions of paths. We represent $G[S]$ on track 0 (the *even* track). Let $v_1, v_2, \ldots$ be the ordering of S produced by taking $V_0, V_2, \ldots$ in order, each in the order in which we generated it. Let the interval for v_i contain point i on track 0. If v_i is adjacent to v_{i-1}, enlarge the interval to include $[i - 4/3, i]$. Similarly, include $[i, i + 4/3]$ if v_i is adjacent to v_{i+1}.

Similarly represent $G[S']$ by intervals on track 1 (the *odd* track). We have used only one interval per vertex, leaving another interval available for use on the other track.

For $i > 0$, each vertex in V_i has a neighbor in V_{i-1}, and all remaining edges have this form. A vertex in V_i cannot be adjacent to two nonconsective vertices in V_{i-1}; these edges plus the paths from these vertices to r would enclose the vertices between them in the indexing of V_{i-1}. Similarly, no two vertices of V_i can have the same pair of neighbors in V_{i-1}; one would be enclosed by the paths generated from the other.

For each $v \in V_i$, we now add one interval for v in the track with parity opposite to i, representing the edges from v to V_{i-1}. If v has one neighbor x in V_{i-1}, we place a small interval for v in the displayed portion of $f(x)$.

In the remaining case, v has two neighbors x, y in V_{i-1}. If $xy \in E(G)$, then we place a small interval for v in the displayed portion of xy. If $xy \notin E(G)$, then x and y have consecutive but non-overlapping intervals in the representation of $G[V_{i-1}]$. No vertex has been assigned any point between these intervals, and we may assign an interval to v that overlaps both. Since x and y cannot have another common neighbor in V_i, this region is not requested by another vertex. Thus we have completed a 2-track representation of G.

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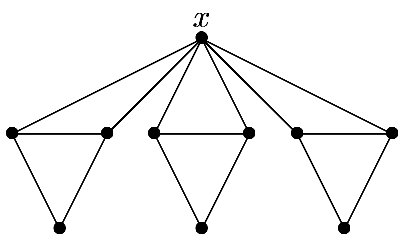
Example 1. *Not every outerplanar graph is a union of two forests of caterpillars.* Let H be the graph consisting of three copies of the "kite" $K_4 - e$ identified at a single vertex x that has degree 2 in each kite. Form G from $5H$ by adding a single vertex w adjacent to the five copies of x. Note that H and G are outerplanar.

We claim that every decomposition of H into two caterpillar forests F_1, F_2 assigns each forest at least one edge incident to x. If all such edges lie in F_1, then the edges joining 3-valent vertices in each kite lie in F_2, to avoid triangles in F_1. Now each vertex of degree two in H is incident to one edge of F_1 (to avoid triangles in F_2 and 4-cycles in

F_1. Now F_1 has an induced subgraph that is a subdivision of $K_{1,3}$; this is the forbiddent subtree for caterpillars.

In a decomposition of G, at least three edges incident to w lie in the same caterpillar. Incident to each of these copies of x we have at least one edge of that copy of H in the same caterpillar. Now again we have the forbidden subgraph. ■



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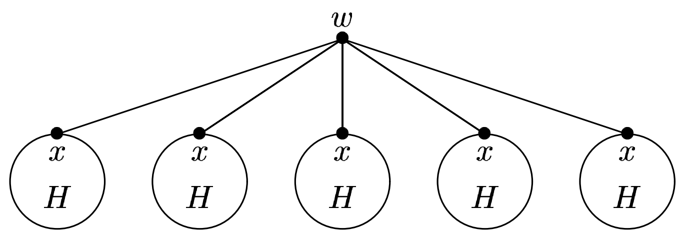

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THEOREM 2. Every outerplanar graph decomposes into three caterpillar forests, and this is sharp.

Proof: In light of Example 1, it suffices to construct such a decomposition. Form the sets V_i as in the proof of Theorem 1. Partition them into three groups, by the congruence class of i modulo 3. The forest F_k will contain $G[V_i]$ for each i congruent to k modulo 3. Let $c(i)$ denote the congruence class of i modulo 3.

As observed in the proof of Theorem 1, 1) each vertex of V_i has a most two neighbors in V_{i-1}, 2) if it has two, they are consecutive, and 3) no two vertices in V_i can have the same pair of neighbors in V_{i-1}. We assign the edges between V_i and V_{i-1} to $F_{c(i)}$ or $F_{c(i-1)}$. Consider $v \in V_i$. If v has one neighbor in V_{i-1}, we assign the edge to $F_{c(i-1)}$. If v has two neighbors in V_{i-1}, we assign one edge to $F_{c(i-1)}$ (involving the earlier-indexed vertex) and the other to $F_{c(i)}$.

Although $F_{c(i)}$ may acquire some edges from V_i to V_{i-1}, it acquires at most one such edge incident to each vertex. The edges it acquires from V_{i+1} have distinct endpoints in V_{i+1}. Since $F_{c(i)}$ acquires no edges between V_{i+1} and V_{i+2} or between V_{i-2} and V_i, each component of $F_{c(i)}$ is a caterpillar. ■

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