TOTAL INTERVAL NUMBER
FOR GRAPHS WITH BOUNDED DEGREE

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Abstract. The total interval number of an $n$-vertex graph with maximum degree $\Delta$ is at most $(\Delta + 1/\Delta)n/2$, with equality if and only if every component of the graph is $K_{\Delta, \Delta}$. If the graph is also required to be connected, then the maximum is $\Delta n/2 + 1$ when $\Delta$ is odd, but when $\Delta$ is even it exceeds $|\Delta + 1/(2.5\Delta + 7.7)|n/2$ for infinitely many $n$.

Given sets \{${S_v: v \in V}$\}, the intersection graph of the collection of sets is the simple graph with vertex set $V$ such that $v$ is adjacent to $v$ if and only if $S_u \cap S_v \neq \emptyset$. The family of sets is an intersection representation of its intersection graph. The interval graphs are the intersection graphs representable by assigning each vertex a single interval on the real line. More generally, we allow a representation $f$ to assign each vertex a union of intervals on the real line; if $G$ is the intersection graph of this collection, then $f$ is a multiple-interval representation of $G$. Let $\#f(v)$ be the number of disjoint intervals whose union is $f(v)$. If $\#f(v) = k$, we say that $f(v)$ consists of $k$ intervals or that $v$ is assigned $k$ intervals.

We may try to make the representation of $G$ “efficient” by minimizing $\max_{v \in V} \#f(v)$ or $\sum_{v \in V} \#f(v)$. The interval number $i(G)$ of a graph $G$ is the minimum of $\max_{v \in V(G)} \#f(v)$ over all multiple-interval representations of $G$. Interval number has been studied since 1979, beginning with [7] and [2]. The total interval number $I(G)$ of a graph $G$ is the minimum of $\sum_{v \in V(G)} \#f(v)$ over all multiple-interval representations of $G$. Although introduced in [2], total interval number was not studied until Aigner and Andreae [1] obtained extremal results for some fundamental families. Further results appear in [3,4,5,6].

† Research supported in part by grant 93-01-01486 of the Russian Foundation for Fundamental Research and the grant RPY300 of the International Science Foundation and Russian Government.
‡ Research supported in part by NSA/MSP Grants MDA904-90-H-4011 and MDA904-93-H-3040.

AMS codes: 05C35, 05C38
Keywords: total interval number, interval representation, vertex degree, trail cover
Running head: TOTAL INTERVAL NUMBER AND DEGREE
In this paper we prove that $I(G) \leq (\Delta + 1/\Delta)n/2$ for graphs with maximum degree at most $\Delta$, which is best possible. The proof yields a polynomial algorithm for producing a representation that satisfies the bound. Kratzke and West [5] proved that if $G$ contains a collection of $t$ pairwise edge-disjoint trails that together contain an endpoint of every edge of $G$, then $I(G) \leq e(G) + t$, where $e(G) = |E(G)|$. Such a collection of trails is a trail cover of size $t$, generalizing the notion of vertex cover; in this paper, “covering $e$” means “containing an endpoint of $e$”. Let $t(G)$ denote the minimum size of a trail cover. If $G$ is triangle-free, then $I(G) = e(G) + t(G)$; a simple counting argument [5] establishes the lower bound. The graph $mK_{\Delta,\Delta}$ has $m$ components that are complete bipartite graphs; it is regular and triangle-free, with $n = 2m\Delta$ vertices, and its minimum trail covers have size $m$. Hence $I(mK_{\Delta,\Delta}) = m\Delta^2 + m = (\Delta + 1/\Delta)n/2$, and our bound is best possible. We also prove that these are the only graphs achieving the bound. The proof yields a polynomial-time algorithm to achieve the bound.

Connected Graphs

Before proving the bound for general graphs, we discuss the more restricted class of connected graphs. For $\Delta$ even, the maximum of $I(G)$ in terms of $n$ and $\Delta$ is $\Delta n/2 + 1$. For $\Delta$ odd, we provide constructions where the excess over $\Delta n/2$ is linear in $n$.

**PROPOSITION** Suppose $\Delta$ is even. Among connected $n$-vertex graphs with maximum degree $\Delta$, the maximum of the total interval number is $\Delta n/2 + 1$.

**Proof:** Suppose $G$ is connected and has maximum degree $\Delta$. If $G$ is Eulerian, then $I(G) \leq e(G) + 1 \leq \Delta n/2 + 1$, with equality if $G$ is triangle-free and regular. If $G$ is not Eulerian, suppose $G$ has $2k$ vertices of odd degree. Since $G$ is connected, we can decompose $E(G)$ into $k$ trails, so $t(G) \leq k$. Since each vertex of odd degree has degree less than $\Delta$, we have $2e(G) \leq \Delta n - 2k$, and hence $I(G) \leq e(G) + k \leq \Delta n/2$. 

In addition to the $\Delta$-regular triangle-free graphs, equality holds also for $\Delta$-regular graphs in which every vertex belonging to a triangle is a cut-vertex. When $\Delta$ is odd, the bound of $\Delta n/2 + 1$ no longer holds; we provide a construction.

**PROPOSITION** Suppose $\Delta$ is odd and at least 3. Among connected $n$-vertex graphs with maximum degree $\Delta$, the maximum total interval number exceeds $[\Delta + 1/(2.5\Delta + 7.7)]n/2$ for infinitely many $n$.

**Proof:** We prove the claim by using copies of a triangle-free graph $H$ with degree sequence $(\Delta, \ldots, \Delta, \Delta - 1)$ to construct a $\Delta$-regular triangle-free connected graph $G$ with $n$ vertices such that $t(G) \geq [1/(2.5\Delta + 7.7)]n/2$. Begin with a caterpillar $C$ consisting of a path with $k + 2$ vertices and $\Delta - 2$ leaves attached to each interior vertex of the path. For each of the $k(\Delta - 2) + 2$ leaves of $C$, we provide a copy of $H$ and identify its vertex of degree $\Delta - 1$ with that leaf.
The resulting graph $G$ is triangle-free and $\Delta$-regular, so $I(G) = \Delta n/2 + t(G)$. Because each edge of $C$ is a cut-edge of $G$, every trail cover of $G$ has an endpoint in each copy of $H$. Hence $t(G) \geq [k(\Delta - 2) + 2]/2$, and in fact $t(G) = [k(\Delta - 2) + 2]/2$. If $H$ has $n'$ vertices, then $n = [k(\Delta - 2) + 2]n' + k$. We obtain $n < 2t|n' + 1/(\Delta - 2)|$, and hence $t(G) > \frac{1}{n'+1/(\Delta-2)}(n/2)$.

It remains to construct a suitable $H$ with $n'$ as small as possible. When $\Delta = 3$, we form $H$ by subdividing one edge of $K_{3,3}$; here $n' = 7$ and $t(G) > \frac{1}{7}(n/2)$. For larger $\Delta$, consider the lexicographic composition $F_s = C_5|K_s$, expanding each vertex of a 5-cycle into an independent set of size $s$. The graph $F_s$ is $2s$-regular and triangle-free, and for $s \geq 2$ it has a pair of easily described edge-disjoint Hamiltonian cycles. When $(\Delta + 1)/2$ is odd, we form $H$ by deleting from $F_{(\Delta+1)/2}$ the odd-indexed edges on one Hamiltonian cycle. Since $(\Delta + 1)/2$ is odd, this reduces one vertex degree from $(\Delta + 1)$ to $(\Delta - 1)$ and the others to $\Delta$. When $(\Delta + 1)/2$ is even, we form $H$ by deleting from $F_{(\Delta+3)/2}$ the odd-indexed edges on one Hamiltonian cycle and all edges on another Hamiltonian cycle. Since $n' = 2.5(\Delta + 3)$ is odd, we again obtain the desired degree sequence. Here $\Delta \geq 7$, which yields the 7.7 in the statement of the result.

\[ \square \]

The Main Result

We consider an $n$-vertex graph $G$ with maximum degree $\Delta$ and wish to prove that $I(G) \leq (\Delta + 1/\Delta)n/2$. We may assume that $G$ has no isolated vertices, because such vertices require no intervals; when $f(v) = \emptyset$ and the intersection graph is taken, $v$ becomes an isolated vertex.

Our approach is to select a set of edge-disjoint trails $T_1, \ldots, T_k$ to cover $E(G)$, in a greedy manner subject to various conditions; each new trail contains some previously uncovered edge. We partition $E(G)$ into sets associated with the trails; the set $E_i$ associated with $T_i$ consists of $E(T_i)$ and the newly-covered edges that do not belong to later trails. We will also associate a set $S_i \subseteq V(G)$ with each trail (the union of closed neighborhoods of certain vertices of the trail); these sets will be pairwise disjoint. We can represent $E_i$ using $|E_i| + 1$ intervals, fewer if $E_i$ contains a triangle with at most one edge on $T_i$.

When $G$ has maximum degree $\Delta$, we have $e(G) \leq \Delta n/2$. If $e(G) = \Delta n/2 - k$, we will use $|E_i| + 1$ intervals for at most $k + n/(2\Delta)$ trails $T_i$. We do this by ensuring that we use an extra interval for $T_i$ only when there exists $\alpha_i \in \{0, 1, 2\}$ such that $|S_i| \geq (2 - \alpha_i)\Delta$ and the degrees of two new vertices of $T_i$ sum to at most $2\Delta - \alpha_i$. If the values of $\alpha_i$ over the $s$ trails using an extra interval sum to $r$, then we have $I(G) \leq e(G) + s$ and $e(G) \leq \Delta n/2 - r/2$. Hence $I(G) \leq \Delta n/2 + (2s - r)/2$. Also we have associated $(2s - r)\Delta$ vertices with these trails. Since $(2s - r)\Delta \leq n$, we have the desired bound. The remainder of the proof consists of showing that we can choose the trails to ensure these conditions.

We use “open” trails: trails with two distinct endpoints. We say that a trail is closable if its endpoints are adjacent via an edge not belonging to the trail. When $T$ is closable, we let $T'$ denote the closed trail formed by adding the edge between the endpoints of $T$. 
**THEOREM** Every simple graph with $n$ vertices and maximum degree $\Delta$ has total interval number at most $(\Delta + 1/\Delta)n/2$. Furthermore, equality holds only when every component is $K_{\Delta, \Delta}$.

**Proof:** We select a sequence of pairwise edge-disjoint open trails $T_1, \ldots, T_k$ in a greedy manner. The new vertices of $T_i$ are the vertices in $T_i$ that do not appear in $T_1, \ldots, T_{i-1}$ and that cover at least one edge not covered by vertices of earlier trails. The new edges of $T_i$ are the previously uncovered edges that are covered by new vertices of $T_i$. Note that an edge of $T_i$ is new (for $T_i$) if and only if both its endpoints are new.

We choose each $T_i$ to be an open trail having endpoints that are new. Among these, we choose $T_i$ with maximum number of new vertices. Among these, we choose $T_i$ to be closable, if such a candidate is available. Among the remaining candidates for $T_i$, we choose $T_i$ with minimum length.

The sequence ends when all edges are covered. The set $E_i$ of edges associated with $T_i$ is $E(T_i)$ together with the new edges that do not belong to later trails. By construction, these edge sets are disjoint. We postpone the definition of the vertex set $S_i$ associated with $T_i$.

**Claim 1:** If $T_i$ is not closable and has endpoint $v$, then only one edge incident to $v$ belongs to $T_i$. Otherwise, we delete the initial portion of $T_i$ up to the next appearance of a new vertex other than the other endpoint of $T_i$ (this may be $v$ again). The shorter trail $T$ is open and has the same new set as $T_i$; it may be closable, but since $T_i$ is not closable, we would in either case choose $T$ in preference to $T_i$.

**Claim 2:** If the vertices of $T_i$ are not all new, then $T_i$ is not closable and the end edges of $T_i$ are new. Let $x$ be the first vertex of $T_i$ that is not new, belonging to an earlier trail $T_j$. If $T_i$ is closable, then $T_j$ can absorb the closed trail $T_i'$ to enlarge its new set. If $T_i$ is not closable, then the first edge of $T_i$ is new unless it is $ux$. By the maximality of the new set, every neighbor of $u$ along a new edge belongs to $T_i$. If $v$ is the first such vertex on $T_i$, then the $u, v$-portion of $T_i$ together with the edge $uv$ forms a closed trail containing $x$ that can be absorbed by $T_j$ to enlarge its new set.

**Claim 3:** If $T_i$ is closable, then no vertex of $T_i$ appears in another trail or has a neighbor in a later trail. By Claim 2, every vertex of $T_i$ is new. If $w$ is a vertex of $T_i$ that equals or is adjacent to a vertex $w'$ of a later trail $T_j$, then we can traverse $T_i'$ starting at $w$, enter $T_j$ at $w'$, and continue to an end of $T_j$, replacing $T_i$ by a trail with at least two more new vertices.

**Claim 4:** If $T_i$ is not closable, then there is no edge to a later trail from an endpoint of $T_i$ or from its neighbor along $T_i$. Any such edge permits an extension of $T_i$ (or of $T_i$ minus its endpoint) using a portion of $T_j$ that has at least two new vertices, thereby creating a trail with more new vertices than $T_i$.

The start vertices of $T_i$ are its endpoints if $T_i$ is not closable, or all of its vertices if $T_i$
is closable. By Claim 3, every start vertex of $T_i$ is a new vertex of $T_i$.

**Claim 5:** No vertex of $T_i$ is adjacent to two start vertices of later trails, or to a start vertex of $T_i$ and a start vertex of a later trail. Suppose $w \in V(T_i)$ has neighbors $x, y$ that are start vertices of $T_j, T_k$, respectively, with $i \leq j \leq k$ and $i \neq k$. By Claim 3, $T_i$ is not closable. By the “newness” of start vertices (and by Claim 4 if $i = j$), $wx, wy$ do not belong to $E(T_i)$. If $j = k$, $T_i$ could thus absorb a portion of $T_j$ that contains a new vertex, giving $T_i$ more new vertices. Hence we may assume $i \leq j < k$. In this case, $wy \notin E(T_j)$, since $y$ is new in $T_k$. If $T_j$ is closable, then $j > i$ and Claim 3 yields $wx \notin E(T_j)$. If $T_j$ is not closable, then Claim 1, Claim 4 and the edge $wy$ imply that $wx \notin E(T_j)$. Now $T_j$, which we can view as ending at $y$, can be extended via $w$ to absorb at least two new vertices from $T_k$.

**Claim 6:** If $u, v$ are start vertices of $T_i, T_j$ with $i < j$, then $u, v$ are nonadjacent and have no common neighbor. By Claims 3 and 4, a start vertex of $T_i$ has no neighbor in a later trail. No start vertex of $T_i$ has a neighbor outside all trails, because such a neighbor could be used to enlarge the new set of $T_i$. By Claim 5, $u$ and $v$ have no common neighbor in trail $T_i$ or earlier.

**Claim 7:** If $E_i$ contains a triangle with at most one edge on $T_i$, then $E_i$ can be represented using $|E_i|$ intervals. If the vertices of $T_i$ are $v_1, \ldots, v_n$ in order (with repetition), then we represent $T_i$ by assigning the interval $(j - 2/3, j + 2/3)$ to $v_j$. This uses $e(T_i) + 1$ intervals. For each additional edge $e \in E_i$ that is not in the triangle, suppose $e = xv_j$ where $v_j$ is a new vertex of $T_i$. We represent $e$ by adding a small interval for $x$ within $(j - 1/3, j + 1/3)$ (intersecting only the interval for $v_j$). If the triangle in $E_i$ contains an edge $v_jv_{j+1}$ of $T_i$, then we add an interval for their common neighbor in $E_i$ within $(j + 1/3, j + 2/3)$, gaining two edges for one interval. If it contains no edge of $T_i$, we select some $v_j \in V(T_i)$ on the triangle and add a common interval for the other two vertices of the triangle within $(j - 1/3, j + 1/3)$, gaining three edges for two intervals. In total, we have used $|E_i|$ intervals.

Now, let $S_i$ consist of the start vertices of $T_i$ and their neighbors. By Claim 6, the sets $S_i$ are pairwise disjoint. Choose two start vertices $u, v$ in $T_i$ with the minimum degree sum. If $d(u) + d(v) \leq 2\Delta - 2$, then by the discussion before the theorem statement we do not need to save an interval for $T_i$. If $d(u) + d(v) = 2\Delta - 1$, then one of $u, v$ has degree $\Delta$ and we have $|S_i| > \Delta$.

Hence we may assume that every start vertex of $T_i$ has degree $\Delta$. By the computation before the theorem statement, it remains only to show that $|S_i| \geq 2\Delta$ if $E_i$ does not contain a triangle with at most one edge on $T_i$. If some pair of start vertices on $T_i$ has no common neighbor, then $|S_i| \geq 2\Delta$, so we may assume that every pair of start vertices has a common neighbor.

Suppose first that $T_i$ is not closable. Let $u, v$ be the endpoints of $T_i$, and let $w$ be a common neighbor; by Claim 5 $w$ does not belong to an earlier trail. If neither of $\{uw, vw\}$ belongs to $T_i$, then $T_i$ can be extended by $ww$ to obtain a closable trail with the same new
set as $T_i$, which would be preferred to $T_i$. This includes the case where $u, v$ are adjacent and $T_i$ has length 1. In the remaining case, $u, v$ are nonadjacent and any common neighbor of them is adjacent to one of them using an end edge of $T_i$. There are at most two such common neighbors. Hence $|S_i| \geq 2 + 2\Delta - 2 = 2\Delta$, as desired.

Finally, suppose that $T_i$ is closable, which requires at least three vertices, each pair of which has a common neighbor. By Claims 3 and 5, the common neighbors of vertices in $T_i$ also lie in $T_i$. Furthermore, every edge of $T'_i$ forms a triangle only using two other edges of $T'_i$; otherwise, we can use the endpoints of that edge as the endpoints of $T_i$ and use the common neighbor to form a triangle having at most one edge on $T_i$.

Since $T'_i$ forms a connected subgraph of $G$, it has a vertex $w$ that is not a cut-vertex of $T'_i$. Deleting from $T'_i$ any set of edges incident to $w$ leaves a connected subgraph, except possibly for isolating $w$. Every edge $wv$ incident to $w$ in $T'_i$ lies on a triangle in $T'_i$; let $u$ be the third vertex of this triangle. Deleting \{wv, uw\} from $T'_i$ leaves a subgraph having a $u, v$-Eulerian trail $T$. Now $u, v, w$ form a triangle with only one edge on $T$. Furthermore, every edge of $E_i$ is incident to at least one vertex of $T$, because when $T_i$ is closable every edge of $E_i$ has both endpoints on $T_i$. By the construction in Claim 7, we can use represent $E_i$ using only $|E_i|$ intervals, saving one for the edges \{wv, uw\}.

We have resolved all cases, and the proof of the bound is complete. Next we consider how equality may be achieved. We may assume that $G$ is connected. It suffices to show that if $G \neq K_{\Delta, \Delta}$, then we save an extra interval for $T_i$.

If $T_i$ is closable, then Claim 3 implies that $V(T_i) = V(G)$, and hence $E_i = E(G)$. If $G$ is not $\Delta$-regular or if $n > 2\Delta$, then $|E(G)| + 1 < (\Delta + 1/\Delta)n/2$, and we are done. If $G$ is $\Delta$-regular and $\Delta \geq n/2$, then $G$ is Hamiltonian, by Dirac’s Theorem. If $G \neq K_{\Delta, \Delta}$, then $G$ has a triangle, by Turán’s Theorem. By the choice of $T_i$ to minimize length, $T'_i$ is a Hamiltonian cycle. If $T'_i$ uses any edge of the triangle, then we delete that edge from $T'_i$ to obtain $T_i$. Hence we can choose $T_i$ so that we have a triangle with at most one edge on $T_i$. By Claim 7, we can now represent $E(G)$ using $|E(G)| = \Delta n/2$ edges.

If $T_i$ is not closable, recall the computation of our bound on $I(G)$. We have $I(G) \leq \Delta n/2 + (2s - r)/2$, where there are $s$ trails $T_i$ using $|E_i| + 1$ intervals and $r = \sum \alpha_i$, with $2\Delta - \alpha_i$ being the sum of the degrees of the chosen vertices on $T_i$. We proved the bound by associating $(2s - r)\Delta$ vertices with these trials, since then $(2s - r)\Delta \leq n$. We have strict inequality if some $T_i$ uses only $|E_i|$ intervals (since its start vertices are not in the sets $S_j$ associated with other trails) or if for some $T_i$ the associated set $S_i$ has more than $2\Delta - \alpha_i$ vertices.

Now consider $T_i$, with endpoints $u, v$. If $d(u) + d(v) < 2\Delta$, then $|S_i| > (2 - \alpha_i)\Delta$ immediately, so we may assume $d(u) + d(v) = 2\Delta$. By Claim 4, all neighbors of $v$ lie in $T_i$. By Claim 1, only one edge incident to $v$ belongs to $T_i$. Since our greedy selection prefers closable trails, this implies that $u, v$ are not adjacent (unless $uv \in E(T_i)$, in which case $\Delta = 1$ and $G = K_{\Delta, \Delta}$). Suppose the vertices of $T_i$ are $u = x_1, x_2, \ldots, x_m = v$ in order. If $v$ is adjacent to both $x_i$ and $x_{i+1}$ for some $i < m - 2$, then it forms a triangle with one edge on $T_i$, and Claim 7 applies. If $i = m - 2$, then because $v$ has no neighbors on later
trails, we can again save an interval for this triangle. Hence we may assume that \( v \) does
not have consecutive neighbors on \( T_i \). If \( v \) is adjacent to \( x_i \) for \( i < m - 1 \), then \( x_{i+1} \) has no
neighbor \( w \) in another trail \( T_j \). Otherwise, we could follow \( x_1, \ldots, x_i, v, x_{m-1}, \ldots, x_{i+1}, w \)
and continue in \( T_j \) to enlarge the new set of \( T_1 \). Since the successors on \( T_1 \) of neighbors of
\( v \) have no neighbors in later trails, they appear in no later \( S_j \), and we can add them to \( S_1 \).
Now \( S_1 \) consists of at least the neighbors of \( v \), their successors on \( T_1 \), and \( u \), which totals
\( 2\Delta + 1 \) vertices.

The alterations that are used to improve trails always increase the new set or decrease
the length (or make it closed); there can be at most \( n^2 \) of these changes for each trail.
Also the search for whether a change is needed takes polynomial time. Hence this proof
can be implemented as a polynomial algorithm to produce a representation that satisfies
the bound.

References


