THE EDGE-BANDWIDTH OF THETA GRAPHS

Dennis Eichhorn, Dhruv Mubayi, Kevin O'Bryant, and Douglas B. West

Abstract. An *edge-labeling* f of a graph G is an injection from $E(G)$ to the set of integers. The *edge-bandwidth* of G is $B'(G) = \min_f\{B'(f)\}$, where $B'(f)$ is the maximum difference between labels of incident edges of G. The *m-theta graph* $\Theta(l_1, \dots, l_m\}$ is the graph consisting of m pairwise internally disjoint paths with common endpoints and lengths $l_1 \le \cdots \le l_m$.

We determine the edge-bandwidth of all m-theta graphs.

1. Introduction

A *labeling* of a graph G is an injection from $V(G)$ to the set of integers. Given a labeling f, let $B(f)$ be the maximum difference between labels of vertices adjacent in G. The *bandwidth* of G, written $B(G)$, is the minimum of $B(f)$ over all labelings f of G.

The concept of bandwidth arose in the study of numerical matrix inversion. If the nonzero entries of a symmetric matrix lie in a band at most k steps from the main diagonal, then performing Gaussian elimination by pivoting only on positions within this band will not introduce nonzero entries outside the band. Ignoring the positions outside the band speeds up the computation. ¿From the matrix, we obtain the adjacency matrix of a graph by replacing nonzero entries with 1's. An optimal bandwidth labeling of the resulting graph yields a simultaneous renumbering of the rows and columns to confine the nonzero entries in the smallest possible band.

Additional applications have arisen more recently in computer science involving scheduling and simulation problems. For example, in a network of processors, certain pairs of processors are able to communicate directly, thus defining a graph. This graph can be used to simulate other graphs on which parallel algorithms may have been developed. A *simulation* f maps each vertex of G to a vertex in a host graph H and specifies for each $xy \in E(G)$ an $f(x), f(y)$-path in H to simulate it. The *dilation* of the simulation is the maximum length among paths in H used to represent edges in G. The bandwidth of a graph G is the minimum dilation among injective simulations of G using a path as the host graph.

Computing the bandwidth of graphs in general is NP-complete [3]. Moreover, this is even true of trees with maximum degree 3 [1]. It is thus of interest to consider this problem on

Typeset by $\mathcal{AMS}$-TEX

restricted classes of graphs. We study a parameter that corresponds to the restriction of the bandwidth parameter to the class of line graphs.

The parameter analogous to bandwidth for edge-labelings is introduced in [2]. An *edge-labeling* f of a graph G is an injection from $E(G)$ to the set of integers. Let $B'(f)$ be the maximum difference between labels of incident edges of G. The *edge-bandwidth* of G, written $B'(G)$, is the minimum of $B'(f)$ over all edge-labelings f of G. Clearly $B'(G)$ is equal to the bandwidth of the line graph of G.

The *m-theta graph* $\Theta(l_1, \ldots, l_m)$ is the graph consisting of m pairwise internally disjoint paths $P_1, \ldots, P_m$ with common endpoints v, w and lengths $l_1 \leq \cdots \leq l_m$, respectively. The *left end-edges* or simply *left ends* of $\Theta(l_1, \ldots, l_m)$ are the edges incident to v. The *right end-edges* or simply *right ends* of $\Theta(l_1, \ldots, l_m)$ are the edges incident to w. Thus, a left end is a right end if and only if the corresponding path has length one.

Determining the bandwidth of theta graphs was a difficult process completed in [4]. This class of graphs is especially interesting for studying bandwidth because none of the standard techniques for obtaining lower bounds yields the correct answer. This remains true for the edge-bandwidth of theta graphs; also, the optimal constructions are far from obvious. In [2], the effect of graph operations like subdivision and contraction of edges on edge-bandwidth was studied; theta graphs were investigated because they showed that many of the results were sharp. In this paper, we determine the edge-bandwidth of all theta graphs.

2. Degenerate Cases

In this section, we determine the edge-bandwidth of some special classes of m-theta graphs. These results simplify the arguments in later sections. Since the edge-bandwidth is 1 when $m = 1$ and is 2 when $m = 2$, we henceforth assume that $m > 2$. Since the m pairwise incident left ends receive m distinct labels, it is always the case that $B'(\Theta(l_1, \ldots, l_m)) \geq m-1$. Equality rarely holds.

Proposition 2.1. *If $B'(\Theta(l_1, \ldots, l_m)) = m - 1$, then $l_1 = \cdots = l_m = 1$.*

Proof. Suppose that $l_m > 1$ and that f is an edge labeling of $\Theta(l_1, \ldots, l_m)$ with $B'(f) = m-1$. We may assume without loss of generality that the minimum label among all those appearing on the ends of the v, w-paths is on a left end. Since the left ends are pairwise incident, the labels on the m left ends fill an interval $[p, p + m - 1]$. Similarly, the labels on the right ends fill an interval $[q, q + m - 1]$. Since the labels on left ends and right ends are not the same set, our assumption yields $p < q$. Thus the v, w-path with label p on its left end has length at least 2. Since this path starts with label p, ends with a label larger than p, and has no label in $[p + 1, p + m - 1]$, some pair of labels on consecutive edges on this path differ by at least m. This contradicts $B'(f) = m - 1$. □

Proposition 2.2. *Suppose that $m > 2$ and $l_m > 1$. If $l_1 = 1 < 2 = l_m$, or if $l_1 = l_{m-1} = s$, then $B'(\Theta(l_1, \ldots, l_m)) = m$.*

Proof. By Proposition 2.1, it suffices to prove the upper bounds. In the two cases above, we assign labels to the edges as illustrated in Figures 1 and 2. □

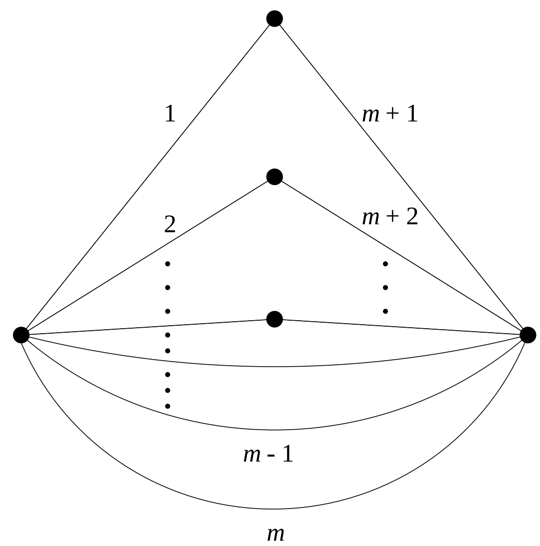

Figure 1. The case $l_1 = 1 < 2 = l_m$

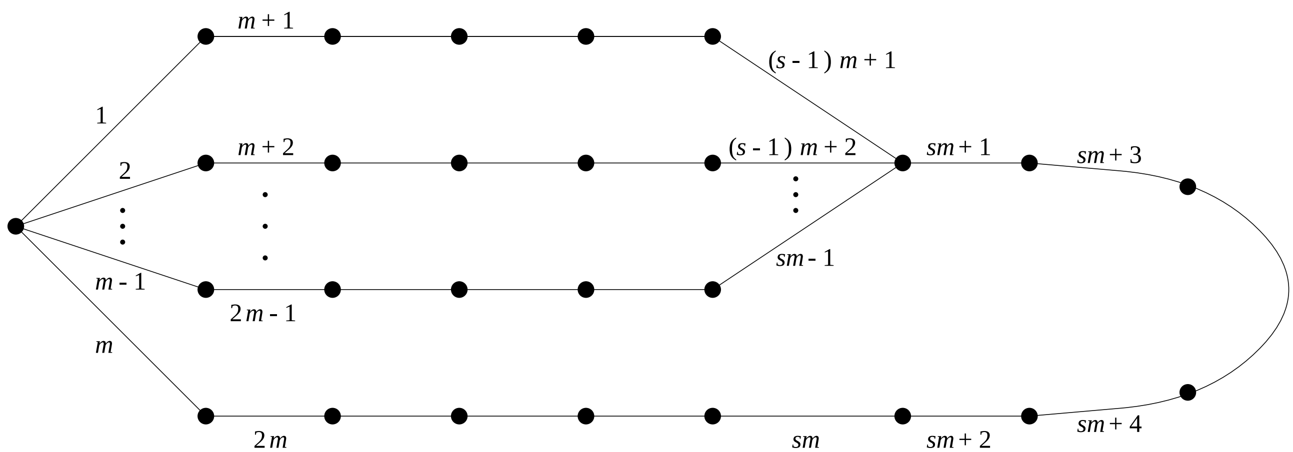

Figure 2. The case $l_1 = l_{m-1} = s$

3. The Lower Bound

In this section we obtain lower bounds on the edge-bandwidth of $\Theta(l_1, \dots, l_m)$. By the results in Section 2, we may restrict our attention to the case where $m > 2$, $l_m > 2$, and $l_1 \neq l_{m-1}$. The edge-bandwidth of such graphs depends on a crucial invariant.

Definition 3.1. Suppose that $m \geq 3$ and $G = \Theta(l_1, \dots, l_m)$, where $l_m > 2$ and $l_1 \neq l_{m-1}$. Let $s = l_1$ and $c = \max\{i : l_i = s\}$. Define $U(G)$ to be the largest integer u such that

$$\frac{(s-1)(3m-2-u)}{2} \geq \left\lceil \sum_{i=c+1}^{u} (l_i - 2) \right\rceil + (s-1)(m-u+c). \qquad \square$$

When $u = c$ and $s > 1$, the inequality reduces to $m - c \geq 2$, which holds for all values of m and c as defined above. When $u = c$ and $s = 1$, both sides of the inequality are 0. When

$u = m$, the summation has value at least $(m-c)(s-1)$, so the right side of the inequality is at least $m(s-1)$, and the inequality cannot hold. Moreover, the right side of the inequality is a nondecreasing function of u in $[c, m]$, since $l_i - 2 \geq s - 1$ for $i > c$, and the left side of the inequality is a nonincreasing function of u. These remarks together imply that $U(G)$ is well defined.

Theorem 3.2. *Let $G = \Theta(l_1, \ldots, l_m)$, where $m \geq 3$, $l_m > 2$ and $l_1 \neq l_{m-1}$. If $U(G)$ is defined as in Definition 3.1, then*

$$B'(G) \geq \left\lceil \frac{3m - 2 - U(G)}{2} \right\rceil .$$

Proof. Consider an edge-labeling f of G with $B'(f) = k$. We may assume without loss of generality that the minimum label among all those appearing on the ends of the v, w-paths is on a left end. Let p be the maximum label that appears on the left ends, and let q be the minimum label that appears on the right ends. We give lower bounds in the cases $p \geq q$ and $p < q$.

Case 1: $p \geq q$. Consider the interval $I = [p-k, q+k]$, which contains $2k+q-p+1$ integers. Each v, w-path with length at least two has at least two labels from I, since the labels on both ends come from I. Let $I' = [q, p]$; note that $I' \subseteq I$. By the choice of p and q, every v, w-path of length one must contain a label from I'. If a v, w-path with length at least three has exactly two labels from I, then it must also contain a label from I', since avoiding this interval forces labels of at least two consecutive edges on this path to differ by more than k. The number v, w-paths with length at least three having exactly two labels from I is therefore at most $p - q + 1 - d_1$, where d_j is the number of paths of length j.

By counting one label for every path of length one, two labels for every path of length at least two, and an additional label for every path of length at least three that must have a third label from I, we obtain

$$d_1 + 2(m - d_1) + [m - d_1 - d_2 - (p - q + 1 - d_1)] \leq |I| = 2k + q - p + 1,$$

which simplifies to

$$k \geq \frac{3m - 2 - d_1 - d_2}{2} \tag{1}$$

in the case $p \geq q$.

Recall from Definition 3.1 that $s = l_1$ and $c = \max\{i : l_i = s\}$. When $s = 1$, it follows immediately from Definition 3.1 that $d_1 + d_2 = U(G)$. When $s = 2$, $d_1 + d_2 = d_2 = c \leq U(G)$ by the argument following Definition 3.1. When $s > 2$, $d_1 + d_2 = 0 < U(G)$. Thus, in the case $p \geq q$,

$$k \geq \left\lceil \frac{3m - 2 - U(G)}{2} \right\rceil$$

as required.

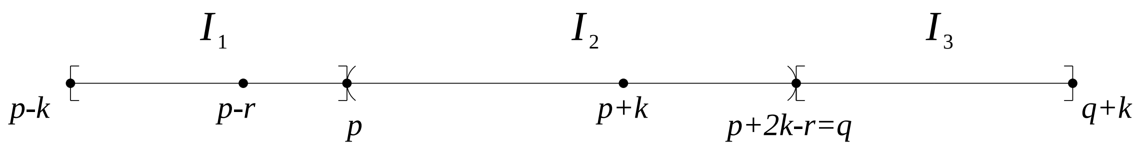

Figure 3. The intervals I_1, I_2, and I_3 when $s' = 2$ and $p < q$

Case 2: $p < q$. Set $I_1 = [p-k, p]$, $I_2 = (p, q)$, and $I_3 = [q, q+k]$ (see Figure 3). Recall from Definition 3.1 that $s = l_1$ and $c = \max\{i : l_i = s\}$. The left end of each v, w-path has a label from I_1, and the right end of each v, w-path has a label from I_3. Thus, each v, w-path has at least two labels from $I_1 \cup I_3$ (note that since $p < q$, we have $s \geq 2$). Let $\mathcal{P}$ denote the set of v, w-paths, let $\mathcal{P}' \subseteq \mathcal{P}$ be the set of paths with exactly two labels outside I_2, let $\mathcal{P}'' \subseteq \mathcal{P}$ denote the set of all paths of length s, let $u = |\mathcal{P}'|$, and let $c' = |\mathcal{P}' \cap \mathcal{P}''|$.

The intervals I_1 and I_3 together contain $2k + 2$ integers. On the other hand, these intervals contain at least three labels from each path in $\mathcal{P} - \mathcal{P}'$ since having a label to the left of I_1 or to the right of I_3 forces the use of an extra label in I_1 or I_3, respectively. The intervals I_1 and I_3 together also contain two labels from each path in $\mathcal{P}'$. Thus

$$3(m-u) + 2u \leq |I_1 \cup I_3| = 2k + 2,$$

which gives

$$k \geq \frac{3m - 2 - u}{2}. \tag{2}$$

As with all v, w-paths, the labels on the v, w-paths of length s begin in I_1 and end in I_3. Since they do not increase or decrease by more than k with each step, we have $q - p \leq (s-1)k$. Suppose that $q - p = s'k - r$, where $0 < s' < s$ and $0 \leq r < k$. Exactly $l - 2$ labels from every path of length l in $\mathcal{P}' - \mathcal{P}''$ lie in I_2. Every path not in $\mathcal{P}' - \mathcal{P}''$ must use at least $s' - 1$ labels from I_2. A v, w-path that uses exactly $s' - 1$ labels from I_2 must use a label from $[p - r, p]$, and hence the number of such paths is at most $r + 1$. Thus the $m - u + c'$ v, w-paths not in $\mathcal{P}' - \mathcal{P}''$ use at least $(m - u + c')s' - (r + 1)$ labels from I_2. Letting $l(P)$ denote the length of the path P, we obtain

$$\left[\sum_{P \in \mathcal{P}' - \mathcal{P}''} (l(P) - 2)\right] + [(m - u + c')s' - (r + 1)] \leq |I_2| = s'k - r - 1.$$

Rearranging this inequality and using the inequality $s' \le s-1$ yields

$$\begin{aligned} k &\ge \frac{1}{s-1}\left[\sum_{P\in\mathcal{P}'-\mathcal{P}''}(l(P)-2)\right]+m-u+c' \\ &\ge \frac{1}{s-1}\left[\sum_{i=c+1}^{u+c-c'}(l_i-2)\right]+m-u+c' \\ &\ge \frac{1}{s-1}\left[\sum_{i=c+1}^{u}(l_i-2)\right]+m-u+c. \qquad (3) \end{aligned}$$

Combining (2) and (3) we have

$$\begin{aligned} k &\ge \max\left\{\frac{3m-2-u}{2}\ ,\ \frac{\sum_{i=c+1}^{u}(l_i-2)}{s-1}+m-u+c\right\} \\ &\ge \min_u \max\left\{\frac{3m-2-u}{2}\ ,\ \frac{\sum_{i=c+1}^{u}(l_i-2)}{s-1}+m-u+c\right\} \end{aligned}$$

Now, we wish to choose u so that the maximum is minimized. A brief analysis of integer parts shows that we may choose u to be the largest integer in $[0,m]$ such that $(3m-2-u)/2 \ge \sum_{i=c+1}^{u}(l_i-2)/(s-1)+m-u+c$, i.e., $u=U(G)$. Thus, in the case $p<q$,

$$k \ge \left\lceil \frac{3m-2-U(G)}{2} \right\rceil$$

as required. □

4. A Sharp Upper Bound; the "Step-Through" Construction

In this section we show that the lower bound in Theorem 3.2 is sharp for all theta graphs by exhibiting an edge-labeling of $\Theta(l_1,\dots,l_m)$ that achieves this lower bound. The following definition is introduced to simplify our construction.

Definition 4.1. Let P be a path with consecutive edges $e_1,\dots,e_l$. Suppose that $l\ge 2$ and let z be a positive integer. A *lower z-bent edge-labeling* f of P is defined as follows: First $f(e_l)$ and $f(e_1)$ are chosen so that $f(e_l)-f(e_1)>z$. Next, index the edges also as $e'_1,\dots,e'_l$ so that $(e'_1,e'_2,\dots)=(e_l,e_1,e_{l-1},e_2,\dots)$. If $l>2$, then let $f(e_{l-1})=f(e_1)-z$. For $4\le i\le l$, let $f(e'_i)=f(e'_{i-1})-z$.

Example 4.2. Figure 4 illustrates a lower 4-bent edge-labeling of a path of length 9 (note that as long as the condition $f(e_9)-f(e_1)>4$ is satisfied, the value of $f(e_9)$ has no effect on the other labels).

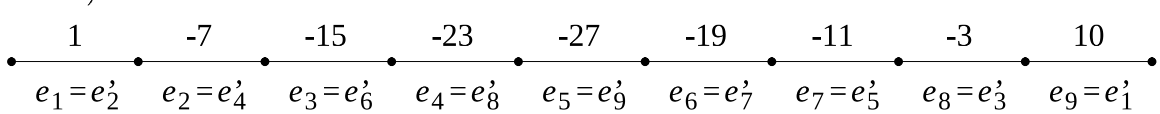

Figure 4.

Lemma 4.3. *If f is a lower z-bent edge-labeling of a path with edges $e_1, \dots e_l$, then $B'(f) \leq z + f(e_l) - f(e_1)$. Furthermore, $f(e_i) \equiv f(e_1) \pmod z$ for $1 \leq i \leq l-1$.*

Proof. Apart from the pair $(f(e_{l-1}), f(e_l))$, the labels on consecutive edges of P differ by $2z$ except for one difference of z in the middle. Since $z < f(e_l) - f(e_1)$ and $f(e_{l-1}) = f(e_1) - z$, we have $B'(f) = f(e_l) - f(e_{l-1}) = z + f(e_l) - f(e_1)$. □

Remark 4.4. *Upper z-bent edge-labelings* are defined analogously. Instead of the labels first decreasing and then increasing, the labels first increase by $2z$ and then decrease by $2z$ with e_1 getting the smallest label on the path. We omit the formality of defining this rigorously.

Theorem 4.5. *Let $G = \Theta(l_1, \dots, l_m)$, where $m \geq 3$, $l_m > 2$ and $l_1 \neq l_{m-1}$. If $U(G)$ is defined as in Definition 3.1, then*

$$B'(G) = \left\lceil \frac{3m - 2 - U(G)}{2} \right\rceil.$$

Proof. The lower bound follows from Theorem 3.2. The upper bound is established by exhibiting an edge-labeling f with $B'(f) = k = \left\lceil \frac{1}{2}(3m - 2 - U(G)) \right\rceil$. We first specify a collection of intervals and then specify labels on edges within these intervals. With $u = U(G)$, set

$$x = \left\lfloor \frac{m-u}{2} \right\rfloor \qquad \text{and} \qquad z = \left\lceil \frac{m-u}{2} \right\rceil.$$

We will obtain the desired bound on $B'(G)$ by noting that

$$k = 2z + x + u - 1. \tag{5}$$

Recall from Definition 3.1 that $s = l_1$ and $c = \max\{i : l_i = s\}$; also let $a_j = jk$. Let $I_0 = [c, a_1]$ and $I_s = (a_s + c, a_{s+1} + 1]$. For $1 \leq j \leq s-1$, let $I_j = (a_j + c, a_{j+1}]$.

For $1 \leq j \leq s-1$, partition I_j into three intervals:

$$X_j = (a_j + c, a_j + c + x], \qquad Y_j = (a_j + c + x, a_{j+1} - z], \qquad Z_j = (a_{j+1} - z, a_{j+1}].$$

Partition the paths $P_1, \dots, P_m$ into four sets $\mathcal{P}_T$, $\mathcal{P}_U$, $\mathcal{P}_X$, $\mathcal{P}_Z$, where

$$\mathcal{P}_T = \{P_1, \dots, P_c\},$$

$$\mathcal{P}_U = \{P_{c+1}, \dots, P_u\},$$

$$\mathcal{P}_X = \{P_{u+1}, \dots, P_{u+x}\},$$

$$\mathcal{P}_Z = \{P_{u+x+1}, \dots, P_m\}.$$

For $P_i \in \mathcal{P}_T$, let the labels on the edges of path P_i be $a_1 + i, a_2 + i, \dots, a_s + i$, in order from left to right. The labels on paths from $\mathcal{P}_Z$ will begin in I_0, then go below this interval, and then travel back up through $I_0, \dots, I_{s-1}$ and terminate in I_s. The labels on paths from $\mathcal{P}_X$ will begin in I_0, then go all the way above I_s, and finally end in I_s. The labels on paths from

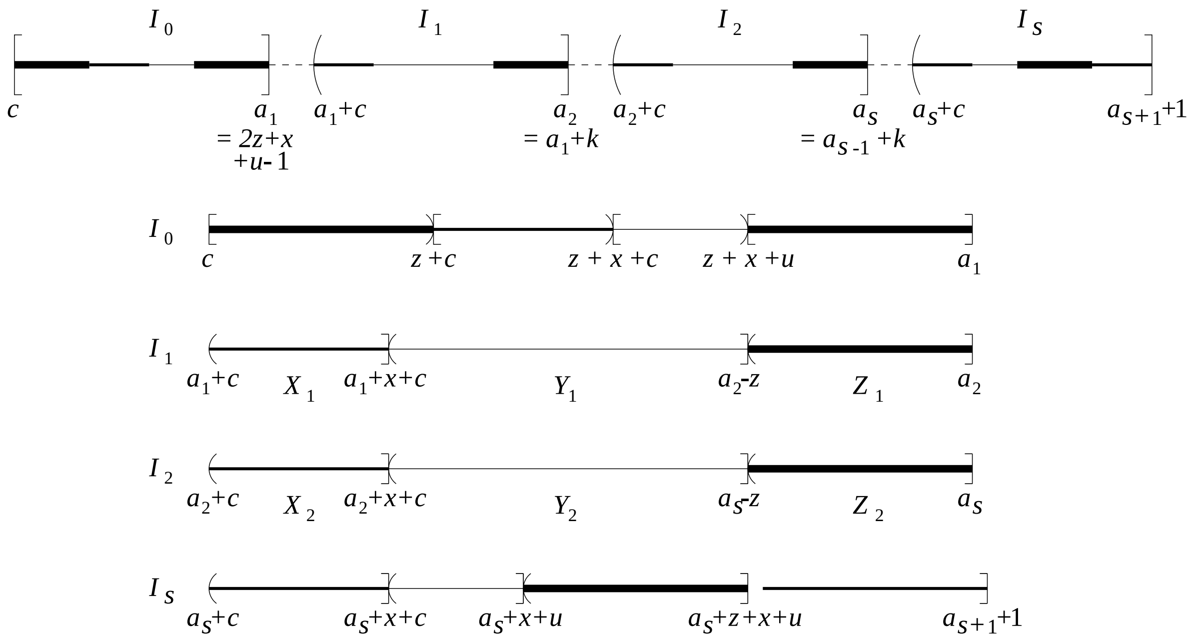

Figure 5. The "step-through" construction when $s = 3$, $x < z$

$\mathcal{P}_U$ will go neither above I_s nor below I_0; they will "step through" the middle in $I_0, \ldots, I_s$ (see Figure 5).

We first describe the labeling of paths from $\mathcal{P}_Z$. Suppose P is a path from $\mathcal{P}_Z$ with consecutive edges $e_1, e_2, \ldots, e_l$ from left to right. Assign an integer $q_P \in [c, z + c)$ to P (different paths are assigned distinct integers). Let $f(e_{l-s}) = z + x + u - c + q_P$. If $l > s + 1$, then let $f(e_1) = q_P$ and denote the $e_1, \ldots, e_{l-s}$ portion of P by P'. Let f restricted to P' be a lower z-bent edge-labeling as described in Definition 4.1. Lemma 4.3 and (5) together imply that labels on adjacent edges of P' differ by at most $z + f(e_{l-s}) - f(e_1) = k - c + 1 \leq k$. Also, each of these labels is congruent to $q_P \pmod z$. Thus the paths from $\mathcal{P}_Z$ receive disjoint sets of labels, and therefore the labeling of these paths is injective. For $1 \leq j \leq s - 1$, let $f(e_{l-s+j}) = f(e_{l-s}) + jk$, and let $f(e_l) = a_s + x + u - c + 1 + q_P$. The difference between adjacent labels on P is at most k. Observe that $f(e_{l-s+j}) \in Z_j$ for $1 \leq j \leq s - 1$.

The labels on paths from $\mathcal{P}_X$ are obtained similarly to those on paths from $\mathcal{P}_Z$, except that we begin labeling from the other direction. The interval I_s plays the role that I_0 played. Instead of using integers less than c for a lower z-bent edge-labeling, we use integers greater than $a_{s+1} + 1$ for an upper x-bent labeling. Instead of using the intervals $Z_1, \ldots, Z_{s-1}$, we use the intervals $X_1, \ldots, X_{s-1}$.

It remains to label the $u - c$ paths from $\mathcal{P}_U$. We label these paths using integers from the intervals $I_0, Y_1, Y_2, \ldots, Y_{s-2}, Y_{s-1}, I_s$. Each of the left ends of paths from $\mathcal{P}_U$ will have labels from $[z + x + c, z + x + u) \subseteq I_0$. The right ends of these paths will have labels from $(a_s + x + c, a_s + x + u] \subseteq I_s$. The remaining labels on each of these paths come from the intervals $Y_1, \ldots, Y_{s-1}$. Since each Y_j contains $k - (x + z + c) = z + u - c - 1$ elements, these intervals together contain $(s - 1)(z + u - c - 1)$ elements. On the other hand, we require $\sum_{i=c+1}^{u}(l_i - 2)$

labels. By the definition of u,

$$\sum_{i=c+1}^{u}(l_i-2) \le \frac{(s-1)(3m-2-u)}{2} - (s-1)(m-u+c)$$
$$\le (s-1)\left(\frac{m-u}{2}+u-c-1\right) \le (s-1)(z+u-c-1),$$

so we have enough labels.

Consider $P \in \mathcal{P}_U$, with edges $e_1,\dots,e_l$ from left to right. Assign an integer $q_P \in (z+x+c, z+x+u]$ to P with distinct integers assigned to distinct paths. Let $f(e_1)=q_P$ and $f(e_l)=a_s-z+1+q_P \in I_s$. The integers $a_j-z+1+q_P \in Y_j$ $(0<j<s)$ will all be used to label the edges of P. Let Y' be the set of integers thus used for all paths from $\mathcal{P}_U$. Since $l \ge s+1$, even after using all these labels, the edges of each $P \in \mathcal{P}_U$ require $l-(s+1)$ more labels. These labels can be chosen arbitrarily from $(\cup_{j=1}^{s-1} Y_j) - Y'$. We label in this way, making sure only that $f(e_j) < f(e_{j'})$ whenever $j<j'$. It is readily observed that the labels on adjacent edges of P differ by at most k.

Since labels on adjacent edges of the v,w-paths of length s differ by at most k, we have

$$B'(f) = k = 2z+x+u-1 = \left\lceil \frac{3m-2-u}{2} \right\rceil. \qquad \square$$

References

[1] M. R. Garey, R.L. Graham, D.S. Johnson, D.E. Knuth, *Complexity results for bandwidth minimization*, SIAM J. Appl. Math. **34** (1978), 477–495.

[2] T. Jiang, D. Mubayi, A. Shastri, D. B. West, *Edge-Bandwidth of graphs*, SIAM J. Discrete Math. (to appear).

[3] Papadimitriou, *The NP-completeness of the bandwidth minimization problem*, Computing **16** (1976), 263–270.

[4] G. W. Peck, A. Shastri, *Bandwidth of theta graphs with short paths*, Discrete Math. **142** (1992), 177–187.

Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, Illinois 61801
E-mail address: eichhorn@math.uiuc.edu

Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, Illinois 61801
E-mail address: obryant@math.uiuc.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, GA, 30332-0160
E-mail address: mubayi@math.gatech.edu

Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, Illinois 61801
E-mail address: west@math.uiuc.edu