THE SUPERREGULAR GRAPHS

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Abstract. A regular graph is *superregular* if it has no vertices or if the subgraphs induced by the neighbors and by the nonneighbors of each vertex are superregular. The superregular graphs are precisely the disjoint union of m isomorphic cliques, the Cartesian product of two isomorphic cliques, the five-cycle, and the complements of these graphs.

1. INTRODUCTION

In problem 6617 of the *American Mathematical Monthly* [4], Andrew Vince requested a characterization of special regular graphs he called "superregular graphs". Given a graph G, let $N(x) = \{v \in V(G-x)\colon\ vx \in E(G)\}$ and $\bar{N}(x) = \{v \in V(G-x)\colon\ vx \notin E(G)\}$, and let $G[U]$ denote the subgraph of G induced by vertex set U. The *subconstituents* of G are the graphs arising as $G[N(x)]$ or $G[\bar{N}(x)]$ over all $x \in V(G)$. A regular graph is *superregular* if it has no vertices or if all its subconstituents are superregular. Let $\mathbf{S}$ be the class consisting of aK_b for all $a, b \geq 0$ (disjoint union of isomorphic cliques), $K_m \square K_m$ for all $m \geq 0$, (Cartesian product of two isomorphic cliques), the graph C_5, and the complements of these graphs. In this note, we prove that G is superregular if and only if $G \in \mathbf{S}$.

Since the the subconstituents of G are the complements of the subconstituents of its complement $\bar{G}$, the set of superregular graphs is closed under complementation. As discussed in Lemma 1, the subconstituents of graphs in $\mathbf{S}$ belong to $\mathbf{S}$; hence by induction all graphs in $\mathbf{S}$ are superregular. Reflecting the recursive definition, we also use induction to prove that no other graph is superregular, by proving that a superregular graph whose subconstituents all belong to $\mathbf{S}$ must also belong to $\mathbf{S}$.

A k-regular graph is *strongly regular* with parameters (k, λ, μ) if every adjacent pair of vertices has λ common neighbors and every nonadjacent pair of vertices has μ common neighbors. Chapter 2 of [2] studies such graphs at length. We prove that superregular graphs are strongly regular. The parameters of a strongly regular graph of order n satisfy $(n-k-1)\mu = k(k-\lambda-1)$; this integrality condition helps us to eliminate candidates outside $\mathbf{S}$.

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If G is strongly regular, then $G[N(x)]$ is regular, and regularity of $G[\bar{N}(x)]$ follows then from strong regularity of $\bar{G}$. Hence all subconstituents of a strongly regular graph are regular. Cameron, Goethals, and Seidel [1] (see also Chapter 8 of [2]) studied strongly regular graphs for which *some* vertex has strongly regular subconstituents; every superregular graph satisfies this property. Using graph eigenvalues, they obtain a necessary condition for this property. Here we present a self-contained characterization of the more restricted class of superregular graphs, using only elementary graph-theoretic and number-theoretic arguments. We also note that, not long after this paper was written, R. Maddox of Pepperdine University independently proved the same result.

2. PROPERTIES OF SUPERREGULAR GRAPHS

We use n for the order of G. The *degree* of a regular graph is the degree of its vertices; we use k for the degree of G. We write K_m^2 for the Cartesian product $K_m \square K_m$; this graph consists of m^2 vertices in a square grid, with vertices adjacent when they lie in the same row or in the same column.

LEMMA 1. Every graph in **S** is superregular.

Proof: Each $G \in \mathbf{S}$ is regular and vertex-transitive; choose $x \in V(G)$. By induction on $a+b$, we have superregularity for $G = aK_b$, since $G[N(x)] = K_{b-1}$ and $G[\bar{N}(x)] = (a-1)K_b$. For $G = K_m^2$, we also apply induction, since $G[N(x)] = 2K_{m-1}$ and $G[\bar{N}(x)] = K_{m-1}^2$. Finally, for $G = C_5$, $G[N(x)] = 2K_1$ and $G[\bar{N}(x)] = K_2$. □

The disconnected superregular graphs are easy to characterize.

LEMMA 2. If G is superregular and disconnected, then $G = aK_b$ with $ab > 1$.

Proof: If some component of G is not a clique, then we may choose vertices x, y, z such that y has distance two from x and z belongs to another component. This implies $d_{G[\bar{N}(x)]}(y) < k = d_{G[\bar{N}(x)]}(z)$, which contradicts the regularity of $G[\bar{N}(x)]$. If every component of G is a clique, then regularity of G requires equal sizes. □

LEMMA 3. If G is superregular, then each $G[N(x)]$ has the same degree, and each $G[\bar{N}(x)]$ has the same degree.

Proof: If x, y are adjacent, then $d_{G[N(y)]}(x) = |N(x) \cap N(y)| = d_{G[N(x)]}$. Hence the degree of $G[N(v)]$ is the same for all v in a single component of G. If G is disconnected, then $G = mK_p$, and the statement holds across all components. Since $G[\bar{N}(x)]$ is the complement of $\bar{G}[N(x)]$, the second statement follows by applying the first to $\bar{G}$. □

We can now define s, t as additional parameters of a superregular graph, where s is the degree of each $G[N(x)]$ and t is the degree of each $G[\bar{N}(x)]$. We obtain strong regularity of G and a "balance equation" relating the parameters.

LEMMA 4. If G is superregular, then G is strongly regular, and

$$(n-k-1)(k-t) = k(k-s-1).$$

Proof: If x, y are non-adjacent, then the t-regularity of $G[\bar{N}(x)]$ implies that x and y have $k-t$ common neighbors. We have observed that adjacent pairs have s common neighbors. Hence G is strongly regular, with parameters $\lambda = s$ and $\mu = k-t$. For any strongly regular graph, $(n-k-1)\mu = k(k-\lambda-1)$; both sides count the edges between $N(x)$ and $\bar{N}(x)$. □

For $G \in \mathbf{S}$, the *Type* of G is A, A', B, B', C, if G is $aK_b, \overline{aK_b}, K_m^2, \overline{K_m^2}, C_5$, respectively. These have degrees $b-1, (a-1)b, 2(m-1), (m-1)^2, 2$, respectively.

The inductive step of the main theorem becomes tractable if we can assume that the subconstituents arising as $G[N(x)]$ are all the same and that the subconstituents arising as $G[\bar{N}(x)]$ are all the same. This follows from the next lemma.

LEMMA 5. The set $\mathbf{S}$ has at most one graph of order n and degree k.

Proof: Suppose $\mathbf{S}$ has two graphs G, G' with the same order and degree. Within a Type, the formulas for order and degree allow at most one graph with order n and degree k, so we may assume G, G' have different Types. Use of complementation reduces the cases to those below. There is only one n-vertex graph for $n \le 1$, so we may assume $n \ge 2$.

Type C: Suppose $G = C_5$. Since 5 is not square, we may assume Type A or A' for G'. With $ab = 5$, G' has degree 0 or 4, but C_5 is 2-regular, so the degree condition fails.

Types B, B': Suppose $G = K_m^2$ and $G' = \overline{K_m^2}$. The degree condition $2m-2 = (m-1)^2$ requires $m = 3$. When $m = 3$, G and G' do have the same order and degree, but they are the same graph; K_3^2 is self-complementary.

Types A, B: Suppose $G = aK_b$ and $G' = K_m^2$. We have $m^2 = ab$ and $2m-2 = b-1$. This leads to $2m = b+1$ and $(b+1)^2 = 4m^2 = 4ab$. The right side of the last expression is divisible by b, but the left side is not.

Types A, B': Suppose $G = aK_b$ and $G' = \overline{K_m^2}$. We have $m^2 = ab$ and $(m-1)^2 = b-1$, with $m \ge 2$. If $a = 1$, then $m^2 - 1 = (m-1)^2$, which requires $m = 1$. If $a \ge 2$, then $m^2/2 \ge b$ and $m^2/2 - 1 \ge (m-1)^2$, which requires $m \le 2$. Hence $G = 2K_2$ and $G' = \overline{K_2^2}$, but these are the same graph.

Types A, A': Suppose $G = aK_b$ and $G' = \overline{cK_d}$. We have $ab = cd$ and $b-1 = (c-1)d$. Hence $d|(b-1)$, which implies $\gcd(d,b) = 1$. Now $ab = cd$ implies $d|a$ and hence $d \le a$. Also, $b-1 = (c-1)d$ implies $c \le b$. From $ab = cd$, $d \le a$, and $c \le b$, we obtain $d = a$ and $c = b$. But now $b-1 = (c-1)d$ implies $d = 1$. This yields $G = K_b$ and $G' = \overline{bK_1}$, but these are the same graph. □

COROLLARY 6. If G is a superregular graph with all subconstituents in $\mathbf{S}$, then $\{G[N(x)]\colon x \in V(G)\}$ and $\{G[\bar{N}(x)]\colon x \in V(G)\}$ have only one element each.

Proof: By Lemma 3, the parameters s and t are fixed for all x. The graph $G[N(x)]$ has order k and degree s, and the graph $G[\bar{N}(x)]$ has order $n-k-1$ and degree t. By Lemma 5, only one graph in $\mathbf{S}$ meets each set of requirements. □

For a superregular graph G with all subconstituents in $\mathbf{S}$, we henceforth use $H(G)$ to denote the unique element of $\{G[N(x)]\}$ and $H'(G)$ to denote the unique element of $\{G[\bar{N}(x)]\}$. We next establish a key relationship between $H(G)$ and $H'(G)$. Every use of "containment" or "appearance" of subgraphs refers to the induced subgraph relation.

LEMMA 7. If G is a superregular graph with all subconstituents in $\mathbf{S}$, then $H(G)$ contains $H(H'(G))$, and $H'(G)$ contains $H'(H(G))$.

Proof: If x, y are not adjacent, then the subgraph of G induced by $N(x) \cap \bar{N}(y)$ is $H(G[\bar{N}(y)]) = H(H'(G))$ and appears in $G[N(x)] = H(G)$. If x, y are adjacent, then the subgraph of G induced by $N(x) \cap \bar{N}(y)$ is $H'(G[N(x)]) = H'(H(G))$ and appears in $G[\bar{N}(y)] = H'(G)$. □

Since every subconstituent of a graph in $\mathbf{S}$ belongs to $\mathbf{S}$, Lemma 7 requires inclusions between induced subgraphs belonging to $\mathbf{S}$. Many such inclusions are impossible.

LEMMA 8. If $p \geq 3$, then aK_b and $\overline{aK_b}$ do not contain K_p^2 or $\overline{K_p^2}$. If $p = 2$, then aK_b does not contain K_p^2 and $\overline{aK_b}$ does not contain $\overline{K_p^2}$. If $c, d \geq 2$, then aK_b does not contain $\overline{cK_d}$, and $\overline{aK_b}$ does not contain cK_d.

Proof: A disjoint union of cliques contains no P_3, and a complete multipartite graph contains no $\overline{P_3}$. The self-complementary graph K_3^2 contains both P_3 and $\overline{P_3}$. If $c, d \geq 2$, then $\overline{cK_d}$ contains P_3 and cK_d contains $\overline{P_3}$. □

LEMMA 9. If aK_b appears in K_p^2, then $a \leq 2p/(b+1)$, with equality achievable for $b > 1$ only if a is even.

Proof: Let H be an appearance of aK_b in K_p^2. The cliques and independent sets in K_p^2 have at most p vertices, so we may assume $b > 1$. Of the components of H, suppose i are confined to rows and j are confined to columns in the p by p grid of vertices. We then have $i + j = a$, $ib + j \leq p$, and $i + jb \leq p$, since no two components of H can intersect a common row or column. This implies $a(b+1) \leq 2p$. Furthermore, if $i \neq j$, then $ib + j \neq i + jb$, and we conclude that $a(b+1) < 2p$. □

3. PROOF OF THE MAIN RESULT

THEOREM 10. A graph G is superregular if and only if $G \in \mathbf{S}$.

Proof: We use induction on n. Suppose G is superregular (with $n \geq 1$) and every subconstituent of G is in $\mathbf{S}$; we need only prove $G \in \mathbf{S}$. Let $H = H(G)$ and $H' = H'(G)$. The degrees of G, H, H' are k, s, t, respectively. If H' has no vertices, then $G = K_{k+1} \in \mathbf{S}$. From the balance equation and Lemma 2, we know that $(H = K_k) \Leftrightarrow (s = k-1) \Leftrightarrow (G \text{ is disconnected}) \Leftrightarrow (G = aK_b)$. We eliminate these degenerate cases; in particular, we may assume that H is not a clique, and by complementation we may assume that H' is not an independent set.

We name cases by the Type of H and subcases by the Type of H and the Type of H'. Because $H(\bar{G}) = \overline{H'(G)}$ and $H'(\bar{G}) = \overline{H(G)}$, we need only consider one of Subcase AB and Subcase $B'A'$, etc. Except in Case C, we always consider an arbitrary edge $xy \in E(G)$ and set $X = N(x)$, $Y = N(y)$, and $Z = \bar{N}(x) \cap \bar{N}(y)$. As illustrated in Fig. 1, we have $G[X \cap Y] = H(H(G))$ (of order s), $G[X - Y] = G[Y - X] = H'(H(G))$ (of order $k - s - 1$), and $G[Z] = H'(G) - V(H'(H(G)))$ (of order $n - 2k + s$). We write $X \triangle Y$ for $(X - Y) \cup (Y - X)$.

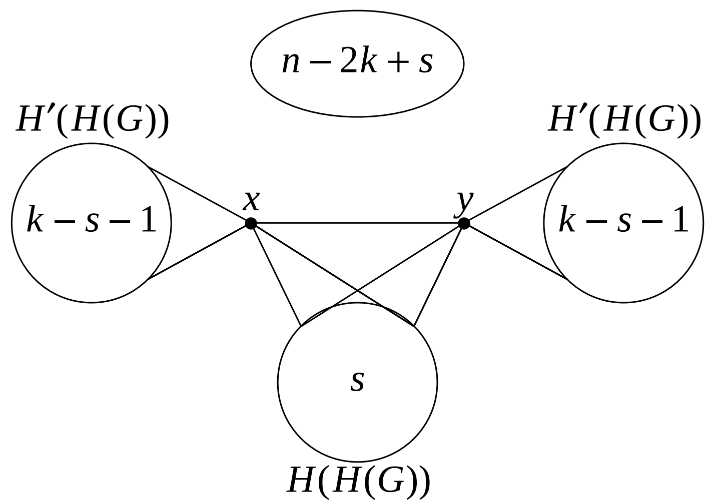

Fig. 1. Structure of a superregular graph.

Case C: $H = C_5$. Here $k = 5$ and $s = 2$. From the balance equation, $(n-k-1)(k-t) = k(k-s-1) = 10$. If $(n-k-1, k-t) = (5,2)$, then $(n,k) = (11,5)$, but no 11-vertex graph is 5-regular. If $(n-k-1, k-t) = (10,1)$, then $(n-k-1, t) = (10,4)$. This requires $H' = 2K_5$, because $2K_5$ is a 10-vertex 4-regular graph in $\mathbf{S}$ and Lemma 6 allows no other. This contradicts Lemma 7, since $H(2K_5) = K_4$ does not appear in C_5.

Case A: $H = aK_b$. By non-degeneracy, $a \geq 2$. We have $G[X \cap Y] = K_{b-1}$ and $G[X - Y] = G[Y - X] = (a-1)K_b$. Since $G[X] = G[Y] = H$, there are no edges between $X \cap Y$ and $X \triangle Y$. Hence the neighbors in Z of any vertex in $X \cap Y$ induce $(a-1)K_b$. If $b \geq 2$, the resulting inequality $(a-1)b \leq |Z|$ reduces to $2(a-1)b \leq h'$, where $h' = n - k - 1$ is the order of H'; call this "Inequality A". We consider various cases for H'.

Subcase AA**:** $H' = cK_d$. By Lemma 7, $K_{d-1} \subseteq aK_b$ and $(a-1)K_b \subseteq cK_d$. Hence $a-1 \le c$ and $d-1 \le b \le d$. The balance equation requires $cd(ab-d+1) = (ab)(a-1)b$. If $d = b$, this becomes $c(a-1)b + c = a(a-1)b$ and then $c = (a-c)(a-1)b$. Since both sides must be positive, we have $a - c \ge 1$. We conclude that $c = a-1$ and $1 = b = d$. This implies $|Z| = (cd+1) - ab + b - 1 = 0$ and $G = K_{a,a} = \overline{2K_a} \in \mathbf{S}$.

If $d = b+1$, alternatively, the balance equation becomes $c(b+1)(a-1)b = ab(a-1)b$. From $c(b+1) = ab$ we obtain $c < a$, which forces $c = a-1$ and $a - b = 1$. Now $H = aK_{a-1}$. If $a \ge 3$, then $b \ge 2$, and Inequality A yields $2(a-1)b \le h' = cd = (a-1)(b+1)$. This requires $b \le 1$, so we may assume $a = 2$. In this case $|Z| = |X-Y| = |Y-X| = 1$ and $G = C_5 \in \mathbf{S}$.

Subcase AB**:** $H' = K_m^2$. Non-degeneracy yields $a \ge 2$ and $m \ge 2$. By Lemma 7, $2K_{m-1} \subseteq aK_b$ and $(a-1)K_b \subseteq K_m^2$. Hence $m-1 \le b \le m$. Also, Lemma 9 implies $a - 1 \le 2m/(b+1)$. Together, these yield $a \le 3$. If $a = 3$, then $b = m-1$, and $G[X-Y] = G[Y-X] = 2K_{m-1}$. Since $2K_{m-1}$ appears in $K_m^2 = H'$ only as the neighborhood of a single vertex, we conclude that Z induces $K_1 + K_{m-1}^2$, with the isolated vertex z adjacent to all of $Y - X$. If $m \ge 3$, then the isolated vertex in $G[Z]$ is unique, and interchanging X and Y in the argument forces it also to be adjacent to all of $X - Y$. If $m = 2$, then $H' = K_{2,2}$ again forces vertices in Z adjacent to all of $X \triangle Y$. Since $4m - 4 \le k = 3m - 3$ is impossible, we conclude that $a = 2$.

With $a = 2$, the inequality $a - 1 \le 2m/(b+1)$ forbids $b = m-1$, so we conclude that $b = m$ and $H = 2K_m$. Now $G[X \cap Y] = K_{m-1}$, $G[X-Y] = G[Y-X] = K_m$, and $G[Z] = K_m^2 - V(K_m) = K_m \,\square\, K_{m-1}$. We arrange Z as m row cliques and $m-1$ column cliques; either of $\{X-Y, Y-X\}$ serves as an additional column to complete the grid. Recall that $X \cap Y$ has no edges to $X \triangle Y$. Since $H = 2K_m$, each vertex of $X \cap Y$ is adjacent to one m-clique in Z, and these columns in Z are distinct. Hence $G[Z \cup (X \cap Y)] = K_{m+1}^{m-1}$. Now each $z \in Z$ has one neighbor in each of $X - Y$, $Y - X$, and $X \cap Y$. Since $H = 2K_m$, two of these must be adjacent, and the only possible such edge joins $X - Y$ and $Y - X$. Since each vertex in these sets needs one additional neighbor, we conclude that $G[Z \cup (X \triangle Y)] = K_m^{m+1}$. Including x and y, we have proved that $G = K_{m+1}^2$, which belongs to $\mathbf{S}$.

Subcase AB'**:** $H' = \overline{K_m^2}$. Non-degeneracy yields $m \ge 2$. Lemma 7 implies $\overline{K_{m-1}^2} \subseteq aK_b$. Lemma 8 then forbids $m \ge 4$. If $m = 2$, then $H' = 2K_2$, which falls into Subcase AA. If $m = 3$, then $H' = K_3^2$, which falls into Subcase AB.

Subcase AA'**:** $H' = \overline{cK_d}$. Non-degeneracy yields $b, d \ge 1$ and $a, c \ge 2$. If $d = 1$, then $H' = K_c$, which falls into Subcase AA. If $b = 1$, then $\bar{G}$ falls into Subcase AA. Hence we may assume $b \ge 2$ and $d \ge 2$. Now Inequality A for G and $\bar{G}$ yields $2(a-1)b \le cd$ and $2(c-1)d \le ab$. Summing and using $a, c \ge 2$, we obtain $ab + cd \le 2b + 2d \le ab + cd$. Equality holds throughout, forcing $a = c = 2$. By Lemma 7, $\overline{(c-1)K_d} \subseteq aK_b$ and $(a-1)K_b \subseteq \overline{cK_d}$. When $a = c = 2$, we obtain $d \le a = 2$. Now Subcase AB applies (and $G = K_3^2$).

Case A'**:** $H = \overline{aK_b}$. If $b = 1$ or $a = 1$, this is covered by Case A, so we may assume $b \ge 2$ and $a \ge 2$. We have $G[X \cap Y] = \overline{(a-1)K_b}$ and $G[X-Y] = G[Y-X] = \overline{K_{b-1}}$.

Now $(X-Y)\cup\{y\}$ forms an independent set in $G[N(x)]=\overline{aK_b}$, which forces $(X\cap Y)\times(X-Y)$ into $E(G)$. Similarly, $X\cap Y$ is completely adjacent to $Y-X$. This establishes $2b+(a-2)b=ab$ neighbors for every vertex in $X\cap Y$, so there is no edge between $X\cap Y$ and Z.

Since $b\geq 2$, we may choose $w\in X-Y$. Already $G[N(w)]$ contains $G[X\cap Y]$ and x. Completing $G[N(w)]=H(G)$ requires $b-1$ additional vertices in $N(w)$ adjacent to all of $X\cap Y$; these can only be in $Y-X$. Since w was arbitrary, $X-Y$ is completely adjacent to $Y-X$. Now every vertex of $X\cup Y$ has ab neighbors, which makes G disconnected unless $Z=\emptyset$. Since every disconnected superregular graph belongs to **S**, we may assume $Z=\emptyset$. Now vertices of $X\cap Y$ have degree $n-1$, which is impossible.

Case B**:** $H=K_p^2$. To avoid earlier cases, H' is K_q^2 or $\overline{K_q^2}$, and $p,q\geq 3$. The balance equation becomes $q^2(p^2-t)=p^2(p-1)^2$, which we write as $1=(p-1)^2/q^2+t/p^2$. The right side exceeds 1 unless $p\leq q$. If $H'=K_q^2$, Lemma 7 implies $2K_{q-1}\subseteq K_p^2$ and hence $q\leq p$, but with $t=2q-2=2p-2$ the balance equation cannot hold. If $H'=K_q^2$, then $t=(q-1)^2$, and the right side again exceeds 1 unless $q\leq p$, but with $q=p$ the only solution is $q=p=2$.

Case B'**:** $H=\overline{K_p^2}$. The remaining possibility is $H'=K_q^2$, with $p,q\geq 3$. Lemma 7 implies $2K_{q-1}\subseteq\overline{K_p^2}$. Since every adjacent pair of vertices in $\overline{K_p^2}$ has exactly two common non-neighbors, this requires $q\leq 3$, and hence $q=3$. The same argument for $\bar{G}$ yields $p=3$. The balance equation becomes $9(9-4)=9\cdot 4$, which is false.

We have considered all possible subconstituents of G in **S**. Whenever the subconstituents allow a superregular graph to be constructed, it belongs to **S**. This completes the inductive proof that every superregular graph belongs to **S**. □

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