Interval Representations of Cliques and of Subset Intersection Graphs[a]

EDWARD R. SCHEINERMAN[b] AND DOUGLAS B. WEST[c]

[b] *Department of Mathematical Sciences*
Johns Hopkins University
Baltimore, Maryland 21205

[c] *Department of Mathematics*
University of Illinois
Urbana, Illinois 61801

INTRODUCTION

In this note, we consider the use of intervals to represent two classes of highly symmetric graphs, in fact with $n!$-fold symmetry. The first are the complete graphs. The second are the graphs whose vertices correspond to the subsets of an n-set, with vertices adjacent if and only if the corresponding sets intersect. The symmetry is important in each discussion. First we define the parameters to be studied.

Consider representing a graph by assigning each vertex v a subset $f(v)$ of the real line, such that vertices are adjacent if and only if the corresponding subsets intersect. If each vertex is assigned a set consisting of at most t intervals, we have a *t-interval representation*. The *interval number* $i(G)$ of a graph G is the minimum t such that G has a t-interval representation. The graphs with interval number 1 are called *interval graphs* and have been thoroughly studied and characterized. If each point of the line appears in sets assigned to at most r vertices of G, the representation has *depth r*. The *depth* r *interval number* $i_r(G)$ is the minimum t such that G has a t-interval representation of depth r. Letting $\Delta(G)$ denote the maximum vertex degree in G and $\omega(G)$ the maximum clique size, note that $i(G) = i_\omega(G) \le \cdots \le i_2(G) \le \Delta(G)$, so all these parameters are well defined. In fact, Griggs and West [1] showed $i_2(G) \le \lceil(\Delta(G) + 1)/2\rceil$. In the next section, we show

$$\left\lceil \frac{1}{2}\left(\frac{n-1}{r-1} + \frac{r}{n}\right)\right\rceil \le i_r(K_n) \le \frac{1}{2}\frac{n}{r-1} + 2\lceil \log_r n\rceil + 1.$$

In other words, the largest clique representable by intervals with depth at most r and multiplicity at most t has about $2t(r-1)$ vertices.

In the third section, we consider the subset intersection graph G_n. The graph G_n has 2^n vertices, one for each subset of $\{1, \ldots, n\}$, with vertices adjacent if the corresponding sets intersect. We show $(\sqrt{n} + 1)/2 \le i(G_n) \le n/2$. We also consider

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another graph parameter for this graph. The *boxicity* of G, denoted $b(G)$, is the minimum number of interval graphs whose intersection (as sets of edges) is G. Equivalently, it is the minimum d such that G is the intersection graph of d-dimensional boxes (products of intervals in each of d coordinates). As with interval number, the boxicity of a graph is as large as the boxicity of any induced subgraph. We show that the boxicity of G_n is much higher than its interval number. In particular, when n is even, the subgraph of G_n induced by the subsets with at most $n/2$ elements has boxicity $\frac{1}{2}\binom{n}{n/2}$. Note that there is no relationship between boxicity and interval number; $K_{m,n}$ has boxicity 2 but interval number $\lceil (mn+1)/(m+n) \rceil$ [8, 11], while the complement of a complete matching on n vertices has interval number 2 but boxicity $n/2$ [7, 8]. Results on both boxicity and interval number, including those used in arguments here, are surveyed in [12].

REPRESENTATIONS OF CLIQUES

Most of the results of this section were initially obtained in the thesis of the first author [9]. A surprisingly accurate lower bound comes from a simple counting argument. This basic argument was used independently in [6] for the purpose of bounding the interval number in terms of the clique number.

THEOREM 1: $i_r(K_n) \geq \left\lceil \frac{1}{2}\left(\frac{n-1}{r-1} + \frac{r}{n}\right) \right\rceil$.

Proof: Count the edges representable by a t-interval representation of depth r. Reading from left to right, the initial endpoint of each interval can introduce at most $r-1$ new edges, except that for the first $r-1$ intervals we have a deficiency of $\sum_{i=1}^{r-1} i = \binom{r}{2}$. To represent K_n, we must have $\binom{n}{2} \leq nt(r-1) - \binom{r}{2}$, which becomes $t \geq \frac{1}{2}\left(\frac{n-1}{r-1} + \frac{r}{n}\right)$. □

When $r = 2$, this bound reduces to $\lceil n/2 \rceil$, which is achievable, since it is well known that the edges of K_n can be decomposed into $\lceil n/2 \rceil$ paths. For $n > r > 2n + \frac{1}{2} - \sqrt{3n^2 - n}$, the bound reduces to 2, which is achievable when $r \geq 3n/8$. To see this, we use the ad hoc representation in FIGURE 1 to show $i_3(K_8) = 2$. Then $i_{3p}(K_{8p}) = 2$ by expanding the set assigned to each vertex into identical sets for p vertices; if $n < 8r/3$, simply delete some of the intervals that were generated. (Note that there is a "redundancy" in FIG. 1 in that intervals for vertices 4 and 5 meet twice. The effect of forbidding redundancy is discussed in [10].)

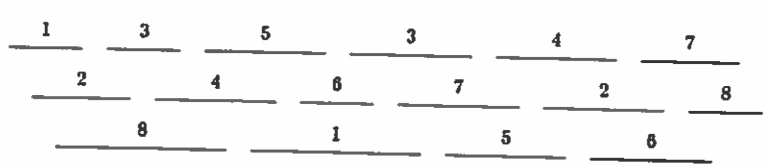

FIGURE 1.

For arbitrary fixed values of r, we have a recursive upper bound construction.

THEOREM 2: $i_r(K_n) \leq i_r(K_{\lceil n/r \rceil}) + \left\lceil \frac{1}{2}\left(\left\lceil \frac{n}{r} \right\rceil + 1\right)\right\rceil$.

Proof: Partition the vertices of K_n into r sets of size at most $\lceil n/r \rceil$. Form a depth r representation for each of the induced cliques. The remaining edges form a complete r-partite graph H. Since H has no $r + 1$-clique, $i_r(H) = i(H)$. As obtained in any of [4, 5, 7] $i(H) \leq \lceil \frac{1}{2}(\lceil n/r \rceil + 1)\rceil$. Putting these representations together yields the desired bound. □

If n is a power of r, we can eliminate the ceiling functions and expand the recurrence to get a closed-form bound. We get

$$i_r(K_n) \leq \sum_{i=1}^{\log_r n} \frac{1}{2}\left(\frac{n}{r^i} + 1\right) \leq \frac{1}{2}\,\frac{n-1}{r-1} + c \log_r n,$$

where $c = 1$ if r is even and $c = \frac{1}{2}$ if r is odd. To get a similar closed-form upper bound in general, we use a slightly weaker recursive bound.

THEOREM 3: $i_r(K_n) \leq \dfrac{n}{2(r-1)} + 2\left\lceil \log_r n \right\rceil + 1.$

Proof: As in the previous proof, partition the vertices of K_n, but use r parts with exactly $\lfloor n/r \rfloor$ vertices and one part S with $|S| < r$. As argued previously, the subgraph induced by vertices outside S has a depth r representation with $t = i_r(K_{\lfloor n/r \rfloor}) + \lceil \frac{1}{2}(\lfloor n/r \rfloor + 1)\rceil$. We can handle the edges involving the remaining vertices by piling up one interval for each, and then placing a small interval for each of the other vertices in the intersection. Since there were less than r leftover vertices, there is a level of depth available for this. This gives the recursive bound $i_r(K_n) \leq i_r(K_{\lfloor n/r \rfloor}) + \lceil \frac{1}{2}(\lfloor n/r \rfloor + 1)\rceil + 1$.

Fixing r and letting $f(n) = i_r(K_n)$, we have $f(n) \leq f(n/r) + \lceil \frac{1}{2}(1 + n/r)\rceil + 1 \leq f(n/r) + n/2r + 2$. Iterating $\lceil \log_r n \rceil$ times yields

$$f(n) \leq \left(\frac{n}{2r} + \frac{n}{2r^2} + \cdots\right) + 2\lceil \log_r n \rceil + 1 \leq \frac{n}{2(r-1)} + 2\lceil \log_r n \rceil + 1. \quad \square$$

For all n, Theorems 1 and 3 show that $i_r(K_n)$ is asymptotic to $n/(2r - 2)$ for fixed r. Although they have the same asymptotic behavior, neither of the bounds in Theorems 1 and 2 is exact. Theorem 2 gives $i_3(K_8) \leq 3$, but we saw earlier that $i_3(K_8) = 2$. For an example where the lower bound of Theorem 1 is not tight, consider $i_6(K_{19})$, where the computation yields $i_6(K_{19}) \geq \lceil 186/97 \rceil = 2$. To show $i_6(K_{19}) \neq 2$, we need an extension of Helly's theorem.

Helly [3] proved that a finite family of pairwise intersecting intervals has a common point, or in other words that a 1-interval representation of K_n has depth at least n. Gyárfás and Lehel [2] extended this to consider pairwise intersecting sets that are each composed of more than one interval. If each set is composed of at most two intervals, they proved there always exist three points such that every set contains at least one of the three points. Consequently, every 2-interval representation of K_n has depth at least $n/3$. Hence every 2-interval representation of K_{19} has depth at least 7, so $i_6(K_{19}) > 2$. (More generally, they showed there exists a finite number $L(t)$ such that, for every finite collection F of pairwise intersecting sets consisting of at most t intervals each, there is a set of $L(t)$ points meeting each set in F. Beyond $L(1) = 1$ and $L(2) = 3$, no values or reasonable bounds are known.)

THE SUBSET INTERSECTION GRAPH

Now consider the graph G_n defined in the Introduction; the intersection graph of the subsets of an n-set. Suppose the elements of the n-set are labeled $\{1, \ldots, n\}$.

THEOREM 4: $(\sqrt{n} + 1)/2 \leq i(G_n) \leq n/2$.

Proof: For the upper bound, there is a simple construction. Assign the subset A intervals $(i - \frac{1}{2}, i + \frac{1}{2}]$ for each $i \in A$, which lets subsets intersect if and only if they are assigned intersecting intervals. The number of intervals assigned to a set is at most its cardinality. However, if $i, i + 1 \in A$, then the intervals $(i - \frac{1}{2}, i + \frac{1}{2}]$ and $(i + \frac{1}{2}, i + \frac{3}{2}]$ form one long interval. Hence the number of intervals assigned to A is actually the number of runs of consecutive elements in A, which is at most $n/2$.

For the lower bound, place the elements in a $\sqrt{n}$ by $\sqrt{n}$ square, and consider the sets formed by the rows and by the columns. Let H_n be the subgraph induced by these $2\sqrt{n}$ sets; we have $i(G_n) \geq i(H_n)$. Since H_n is simply $K_{\sqrt{n}, \sqrt{n}}$, it is well known that $i(H_n) = \lceil(\sqrt{n} + 1)/2\rceil$ [11]. □

Although G_n has many copies of H_n, the gap between the upper and lower bound here cannot be closed solely by considering these, because the upper bound construction shows that the subgraph induced by all sets of size $\sqrt{n}$ has interval number at most $\sqrt{n}$.

To discuss boxicity, let us consider the subgraph G'_n induced by the sets of size at most $n/2$. Of course, $b(G_n) \geq b(G'_n)$, but we can also give a construction to achieve the lower bound on $b(G'_n)$ that does not accommodate addition of the sets of larger size.

THEOREM 5: If n is even, then $b(G'_n) = \frac{1}{2}\binom{n}{n/2}$.

Proof: The boxicity of a complete multipartite graph is the number of nontrivial parts, as proved in [8]. The subgraph of G'_n induced by the sets of size $n/2$ form such a subgraph, with $\frac{1}{2}\binom{n}{n/2}$ parts, which yields the lower bound.

For the upper bound, we provide a dimension for each complementary pair of sets of size $n/2$. For the ith complementary pair, let the two sets be S_i and $\bar{S}_i$. Now, consider an arbitrary set A. In the ith coordinate, assign A the interval $[0, 0]$, $[1, 1]$, or $[0, 1]$ according to whether $A \subseteq S_i$, $A \subseteq \bar{S}_i$, or $A \cap S_i \neq \emptyset \neq A \cap \bar{S}_i$. If A and B intersect, then for each dimension at least one of S_i and $\bar{S}_i$ meets them both in an element of $A \cap B$. If A and B are disjoint, then since G'_n uses only sets of size at most $n/2$, there is an i such that $A \subseteq S_i$ and $B \subseteq \bar{S}_i$ (or vice versa), so that the ith dimension will keep the boxes assigned to A and B from intersecting. □

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