Packing of Steiner trees and S-connectors in graphs

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Aim: Find sharp conditions to guarantee existence of k edge-disjoint S-trees in a graph G.

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- **Tools:** Matroid Union Thm, Submodular functions, a subposet of the lattice of partitions of a set.

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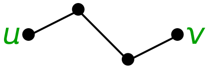

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- S**-trees:** When $\kappa'(G) = 2k-1$ and G is $(2k-1)$-regular and $S = V(G)$, having $\frac{(2k-1)n}{2} \geq k(n-1)$ requires $n \leq 2k$.

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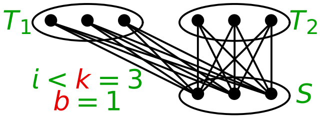

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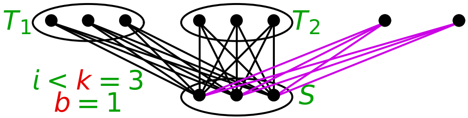

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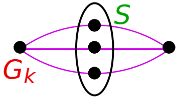

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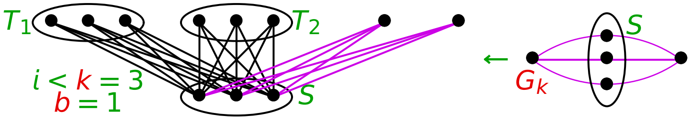

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Now S is $(3k-1)$-edge-connected in G,
and $|T| = (k-1)3b + 2b = (3k-1)b < k(3b-1)$. ■

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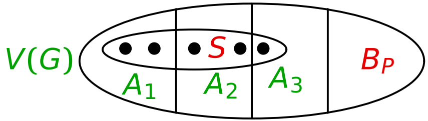

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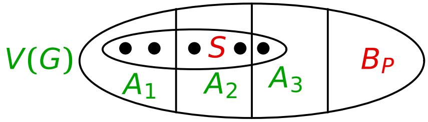

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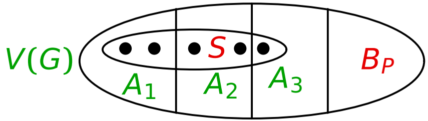

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Def. (k, g)-family = a set of $k + \sum g(v)$ edge-disjoint subgraphs, where k are S-connectors and the others are nontrivial paths ending in S, with $g(v)$ starting at v.

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• Common generalization of Tree Packing Theorem and Hakimi's criterion for orientations w. given outdegrees.

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The paths in a $(0, g)$-family are just edges; orienting the chosen $g(v)$ paths outward from each v gives D. ∎

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The resulting (k, g)-family provides the S-connectors. ■

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A technical theorem guaranteeing the existence of subgraphs with special properties concerning paths.

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Thm. (Main Thm for $S = V(G)$) If $S = V(G) = \{v_1, \ldots, v_n\}$ and SPC holds for S-parity fcn g, then G has edge-disj. subgraphs $H_1, \ldots, H_{n+k}$ such that $d_{H_i}(v_i) = g(v_i)$ for $1 \le i \le n$ and $H_{n+1}, \ldots, H_{n+k}$ are spanning trees.

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The subposet $\mathcal{P}(S)$ of Π_G is closed under join.
For special pairs of elements, the meet in Π_G is in $\mathcal{P}(S)$.

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Pf. $f_g(P) = \sum_{A_i \in P} \delta(A_i) - 2k(|P| - 1) - g(B_P) - 2g(T_P)$.

Contrib. to $\sum_{A_i \in P} \delta(A_i)$ from each edge is submodular.

#Blocks is supermodular (from the partition lattice).

Contrib. of $g(B_P)$ is modular (since $B_{P \wedge P'} = B_P \cup B_{P'}$ and $B_{P \vee P'} = B_P \cap B_{P'}$ when P and P' form a good pair).

Contrib. to $g(T_P)$ from each vertex is supermodular. ■

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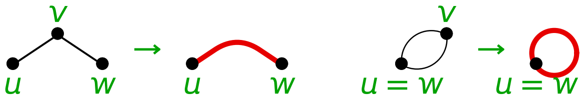
Def. uv, vw-shortcut – delete (copy of) uv and vw, add (extra copy of) uw.



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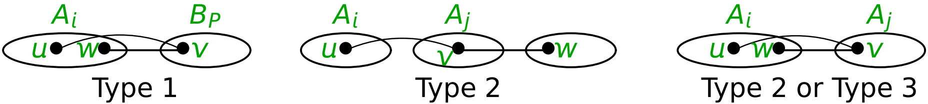

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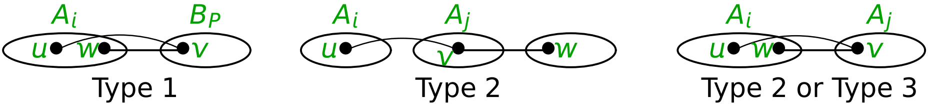
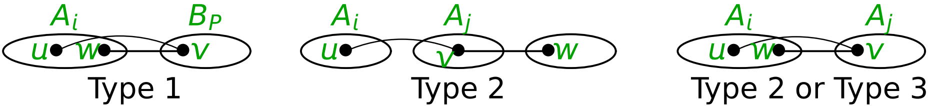


G_w fails if $\exists$ S-partition P such that $f_g(P) \leq -2$ in G_w (since $f_g(P)$ is always even).

Dangerous Sets



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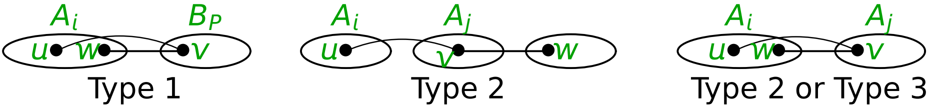

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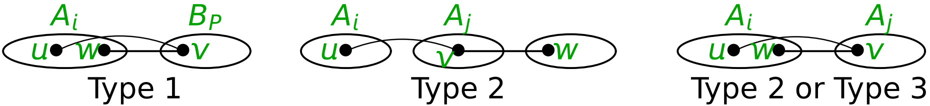

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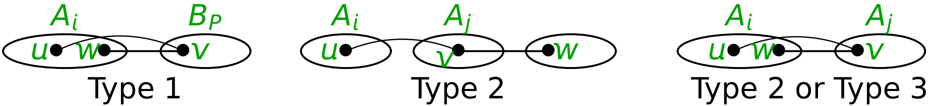

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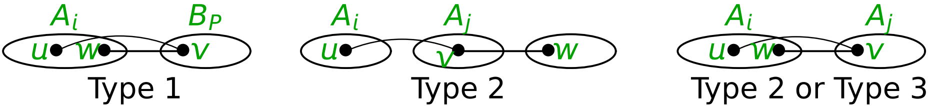

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Application to S-connectors: similar, 3 pages to describe the differences.

The Technical Theorem and Its Use for S-trees

Thm. For $\lambda \ge 6.5$, let S be λk-edge-connected in G. Given $v \in S$ with $d_G(v) = \lambda k$, let $E_0, \dots, E_k$ partition the edges at v, and let $N_i(v) = \{w\colon\ vw \in E_i\}$. If $|E_0| \ge k$, then G has edge-disjoint subgraphs $H_0, \dots, H_k$ such that
(1) $E_i \subseteq E(H_i)$ for $0 \le i \le k$;
(2) $d_{H_0}(s) \ge k$ for all $s \in S$; and
(3) $(S - \{v\}) \cup N_i(v)$ is connected in $H_i - v$ ($1 \le i \le k$).

Pf. In minimal counterexample: (1) $\overline{S}$ is independent.
(2) Vertices of $\overline{S}$ have degree 3 (Mader's Splitting Lemma).
(3) $\exists$ (k, g)-family for special S' and g in special subgraph.
(4) From that (k, g)-family, obtain $H_0, \dots, H_k$. ■

Thm. If S is $6.5k$-edge-connected in G, then G contains k edge-disjoint S-trees.

Pf. Form G' from G: add v and $\lceil 6.5k \rceil$ edges from v to S. Let $S' = S \cup \{v\}$; it is $\lceil 6.5k \rceil$-edge-conn. in G'. Partition edges at v into $E_0, \dots, E_k$ with $|E_0| \ge k$. For G' and S', the theorem above yields $H_0, \dots, H_k$. By (3), $H_1, \dots, H_k$ contain S-trees. ■