Packing of Steiner trees and S-connectors in graphs

Douglas B. West* and Hehui Wu†

June 10, 2011

Abstract

Nash-Williams and Tutte independently characterized when a graph has k edge-disjoint spanning trees; a consequence is that $2k$-edge-connected graphs have k edge-disjoint spanning trees. Kriesell conjectured a more general statement: defining a set $S \subseteq V(G)$ to be *j-edge-connected in G* if S lies in a single component of any graph obtained by deleting fewer than j edges from G, he conjectured that if S is $2k$-edge-connected in G, then G has k edge-disjoint trees containing S. Lap Chi Lau proved that the conclusion holds whenever S is $24k$-edge-connected in G.

We improve Lau's result by showing that it suffices for S to be $6.5k$-edge-connected in G. This and an analogous result for packing stronger objects called "S-connectors" follow from a common generalization of the Tree Packing Theorem and Hakimi's criterion for orientations with specified outdegrees. We prove the general theorem using submodular functions and the Matroid Union Theorem.

1 Introduction

In 1961, Nash-Williams [7] and Tutte [9] independently obtained a necessary and sufficient condition for a graph to have k edge-disjoint spanning trees. A consequence is that every $2k$-edge-connected graph has k edge-disjoint spanning trees. Kriesell [4] conjectured a generalization of this Tree Packing Theorem that seeks edge-disjoint trees containing only a specified subset S of the vertices. Finding the most such trees for given S is the *Steiner-Tree Packing Problem.* Lap Chi Lau [6] gave a partial result toward Kriesell's Conjecture. In this paper, we use a stronger concept called *S-connector* to improve Lau's result.

We use "graph" in the general sense, allowing loops and multi-edges. In a graph G, let S be a set of distinguished vertices called *terminals.* An *S-Steiner-tree* or simply *S-tree* in

*Mathematics Department, University of Illinois, Urbana, IL 61801, west@math.uiuc.edu. Research partially supported by the National Security Agency under Award No. H98230-09-1-0363.

†Mathematics Department, University of Illinois, Urbana, IL, hehuiwu2@illinois.edu. Research partially supported by the Research Experiences for Graduate Students (REGS) program of the University of Illinois.

G is a tree T contained in G such that $S \subseteq V(T)$. An *S-path* is a path in G with both ends in S. *Short-cutting* a u, v-path means replacing its edges with one edge uv. An *S-connector* in G is the union of a family of edge-disjoint S-paths such that short-cutting them yields a connected graph with vertex set S. *In this paper, always* $|S| \geq 2$.

Given a graph G, a vertex set S is *connected in G* if S lies in a single component of G. A set S is *k-edge-connected in G* if S remains connected in every graph obtained by deleting fewer than k edges from G.

Conjecture 1.1 (Kriesell's Conjecture [4]). *If S is $2k$-edge-connected in G, then G contains k edge-disjoint S-trees.*

Known partial results toward Kriesell's Conjecture include the following.

Theorem 1.2 (Kriesell [4]). *If S is $2k$-edge-connected in G, and every vertex outside S has even degree, then G contains k edge-disjoint S-trees.*

Theorem 1.3 (Frank–Király–Kriesell [2]). *If S is $3k$-edge-connected in G, and $G - S$ has no edges, then G contains k edge-disjoint S-trees.*

Theorem 1.4 (Lau [6]). *If S is $24k$-edge-connected in G, then G has k edge-disjoint S-trees.*

We obtain the following improvements.

Theorem 1.5. *If S is $6.5k$-edge-connected in G, then G contains k edge-disjoint S-trees.*

Theorem 1.6. *If S is $10k$-edge-connected in G, then G contains k edge-disjoint S-connectors.*

An S-tree need not be an S-connector. For example, when $|S| \geq 3$, a star whose leaf set is S is an S-tree but not an S-connector. Thus stricter conditions may be needed to guarantee S-connectors. We pose an analogue for S-connectors of Kriesell's Conjecture.

Conjecture 1.7. *If S is $3k$-edge-connected in G, then G contains k edge-disjoint S-connectors.*

We will show that Conjecture 1.7 holds when $G - S$ has no edges; this strengthens Theorem 1.3. For each of these conjectures, infinitely many examples prove sharpness. Sharpness examples for Kriesell's Conjecture are well known. Let G be the graph obtained from $K_{2k,2k}$ by deleting a perfect matching. With $S = V(G)$, the set S is $(2k-1)$-edge-connected in G, since $\kappa'(G) = 2k - 1$. However, G does not have k edge-disjoint S-trees, since k spanning trees would need $k(4k-1)$ edges, while $|E(G)| = (2k)^2 - 2k$. Sharpness for Conjecture 1.7 takes a bit more work.

Example 1.8. To show that Conjecture 1.7 is sharp, we construct an infinite family of graphs G with specified sets S such that S is $(3k-1)$-edge-connected in G but G does not contain k edge-disjoint S-connectors. For $b \in \mathbb{N}$, let S be a set of size $3b$. For $1 \le i < k$, let G_i be a 3-connected 3-regular bipartite graph with partite sets S and T_i. Form the graph G_k by subdividing every edge in a 2-connected 3-regular graph with vertex set T_k of size $2b$, using S as the set of $3b$ vertices of degree 2 added to subdivide the edges.

The graphs $G_1, \ldots, G_k$ all contain the vertex set S; let $G = \bigcup_{i=1}^{k} G_i$. Note that G is bipartite with partite sets S and T, where $T = \bigcup_{i=1}^{k} T_i$. Every vertex of T has degree 3 in G; vertices of S have degree $3k-1$. Any two vertices of S are joined by three internally disjoint paths in $G_1, \ldots, G_{k-1}$ and two in G_k, so S is $(3k-1)$-edge-connected in G.

Finding k edge-disjoint S-connectors in G would require $k(|S|-1)$ edge-disjoint paths passing through vertices of T. Each vertex of T has degree 3 and hence lies in at most one such path. Hence there are at most $|T|$ such paths. We compute $|T| = (k-1)3b + 2b = (3k-1)b$. Comparing $(3k-1)b$ and $k(3b-1)$, we find that not enough paths exist when $b > k$.

In contrast, there is an S-tree in each G_i, so G does have k-edge-disjoint S-trees. □

In the next section, we develop terminology to state our main result, show how it generalizes both the Tree Packing Theorem and Hakimi's Theorem on orientations with specified out-degrees, and prove an important special case that provides the basis step for the general inductive proof. Sections 3 and 4 then complete the proof of the main result, and Sections 5 and 6 apply it to prove our results for S-trees (Theorem 1.5) and S-connectors (Theorem 1.6).

2 Definitions and Special Cases

Stating our main result requires additional terminology and notation.

Definition 2.1. For $S \subseteq V(G)$, write $\overline{S}$ for $V(G) - S$. Write $[A, B]$ for the set of edges in G having endpoints in A and B. Following Lovász, let $\delta(S) = \left|[S, \overline{S}]\right|$.

A partition $A_1, \ldots, A_l$ of a set containing S in $V(G)$ is an *S-partition* if each A_i intersects S. For an S-partition P, we generally write $P = \{A_1, \ldots, A_l\}$ and let $B_P = V(G) - \bigcup_{i=1}^{l} A_i$. Also let T_P be the set of vertices in S that are in blocks of P containing only one vertex of S. We write $|P|$ for the number of blocks in an S-partition P, since P is a set of blocks. Let $\mathcal{P}(S)$ be the set of all S-partitions of G.

Let $\mathbb{N}_0$ be the set of nonnegative integers. Given a graph G, an *S-parity function* is a function $g\colon V(G) \to \mathbb{N}_0$ such that $g(v) \equiv d_G(v) \pmod 2$ for all $v \in \overline{S}$ (there is no restriction on $g(v)$ for $v \in S$). For any vertex set A and function h, let $h(A) = \sum_{v \in A} h(v)$.

In a graph G with terminal set S and S-parity function g, a *g-family* is a set of $g(V(G))$ positive-length paths that can be oriented (from beginning to end) to satisfy the following two properties: (1) each path ends in S, and (2) for each $v \in V(G)$, there are $g(v)$ paths in the family starting at v. A *(k, g)-family* is a set of $k + g(V(G))$ edge-disjoint subgraphs such that k are S-connectors and the others form a g-family. □

Our main result gives a necessary and sufficient condition for existence of a (k, g)-family.

Theorem 2.2. *Let S be a set of terminals in G. If g is an S-parity function on G, then G has a (k, g)-family if and only if $f_g(P) \geq 0$ for all $P \in \mathcal{P}(S)$, where f_g is defined by*

$$f_g(P) = \Big(\sum_{A_i \in P} \delta(A_i)\Big) - 2k(|P| - 1) - g(B_P) - 2g(T_P). \tag{1}$$

We call the condition that $f_g(P) \geq 0$ for all $P \in \mathcal{P}(S)$ the *Strong Partition Condition (SPC)*. The notion of S-parity function enables us to generalize the problem of packing S-connectors in a way (existence of (k, g)-families) that permits a characterization of existence and facilitates the proof of our results about packing of S-trees and S-connectors. The statement of Theorem 2.2 is the reason why we *restrict to $|S| \geq 2$ throughout the paper*. If $|S| = 1$, then every S-partition has one block, so we can make k arbitrarily large without affecting the SPC. However, when $S = \{v\}$ there is only one subgraph that is an S-connector, namely the one subgraph consisting of the vertex v and no edges. We also use the condition $|S| \geq 2$ in Proposition 2.4.

Proposition 2.3. *The SPC is a necessary condition for existence of a (k, g)-family.*

Proof. Consider a (k, g)-family $\mathcal{F}$ in G. For an S-partition P, let $t = \sum_{A_i \in P} \delta(A_i)$. Each S-connector in $\mathcal{F}$ contributes at least $2k(|P| - 1)$ to t. For each vertex v in B_P, the paths starting from v reach S and hence contribute at least $g(v)$ to t. Finally, for $v \in T_P$, the oriented paths starting from v contribute at least $2g(v)$ to t, since they end in some other block of P. Thus $t \geq 2k(|P| - 1) + g(B_P) + 2g(T_P)$, so $f_g(P) \geq 0$. □

The content of Theorem 2.2 is the converse: the Strong Partition Condition suffices for the existence of a (k, g)-family. We show next that the SPC implies a property that is obviously necessary for the existence of a (k, g)-family; hence we will be able to assume this property when we are proving Theorem 2.2. (The stronger inequality $d(v) \geq k + g(v)$ that we obtain in the case $v \in S$ is also necessary for a (k, g)-family.)

Proposition 2.4. *If the SPC holds for an S-parity function g on a graph G, then $g(v) \le d(v)$ for all $v \in V(G)$, where $d(v)$ denotes the degree of v in G.*

Proof. For $v \notin S$, let P be the single-block S-partition $\{V(G) - \{v\}\}$. With $|S| \ge 2$, we have $d(v) - 0 - g(v) - 0 = f_g(P) \ge 0$, so $g(v) \le d(v)$. For $v \in S$, let $P = \{\{v\}, V(G) - \{v\}\}$ (using $|S| \ge 2$). Now $2d(v) - 2k - 0 - 2g(v) \ge f_g(P) \ge 0$, so $d(v) \ge k + g(v)$. □

A natural S-parity function yields a notable application of Theorem 2.2. Given a vertex set $A \subseteq V(G)$, let $n_o(A)$ be the number of vertices of A having odd degree in G.

Theorem 2.5. *Let S be a set of terminals in a graph G. If each $P \in \mathcal{P}(S)$ satisfies*

$$\sum_{A_i \in P} \delta(A_i) \ge 2k(|P| - 1) + n_o(B_P),$$

then G contains k edge-disjoint S-connectors.

Proof. Define an S-parity function by $g(v) = 1$ when v is a vertex of $\overline{S}$ having odd degree in G and otherwise $g(v) = 0$. For $P \in \mathcal{P}(S)$, always $B_P \subseteq \overline{S}$, and hence $g(B_P) = n_o(B_P)$. Also, $g(T_P) = 0$. Hence the difference between the two sides of the specified inequality is $f_g(P)$, and the assumption that it holds is precisely the assumption that the SPC holds for this S-parity function. By Theorem 2.2, G has a (k, g)-family, and hence there are k edge-disjoint S-connectors. □

The condition in Theorem 2.5 is sufficient but not necessary, as seen by adding to such a graph G a large component in which every vertex has odd degree. The case of Theorem 2.5 when no vertex of $\overline{S}$ has odd degree implies Theorem 1.2 in the same way that the Tree Packing Theorem implies that $2k$-edge-connected graphs have k edge-disjoint spanning trees. Indeed, we obtain S-connectors instead of S-trees with the same hypothesis, thereby strengthening Theorem 1.2. Theorem 2.5 also enables us to strengthen Theorem 1.3.

Theorem 2.6. *If S is $3k$-edge-connected in G, and $G - S$ has no edges, then G contains k edge-disjoint S-connectors.*

Proof. Deleting a vertex of degree 1 outside S does not affect the hypothesis, so we may assume that every vertex in $\bar{S}$ has degree at least 2. By Theorem 2.5, it suffices to prove that $\sum_{A_i \in P} \delta(A_i) - n_o(B_P) \ge 2k(|P| - 1)$ for every S-partition P. Since $G - S$ has no edges, $\delta(B_P) \le \sum \delta(A_i)$. Hence $n_o(B_P) \le \frac{1}{3}\delta(B_P) \le \frac{1}{3}\sum \delta(A_i)$, and we have $\sum \delta(A_i) - n_o(B_P) \ge \frac{2}{3}\sum \delta(A_i) \ge 2k|P| > 2k(|P| - 1)$. □

Two other special cases are classical results.

Theorem 2.7 (Nash-Williams [7], Tutte [9]). *A graph G contains k edge-disjoint spanning trees if and only if $\sum_{A_i \in P} \delta(A_i) \geq 2k(|P|-1)$ for every partition P of $V(G)$.*

Proof. Set $S = V(G)$, and make g identically 0. The S-partitions are the partitions of $V(G)$, and the terms in the SPC involving g are always 0. Hence the stated hypothesis is just the SPC for this S and g, and the resulting S-connectors are the spanning trees. □

Theorem 2.8 (Hakimi [3]). *Given a graph G and a function $g\colon V(G) \to \mathbb{N}_0$, there is an orientation D of G such that each vertex v has outdegree at least $g(v)$ in D if and only if for all $T \subseteq V(G)$ there are at least $g(T)$ edges incident to T.*

Proof. Set $S = V(G)$ and $k = 0$. Every S-partition P satisfies $B_P = \varnothing$. Hence the only requirement imposed on $\sum_{i=1}^{l} \delta(A_i)$ in the SPC is from the singleton blocks; the sum must be at least $2g(T_P)$. In fact, the sum counts edges leaving singleton blocks twice, and it counts nothing else when the remainder of $V(G)$ is in one block.

Hence Hakimi's condition implies the SPC, and by Theorem 2.2 a $(0, g)$-family exists. Since $S = V(G)$, the paths can be single edges. Obtain the desired orientation by orienting the $g(v)$ edges chosen for each v outward from v (orient non-chosen edges arbitrarily). □

The special case of Theorem 2.2 when $S = V(G)$ generalizes the Tree Packing Theorem and can be proved using only the Matroid Union Theorem. No special results about S-partitions are needed when S-partitions are just partitions of $V(G)$. We present this proof first because it is needed for the proof of Theorem 2.2, needs no further lemmas, and provides motivation for the definition of f_g.

Given matroids $M_1, \ldots, M_\ell$ defined on the same set E of elements, their *union* M is the hereditary system whose independent sets are $\{\bigcup_{i=1}^{t} I_i \colon\ I_i$ is an independent set in $M_i\}$. The Matroid Union Theorem (Edmonds [1]) states that M is a matroid on E and that the maximum size of an independent set in M is $\min_{X \subseteq E(G)} \left|\overline{X}\right| + \sum_{i=1}^{h} r_i(X)$, where $\overline{X} = E - X$ and $r_i(X)$ denotes the maximum size of a subset of X that is independent in M_i.

In the conclusion of the next theorem, reducing $H_1, \ldots, H_n$ to stars and directing them outward from the centers yields a g-family. When $S = V(G)$, every spanning tree is an S-connector, so $H_1, \ldots, H_{k+n}$ is a (k, g)-family.

Theorem 2.9. *Let $S = V(G) = \{v_1, \ldots, v_n\}$. If the Strong Partition Condition holds for a function $g\colon V(G) \to \mathbb{N}_0$, then G contains edge-disjoint subgraphs $H_1, \ldots, H_{n+k}$ such that $d_{H_i}(v_i) = g(v_i)$ for $1 \le i \le n$ and $H_{n+1}, \ldots, H_{n+k}$ are spanning trees.*

Proof. For $v_i \in V(G)$, let $E(v_i)$ denote the set of edges incident to v_i in G. We introduce matroids $M_1, \ldots, M_{k+n}$ on $E(G)$. Let $M_{n+1}, \ldots, M_{n+k}$ be copies of the cycle matroid of G. For $1 \le i \le n$, let M_i be the matroid on $E(G)$ whose independent sets are $\{X \subseteq E(v_i)\colon |X| \le g(v_i)\}$ (edges not incident to v_i are loops in M_i).

Let $M = \bigcup_{i=1}^{k+n} M_i$; a subset of $E(G)$ is independent in M if and only if it is the disjoint union of sets $X_1, \ldots, X_{n+k}$ such that X_i is independent in M_i for each i. The desired sets exist if and only if M has an independent set of size $k(n-1) + g(V(G))$, in which case the independent sets $X_1, \ldots, X_{n+k}$ decomposing it are the edge sets of the desired subgraphs.

By the Matroid Union Theorem, the maximum size of an independent set in M is $\min_{X \subseteq E(G)} t(X)$, where $t(X) = \left|\overline{X}\right| + \sum_{i=1}^{k+n} r_i(X)$. Hence it suffices to show for each $X \subseteq E(G)$ that $t(X) \ge k(n-1) + g(V(G))$.

If $0 < r_i(X) < g(v_i)$, then deleting $X \cap E(v_i)$ from X shifts the amount $r_i(X)$ from the term for M_i to the term for $\overline{X}$ without increasing other terms. Hence we may restrict our attention to sets X such that $r_i(X) \in \{0, g(v_i)\}$ for $1 \le i \le n$. Given such X, let P be the partition of $V(G)$ whose blocks are the vertex sets of the components of the spanning subgraph of G with edge set X. We express $t(X)$ in terms of P and then apply the SPC.

The set $\overline{X}$ consists of all edges joining blocks of P and possibly some edges within blocks of P. Hence $|\overline{X}| \ge \frac{1}{2} \sum_{A_i \in P} \delta(A_i)$. Note that $B_P = \varnothing$, since $S = V(G)$.

A vertex v_i is a singleton block of P if and only if it has no incident edge in X. Thus $T_P = \{v_i\colon\ r_i(X) = 0\}$. With $r_i(X) \in \{0, g(V(G))\}$, we have $\sum_{i=1}^{n} r_i(X) = g(V(G)) - g(T_P)$. For $i > n$, the rank function of the cycle matroid yields $r_i(X) = n - |P|$.

By these computations, $2t(X) \ge \sum_{A_i \in P} \delta(A_i) - 2k(|P| - n) - 2g(T_P) + 2g(V(G))$. Thus $2t(X) \ge f_g(P) + 2k(n-1) + 2g(V(G))$. By the SPC, $f_g(P) \ge 0$, so the desired independent set and desired subgraphs exist. □

The proof of Theorem 2.2 (Section 4) has many ingredients, including a submodularity inequality for f_g (Section 3), a variant of Mader's Splitting Lemma, and Theorem 2.9. Proving the S-tree result (Theorem 1.5) in Section 5 uses the characterization of (k, g)-families (Theorem 2.2) and Mader's Splitting Lemma. Section 6 presents the analogous argument to prove the S-connector result (Theorem 1.6).

3 S-partitions and submodularity of f_g

We begin by defining a partial order on $\mathcal{P}(S)$. For any S-parity function g, we will prove that the resulting poset is a lattice and that f_g is submodular for special pairs of S-partitions.

If $x \le y$ in a poset $\mathcal{P}$, then x is a *lower bound* for y and y is an *upper bound* for x. If some common upper bound z for x and y satisfies $z \le w$ for every common upper bound w, then z is the *least upper bound* or *join* of x and y, written $x \vee y$. Similarly, the *meet* $x \wedge y$, if it exists, is the *greatest lower bound* of x and y. A *lattice* is a poset in which meets and joins exist for all pairs of elements; a finite lattice has a unique maximal element and a unique minimal element. The *rank* of an element in a poset is one less than the size of a largest chain on which it is the top element. A function ϕ defined on a lattice is *submodular* if $\phi(x \wedge y) + \phi(x \vee y) \le \phi(x) + \phi(y)$ for all elements x and y.

The partition lattice Π_G on $V(G)$ is the poset of all partitions of $V(G)$, ordered by refinement. That is, when Q and Q' are partitions of $V(G)$, we put $Q \le Q'$ in Π_G if for every block $A_i \in Q$, there is a block $A'_j \in Q'$ such that $A_i \subseteq A'_j$. The unique minimal element is the partition into singleton blocks, and in general the rank of a partition Q in $\Pi(G)$ is $|V(G)| - |Q|$, where $|Q|$ denotes the number of blocks of a partition Q.

To define the order relation on $\mathcal{P}(S)$, we map an S-partition P to a partition Q_P of $V(G)$ by defining $Q_P = \{A_1, \dots, A_l, \{b_1\}, \dots, \{b_{|B_P|}\}\}$, where $P = \{A_1, \dots, A_l\}$ and $B_P = \{b_1, \dots, b_{|B_P|}\}$. This mapping is injective; it simply splits B_P into singleton sets and adds them as blocks to P. Define the order relation on $\mathcal{P}(S)$ by putting $P \le P'$ if and only if $Q_P \le Q_{P'}$ in Π_G. This makes $\mathcal{P}(S)$ isomorphic to a subposet $\mathcal{Q}(S)$ of Π_G.

We will study meet and join in $\mathcal{P}(S)$ by relating it to meet and join in $\mathcal{Q}(S)$ as a subposet of Π_G. Let $\wedge_\Pi$ and $\vee_\Pi$ denote the meet and join operations in Π_G. We use two well-known properties of the partition lattice (after subtracting each term from $|V(G)|$, statement (2) becomes the statement that the rank function of Π_G is submodular).

Proposition 3.1. *For partitions Q and Q' of $V(G)$,*
(1) $Q \wedge_\Pi Q' = \{A_i \cap A_j\colon\ A_i \in Q, A_j \in Q'\}$;
(2) $|Q \wedge_\Pi Q'| + |Q \vee_\Pi Q'| \ge |Q| + |Q'|$.

Let the symbols $\wedge$ and $\vee$ without subscripts denote the meet and join in $\mathcal{P}(S)$.

Proposition 3.2. *For $P, P' \in \mathcal{P}(S)$, the meet and join of P and P' are well defined, with*
(1) $P \wedge P' = \{A_i \cap A'_j\colon\ A_i \in P, A'_j \in P', A_i \cap A'_j \cap S \ne \varnothing\}$;
(2) $Q_{P \vee P'} = Q_P \vee_\Pi Q_{P'}$;
(3) $B_{P \vee P'} = B_P \cap B_{P'}$.

Proof. (1) Let $\hat{P} = \{A_i \cap A'_j \colon A_i \in P, A'_j \in P', A_i \cap A'_j \cap S \neq \varnothing\}$. By definition, $\hat{P} \in \mathcal{P}(S)$ and $\hat{P} \leq P, P'$. For any block A'' in any common lower bound P'', exist $A_i \in P$ and $A'_j \in P'$ such that $A'' \subseteq A_i \cap A'_j$. Since $A'' \cap S \neq \varnothing$, we have $A_i \cap A'_j \in \hat{P}$. Hence $P'' \leq \hat{P}$.

(2) Let $Q'' = Q_P \vee_\Pi Q_{P'}$. If $Q'' \notin \mathcal{Q}(S)$, then there exists $A \in Q''$ such that $A \cap S = \varnothing$ and $|A| \geq 2$. For $a \in A$, the block C containing a in Q_P is contained in A. Since $A \cap S = \varnothing$ and P is an S-partition, C must be $\{a\}$. Similarly, $\{a\} \in Q_{P'}$. Now $\{a\}$ is a block in $Q_P \vee_\Pi Q_{P'}$, contradicting $|A| \geq 2$.

Hence $Q'' \in \mathcal{Q}(S)$, making Q'' the least upper bound in $\mathcal{Q}(S)$ for Q_P and $Q_{P'}$. Since $\mathcal{P}(S)$ and $\mathcal{Q}(S)$ are isomorphic, also $P \vee P'$ exists, with $Q_{P \vee P'} = Q_P \vee_\Pi Q_{P'}$.

(3) follows immediately from (2). □

Common lower bounds in $\mathcal{P}(S)$ do not always translate so nicely to $\mathcal{Q}(S)$. Fortunately, they do for the pairs of S-partitions we will need. Two S-partitions $\{A_1, \ldots, A_l\}$ and $\{A'_1, \ldots, A'_l\}$ form a *good pair* if $A_i \cap A'_j \neq \varnothing$ implies $A_i \cap A'_j \cap S \neq \varnothing$.

Proposition 3.3. *If S-partitions P and P' form a good pair, then:*
(1) $Q_{P \wedge P'} = Q_P \wedge_\Pi Q_{P'}$;
(2) $B_{P \wedge P'} = B_P \cup B_{P'}$;
(3) $|P \wedge P'| + |P \vee P'| \geq |P| + |P'|$.

Proof. (1) Since P and P' form a good pair, the expression for their meet simplifies to $P \wedge P' = \{A_i \cap A'_j \colon A_i \in P, A'_j \in P', A_i \cap A'_j \neq \varnothing\}$, which maps to $Q_P \wedge_\Pi Q_{P'}$.

(2) $B_{P \wedge P'}$ and $B_P \cup B_{P'}$ both equal the set of elements outside all $A_i \cap A'_j$.

(3) Note that $|P| = |Q_P| - |B_P|$ and $|P'| = |Q_{P'}| - |B_{P'}|$. Using (2) and Proposition 3.2(3),

$$|B_P| + |B_{P'}| = |B_P \cap B_{P'}| + |B_P \cup B_{P'}| = |B_{P \wedge P'}| + |B_{P \vee P'}|.$$

Now the claim follows from $|Q_P \wedge_\Pi Q_{P'}| + |Q_P \vee_\Pi Q_{P'}| \geq |Q_P| + |Q_{P'}|$ (Proposition 3.1(2)). □

Definition 3.4. Let $G[A]$ denote the subgraph induced by A. Given an S-partition P with blocks $A_1, \ldots, A_l$, assign each edge $e \in E(G)$ a weight $h_P(e)$ by

$$h_P(e) = \begin{cases} 2, & \text{if } e \in [A_i, A_j] \text{ for some } i \text{ and } j; \\ 1, & \text{if } e \in [A_i, B_P] \text{ for some } i; \\ 0, & \text{otherwise.} \end{cases}$$

Grouping the sum by edges yields $\sum_{A_i \in P} \delta(A_i) = \sum_{e \in E(G)} h_P(e)$ for any S-partition P. □

Proposition 3.5. *If $S \subseteq V(G)$ and P and P' form a good pair in $\mathcal{P}(S)$, then*

$$h_{P\wedge P'}(e) + h_{P\vee P'}(e) \le h_P(e) + h_{P'}(e)$$

for all e in $E(G)$. Also, if the endpoints of e lie in different blocks in both P and P', but in the same block in $P \vee P'$, then the two sides of the inequality differ by 2.

Proof. For $uv \in E(G)$, let $W = \{u, v\}$. Note that $h_P(uv) = 2 - |W \cap B_P| - 2t_P(uv)$, where $t_P(uv) = 1$ if $W \subseteq A_i$ for some $A_i \in P$, and otherwise $t_P(uv) = 0$. Since $B_{P\wedge P'} = B_P \cup B_{P'}$ and $B_{P\vee P'} = B_P \cap B_{P'}$, we have $|W \cap B_P| + |W \cap B_{P'}| = |W \cap B_{P\vee P'}| + |W \cap B_{P\wedge P'}|$. Therefore $h_{P\wedge P'}(uv) + h_{P\vee P'}(uv) \le h_P(uv) + h_{P'}(uv)$ if and only if $t_{P\wedge P'}(uv) + t_{P\vee P'}(uv) \ge t_P(uv) + t_{P'}(uv)$. This holds when P and P' form a good pair, since $\max\{t_P(uv), t_{P'}(uv)\} = 1$ implies $t_{P\vee P'}(uv) = 1$, and $t_P(uv) = t_{P'}(uv) = 1$ implies $t_{P\wedge P'}(uv) = 1$.

If u and v lie in different blocks in P and P' but in the same block in $P \vee P'$, then $t_{P\wedge P'}(uv) + t_{P\vee P'}(uv) = t_P(uv) + t_{P'}(uv) + 1$, so the difference between the two sides of the claimed inequality is then 2. □

Lemma 3.6. *Let g be a S-parity function. If P and P' form a good pair in $\mathcal{P}(S)$, then*

$$f_g(P \wedge P') + f_g(P \vee P') \le f_g(P) + f_g(P'). \tag{2}$$

Proof. Let Q be an S-partition. From the definition of f_g and the observation in Definition 3.4 that $\sum_{A_i\in P} \delta(A_i) = \sum_{e\in E(G)} h_P(e)$, we have

$$f_g(Q) = \sum_{e\in E(G)} h_Q(e) - 2k(|Q| - 1) - g(B_Q) - 2g(T_Q). \tag{3}$$

We consider the contributions of these terms to (2). Proposition 3.5 yields

$$\sum_{e\in E(G)} [h_{P\wedge P'}(e) + h_{P\vee P'}(e)] \le \sum_{e\in E(G)} [h_P(e) + h_{P'}(e)].$$

By Proposition 3.3(3),

$$2k(|P \wedge P'| - 1) + 2k(|P \vee P'| - 1) \ge 2k(|P| - 1) + 2k(|P'| - 1).$$

Since $B_{P\wedge P'} = B_P \cup B_{P'}$ and $B_{P\vee P'} = B_P \cap B_{P'}$,

$$g(B_{P\wedge P'}) + g(B_{P\vee P'}) = g(B_P) + g(B_{P'}).$$

For the last term, recall that T_P is the set of vertices in S belonging to blocks in P having no other vertex of S. If $v \in T_P \cup T_{P'}$, then $v \in T_{P\wedge P'}$; if $v \in T_P \cap T_{P'}$, then since P and P' form a good pair, $v \in T_{P\vee P'}$. Summing the contributions made by each vertex yields

$$g(T_{P\wedge P'}) + g(T_{P\vee P'}) \geq g(T_P) + g(T_{P'}).$$

Summing the formulas for all four terms completes the proof of (2). □

When P is an S-partition, with $P = \{A_1, \ldots, A_l\}$, we let $C_P(v)$ denote the member of $\{A_1, \ldots, A_l, B_P\}$ containing v.

Sometimes we will need a stronger inequality than (2), ensuring a difference of 4. For $x \in V(G)$, let $N_G(x) = \{y \in V(G) \colon xy \in E(G)\}$. We write $G - uv$ to mean the graph obtained from G by deleting one copy of the edge uv when uv has multiplicity at least 1.

Lemma 3.7. *Let P and P' be S-partitions that form a good pair. Let uv be an edge such that u and v lie in different blocks in both P and P' but in the same block in $P \vee P'$. If $N_{G-uv}(v)$ intersects both $C_P(u)$ and $C_{P'}(u)$, then $f_g(P) + f_g(P') - f_g(P \wedge P') - f_g(P \vee P') \geq 4$.*

Proof. We showed in proving Lemma 3.6 that the terms in (3) involving g make a nonnegative contribution to $f_g(P) + f_g(P') - f_g(P \wedge P') - f_g(P \vee P')$. Hence it suffices to gain 4 from the other terms.

For each edge e, let $\hat{h}(e) = h_P(e) + h_{P'}(e) - h_{P\wedge P'}(e) - h_{P\vee P'}(e)$. Proposition 3.5 implies that always $\hat{h}(e) \geq 0$ and that the locations of u and v yield $\hat{h}(uv) \geq 2$. It suffices to find another edge e with $\hat{h}(e) \geq 2$ or gain 2 from the term involving the number of blocks.

By the hypothesis on $N(v)$, deleting (one copy of) the edge vu leaves v with a neighbor in each of $C_P(u)$ and $C_{P'}(u)$. Suppose that v still has a neighbor w in $C_P(u) - C_{P'}(v)$ or $C_{P'}(u) - C_P(v)$ (possibly $w = u$). In either case, w and v lie in different blocks in both P and P', and w and u lie in the same block of $P \vee P'$. By hypothesis, this block of $P \vee P'$ also contains v, so Proposition 3.5 yields $\hat{h}(wv) \geq 2$, which suffices.

Therefore, we may assume that the given vertices $w, w' \in N_{G-uv}(v)$ are in $C_P(u) \cap C_{P'}(v)$ and $C_{P'}(u) \cap C_P(v)$, respectively. Since u and v lie in distinct blocks in both P and P', we have $w \neq w'$ (and neither of them is u).

Obtain P'' from P by splitting $C_P(v)$ into $C_P(v) - C_{P'}(u)$ and $C_P(v) \cap C_{P'}(u)$. Since P and P' form a good pair, P'' is an S-partition. Since all intersections of blocks in P'' and P' are intersections of blocks in P and P', also P'' and P' form a good pair, and $P'' \wedge P' = P \wedge P'$.

Furthermore, $P'' \vee P' = P \vee P'$, since $C_{P'}(v)$, $C_P(u)$, and $C_{P'}(u)$ successively put the pairs $\{v,w\}$, $\{w,u\}$, and $\{u,w'\}$ into the same block of $P'' \vee P'$ (using $C_{P''}(u) = C_P(u)$).

Now, since $|P'' \wedge P'| + |P'' \vee P'| - |P''| - |P'| \geq 0$ (by Proposition 3.3(3)) and $|P''| = |P| + 1$, we obtain $|P \wedge P'| + |P \vee P'| - |P| - |P'| \geq 1$. Since it has the coefficient $2k$, this term now provides the additional contribution of 2 that completes the proof. □

Proposition 3.8. *If P is an S-partition and g is an S-parity function, then $f_g(P)$ is even.*

Proof. For $A \subseteq V(G)$, recall that $n_o(A)$ is the number of vertices of A having odd degree in G. Using $B_P \subseteq \overline{S}$ and the definition of S-parity function,

$$\begin{aligned} f_g(P) &= \left(\textstyle\sum_{A_i \in P} \delta(A_i)\right) - 2k(l-1) - g(B_P) - 2g(T_P) \\ &\equiv \left[\textstyle\sum_{i=1}^{l} \left(\sum_{v \in A_i} d_G(v)\right) - 2|E(G[A_i])|\right] + n_o(B_P) \\ &\equiv \left[\textstyle\sum_{i=1}^{l} n_o(A_i)\right] + n_o(B_P) \quad \equiv \quad n_o(V(G)) \quad \equiv \quad 0 \pmod 2. \end{aligned}$$ □

For $X \subseteq \overline{S}$ and $P = (A_1, \ldots, A_l)$, let $P - X = (A_1 - X, \ldots, A_l - X)$. Note that if P is an S-partition, then so is $P - X$. Recall that $[A,B] = \{xy \in E(G)\colon\ x \in A, y \in B\}$.

Proposition 3.9. *If P is an S-partition and $X \subseteq A_i \cap \overline{S}$, where A_i is a block of P, then*

$$f_g(P) - f_g(P - X) \geq \left|[X, \overline{A_i}]\right| - \left|[X, A_i - X]\right|.$$

Proof. Since $f_g(P) = \sum_{i=1}^{l} \delta(A_i) - 2k(|P| - 1) - g(B_P) - 2g(T_P)$, we have

$$\begin{aligned} f_g(P) - f_g(P - X) &= \delta(A_i) - \delta(A_i - X) + g(X) \\ &\geq \delta(A_i) - \delta(A_i - X) \quad = \quad \left|[X, \overline{A_i}]\right| - \left|[X, A_i - X]\right| \end{aligned}$$ □

4 Existence of (k,g)-families

The goal of this section is to prove Theorem 2.2, which states that a (k,g)-family exists if and only if the Strong Partition Condition holds for (G,S,k,g). After proving further properties of good pairs of S-partitions, our inductive proof of the main theorem will use Theorem 2.9 as the basis and a variant of Mader's Splitting Lemma in the induction step.

Let uv and vw be two edges of G. The *uv,vw-shortcut* of G is the graph obtained from G by replacing uv and vw with uw. When u is already adjacent to w, an extra copy of uw

is added; when $u = w$, a double-edge is replaced with a loop. Fix an edge uv with $u \in S$. For $w \in N_{G-uv}(v)$, let G_w denote the uv, vw-shortcut of G. By $G - uv$, we mean the graph obtained from G by deleting one copy of uv; this means that $w = u$ is possible when uv has multiplicity greater than 1 in G.

In order to prove Theorem 2.2 inductively, we will show that if uv is an edge in G with $u \in S$ and $v \notin S$, and G satisfies the Strong Partition Condition (SPC) for an S-parity function g such that $d_G(v) > g(v)$, then there exists $w \in N_{G-uv}(v)$ such that G_w also satisfies the SPC. This is the main technical result of our paper. Mader's Splitting Lemma (Lemma 5.4) is analogous; it guarantees shortcuts that preserve local connectivity conditions.

Definition 4.1. Given $S \subseteq V(G)$, suppose that G satisfies the SPC for an S-parity function g. Fix an edge $uv \in E(G)$ with $u \in S$ and $v \notin S$ such that $d_G(v) > g(v)$. A vertex w is *dangerous* for an S-partition P (relative to uv) if $f_g(P) < 0$ for the graph G_w. Let $D(P) = \{w \in V(G)\colon\ f_g(P) < 0 \text{ for } G_w\}$. □

When $w \in D(P)$, we have $f_g(P) \le -2$ for G_w and $f_g(P) \ge 0$ for G, since $f_g(P)$ is always even (Proposition 3.8). The contributions to $f_g(P)$ for G and G_w differ only in $\sum_{A_i \in P} \delta(A_i)$, which decreases when replacing uv and vw with uw only if $u, w \notin C_P(v)$ (recall that $C_P(x)$ is the member of $\{A_1, \ldots, A_l, B_P\}$ containing x, where $A_1, \ldots, A_l$ are the blocks of P). Since $u \in S$ and $v \notin S$, the ways a decrease can occur are shown in Figure 1. The shortcut decreases $f_g(P)$ by 2 if $v \in B_P$ and $w \in C_P(u)$, by 2 if $v \notin B_P$ and $w \notin C_P(v) \cup C_P(u)$, and by 4 if $v \notin B_P$ and $w \in C_P(u)$. Otherwise, $f_g(P)$ does not change.

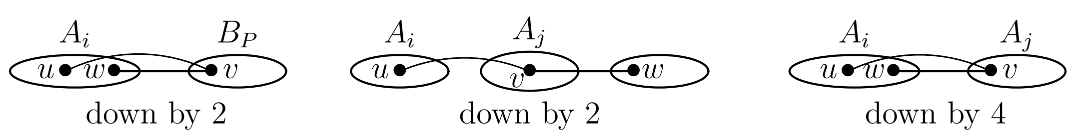

Figure 1: Dangerous locations for w

Vertex w will be dangerous with a decrease of 2 when $f_g(P) = 0$ or a decrease of 4 when $f_g(P) \in \{0, 2\}$. We group the cases as "Types" by the value of $f_g(P)$ and the location of v in P. These types determine the location of all w such that $f_g(P) < 0$ for G_w. For simplicity, write $N'(v)$ for $N_{G-uv}(v)$; thus $N'(v) = N_G(v) - \{u\}$ when uv has multiplicity 1, and otherwise $N'(v) = N_G(v)$. The distinction between Type 2 and Type 3 is that decreasing $f_g(P)$ by 2 instead of 4 is enough when $f_g(P) = 0$, so vertices in all of $N'(v) - C_P(v)$ are dangerous instead of just those in $C_P(u)$. If P is none of these types, then $D(P) = \varnothing$.

Type	$f_g(P)$ for G	location of v	dangerous set $D(P)$
1	0	$v \in B_P$	$N'(v) \cap C_P(u)$
2	0	$v \notin B_P \cup C_P(u)$	$N'(v) - C_P(v)$
3	2	$v \notin B_P \cup C_P(u)$	$N'(v) \cap C_P(u)$

Our goal is to find $w \in N'(v)$ such that w is outside $D(P)$ for every S-partition P; in that case, G_w satisfies the SPC. We will need two lemmas about S-partitions.

With $D(P)$ defined relative to a fixed edge uv, let $\mathcal{M}$ be the set of minimal S-partitions among those with maximal dangerous sets. That is, $P \in \mathcal{M}$ when there is no S-partition P' such that $D(P) \subset D(P')$ or such that $D(P) = D(P')$ and $P' < P$ in $\mathcal{P}(S)$. The next lemma will help us find an S-partition whose dangerous set contains $D(P)$ for all $P \in \mathcal{P}(S)$.

Lemma 4.2. *If $P, P' \in \mathcal{M}$, then P and P' form a good pair.*

Proof. We prove the contrapositive. When P and P' do not form a good pair, there exist $A_i \in P$ and $A'_j \in P'$ such that $\varnothing \neq A_i \cap A'_j \subseteq \overline{S}$. Let $X = A_i \cap A'_j$; we have remarked that $P - X \in \mathcal{P}(S)$. Changing P to $P - X$ splits elements of X from blocks in P (and in $Q(P)$) to become singletons in $Q(P - X)$, so $P - X \leq P$ (also, $P' - X \leq P'$). Hence it suffices to prove $D(P) \subseteq D(P - X)$ or $D(P') \subseteq D(P' - X)$, since then P and P' are not both in $\mathcal{M}$.

Claim $(*)$: If P is Type 1 or 3 and $f_g(P - X) \leq f_g(P)$, then $D(P) \subseteq D(P - X)$ unless $u \in A_i$ and $P - X$ is not Type 2 (and similarly for P'). Since $v \notin C_P(u)$, also $v \notin C_{P-X}(u)$. If $u \notin A_i$, then $C_{P-X}(u) = C_P(u)$, so $D(P) = N'(v) \cap C_P(u) = N'(v) \cap C_{P-X}(u) \subseteq D(P - X)$. Hence $u \in A_i$, so $v \notin A_i$ and $C_{P-X}(v) = C_P(v)$. If $P - X$ is Type 2, then $D(P) \subseteq N'(v) - C_P(v) = N'(v) - C_{P-X}(v) = D(P - X)$.

If $\big|[X, A_i - X]\big| < \delta(X)/2$, then $\big|[X, \overline{A_i}]\big| > \big|[X, A_i - X]\big|$, so $f_g(P) > f_g(P - X)$, by Proposition 3.9. However, the SPC yields $f_g(P - X) \geq 0$, so $f_g(P - X) = 0$ and P is Type 3. By $(*)$, we have $u \in A_i$ and $P - X$ is Type 1. Since P is Type 3, $v \notin B_P$, so $P - X$ being Type 1 requires $v \in X$, which contradicts $u \in A_i$.

This eliminates the case $\big|[X, A_i - X]\big| < \delta(X)/2$, and similarly for A'_j. Since $\big|[X, A_i - X]\big| + \big|[X, A'_j - X]\big| \leq \delta(X)$, the remaining case is $\big|[X, A_i - X]\big| = \big|[X, A'_j - X]\big| = \delta(X)/2$, and $[X, \overline{X}] = [X, (A_i \cup A'_j) - X]$. Also $f_g(P - X) \leq f_g(P)$ and $f_g(P' - X) \leq f_g(P')$ for G, by Proposition 3.9. Since $X \subseteq \overline{S}$, we know $u \notin A_i \cap A'_j$. By symmetry, we may take $u \notin A_i$, and hence P is Type 2 by $(*)$. Thus $f_g(P - X) = f_g(P) = 0$.

If $v \in X$, then $v \notin C_P(u) \cup C_{P'}(u)$ yields $u \notin A_i \cup A'_j$. Since all edges leaving X go to $A_i - X$ or $A'_j - X$, now $[X, \{u\}] = \varnothing$, which contradicts the existence of uv. Hence we may assume $v \notin X$. Since $f_g(P-X) = 0$ and P is Type 2, $v \notin X$ implies $P-X$ is Type 2, so $D(P) = N'(v) - C_P(v) \subseteq N'(v) - C_{P-X}(v) = D(P-X)$. □

We now obtain a single S-partition whose dangerous set contains all dangerous sets.

Lemma 4.3. *There exists an S-partition whose dangerous set contains $\bigcup_{P\in\mathcal{P}(S)} D(P)$.*

Proof. If the dangerous sets for all S-partitions in $\mathcal{M}$ are the same, then every member of $\mathcal{M}$ has the desired property. Suppose $P, P' \in \mathcal{M}$ exist with $D(P) \neq D(P')$. By Lemma 4.2, P and P' form a good pair. Let $\check{P} = P \vee P'$ and $\hat{P} = P \wedge P'$. If $\hat{P}$ is a Type 2 partition, then $D(P) \subseteq N'(v) - C_P(v) \subseteq N'(v) - C_{\hat{P}}(v) = D(\hat{P})$, which contradicts $P \in \mathcal{M}$.

Case 1: u and v lie in the same block of $\check{P}$. By Lemma 3.7 and the SPC, $f_g(P) + f_g(P') \geq f_g(\hat{P}) + f_g(\check{P}) + 4 \geq 4$. Since $D(P), D(P') \neq \varnothing$ requires $f(P), f(P') \leq 2$, we have $f_g(\hat{P}) = f_g(\check{P}) = 0$. Also $f_g(P) = f_g(P') = 2$, so P and P' are both Type 3, and $v \notin B_P \cup B_{P'} = B_{\hat{P}}$. We conclude that $\hat{P}$ is Type 2.

Case 2: u and v do not lie in the same block of $\check{P}$. Suppose first that $f_g(\check{P}) \geq 4$, so both P and P' are Type 3 and $f_g(\hat{P}) = 0$. Also $v \notin B_P \cup B_{P'} = B_{\hat{P}}$, so $\hat{P}$ is Type 2.

Next suppose that $f_g(\check{P}) = 2$. By submodularity, P or P' must be Type 3; let P be Type 3. Hence $v \notin B_P$. Since always $B_{\check{P}} = B_P \cap B_{P'}$ (Proposition 3.2), we obtain $v \notin B_{\check{P}}$.

Hence we may assume that $f_g(\check{P}) = 0$ or that $f_g(\check{P}) = 2$ and $v \notin B_{\check{P}}$. Now $D(\check{P}) \supseteq N'(v) \cap C_{\check{P}}(u) \supseteq N'(v) \cap (C_P(u) \cup C_{P'}(u))$. If neither P nor P' is Type 2, then this last set is $D(P) \cup D(P')$. Since $D(P) \neq D(P')$ and $P, P' \in \mathcal{M}$, neither of $D(P)$ and $D(P')$ contains the other. Hence $D(\check{P})$ strictly contains both, which contradicts $P, P' \in \mathcal{M}$.

If both P and P' are Type 2, then submodularity yields $f_g(\hat{P}) = 0$. Also $v \notin B_P \cup B_{P'} = B_{P\wedge P'}$, so $\hat{P}$ is Type 2. If P (and not P') is Type 2, then $D(P) = N'(v) - C_P(v)$ and $D(P') = N'(v) \cap C_{P'}(u)$. Since u and v are not in the same block of $\check{P}$, the sets $C_P(v)$ and $C_{P'}(u)$ are disjoint. Hence have $D(P') \subset D(P)$, contradicting $P' \in \mathcal{M}$. □

Next we prove an analogue of Mader's Splitting Lemma (Lemma 5.4). Recall that $N'(v) = N_G(v) - \{u\}$ if uv has multiplicity 1, and otherwise $N'(v) = N_G(v)$. When A or B has only one vertex v, we write v instead of $\{v\}$ in the notation $[A, B]$.

Theorem 4.4. *If G satisfies the Strong Partition Condition and has an edge uv with $u \in S$, $v \notin S$, and $d_G(v) > g(v)$, then there is a vertex $w \in N'(v)$ such that G_w satisfies the SPC.*

Proof. By Lemma 4.3, there exists an S-partition P whose dangerous set contains the dangerous sets (relative to uv) for all S-partitions. If no desired vertex w exists, then $D(P) = N'(v)$.

Thus $|[v, C_P(v)]| = 0$. Let P' be the S-partition obtained from P by moving v to $C_P(u)$; note that $|P'| = |P|$ and $T_{P'} = T_P$.

Using the expression for f_g in (1), we have $f_g(P) - f_g(P') = d_G(v) - g(v) > 0$ when P is Type 1, and $f_g(P) - f_g(P') = 2|[v, C_P(u)]| - 2|[v, C_P(v)]| > 0$ when P is Type 2 or Type 3. Since $f_g(P') \geq 0$, this yields $f_g(P) > 0$. Hence P is Type 3.

Since $N'(v) = D(P)$, now $N_G(v) \subseteq C_P(u)$. Since g is an S-parity function, $v \notin S$, and $d_G(v) > g(v)$, we also have $|[v, C_P(u)]| = d_G(v) \geq g(v) + 2 \geq 2$. Now $2 \geq f_g(P) - f_g(P') = 2|[v, C_P(u)]| \geq 4$, a contradiction. We conclude that the desired vertex w exists. □

We can now prove our main result.

Theorem 2.2. *Let S be a set of terminals in G. If g is an S-parity function for G, then G has a (k, g)-family if and only if $f_g(P) \geq 0$ for all $P \in \mathcal{P}(S)$.*

Proof. Proposition 2.3 proves necessity. For sufficiency, we use induction on the total number of vertices and edges, with trivial basis. Theorem 2.9 is the case $S = V(G)$, so we may assume $\overline{S} \neq \varnothing$. We will reduce the claim to a special case where Theorem 2.9 applies.

Let $R = \overline{S} \cap N(S)$. We may assume $R \neq \varnothing$; otherwise, the induction hypothesis applies to $G - \overline{S}$. If $d_G(v) > g(v)$ for some $v \in R$, then choose $u \in N(v) \cap S$. Theorem 4.4 provides $w \in N'(v)$ (for this u and v) such that G_w satisfies the SPC. Since G_w is smaller than G, it has a (k, g)-family. If any of the resulting S-connectors or paths contain the edge uw that is not in G, then replacing that edge with the original uv and vw yields a (k, g)-family in G.

Hence we may assume $d_G(v) = g(v)$ for $v \in R$, by Proposition 2.4. We next reduce to the case $N(v) \subseteq S$ for all $v \in R$. Let $P = \{S\}$; that is, $|P| = 1$ and $B_P = \overline{S}$. Since always $|S| \geq 2$, we have $T_P = \varnothing$, and hence $f_g(P) = |[S, \overline{S}]| - g(\overline{S})$. By the SPC, $|[S, \overline{S}]| \geq g(\overline{S}) \geq \sum_{v \in R} d_G(v)$. However, $|[S, \overline{S}]| \leq \sum_{v \in R} d_G(v)$. We conclude that R is an independent set whose neighbors all lie in S and that $g(v) = 0$ for $v \in \overline{S} - R$.

We argue that in this remaining case $G[S]$ satisfies the SPC. Let $\hat{P}$ be an S-partition of $G[S]$; note that $B_{\hat{P}} = \varnothing$. We may also view $\hat{P}$ as an S-partition of G, in which case we denote it by P, so $B_P = \overline{S}$. Comparing values of f_g for $G[S]$ and G, we have $f_g(\hat{P}) - f_g(P) = g(B_P) - |[S, \overline{S}]|$. Since $g(B_P) = g(R) = |[S, \overline{S}]|$, we have $f_g(\hat{P}) = f_g(P) \geq 0$.

Since $G[S]$ satisfies the SPC, Theorem 2.9 yields $k + g(S)$ edge-disjoint subgraphs of $G[S]$ such that k are S-connectors in $G[S]$ and the others combine into disjoint sets of $g(v)$ edges at v for each $v \in S$. Since $g(v) = 0$ for $v \in \overline{S} - R$ and $g(v) = d_G(v)$ for $v \in R$, adding the edges from R to S as directed paths completes a (k, g)-family for G. □

5 Steiner tree packing

In this section we apply Theorem 2.2 to the problem of packing S-trees. Recall that $E(v)$ denotes the set of edges incident to a vertex v and that a vertex set S is j-edge-connected in a graph G when deleting any set of fewer than j edges leaves S in a single component. Our sufficient condition for k edge-disjoint S-trees uses the following theorem, which is the main technical result of this section and is proved using Theorem 2.2.

Theorem 5.1. *Let k and λk be positive integers with $\lambda \geq 6.5$. Let S be a vertex set that is λk-edge-connected in a graph G. Fix a vertex $v \in S$ with $d_G(v) = \lambda k$. Let $E_0, \ldots, E_k$ be a partition of $E(v)$, and let $N_i = \{w\colon vw \in E_i\}$. If $|E_0| \geq k$, then G has edge-disjoint subgraphs $H_0, \ldots, H_k$ such that*
(1) $E_i \subseteq E(H_i)$ for $0 \leq i \leq k$;
(2) $d_{H_0}(s) \geq k$ for all $s \in S$; and
(3) for $1 \leq i \leq k$, the vertex set $(S - \{v\}) \cup N_i$ is connected in $H_i - v$.

Graphs $H_0, \ldots, H_k$ satisfying the requirements in Theorem 5.1 *properly extend* $E_0, \ldots, E_k$ or *form a proper extension* of $E_0, \ldots, E_k$ in G. By the meaning of "partition", each E_i is nonempty. This notion of proper extension refines the "extension property" used by Lau in [6]. Lau had no special subgraph H_0, and he required $d_{H_i}(s) \geq 2$ for each i and each $s \in S$. In the special case where S is independent, distributing the edges of our H_0 to the other subgraphs yields $H_1, \ldots, H_k$ satisfying his conditions. Lau used only the Nash-Williams Theorem, which we have extended to a condition for (k, g)-families.

Theorem 5.1 immediately yields Theorem 1.5.

Theorem 1.5. *If S is $6.5k$-edge-connected in G, then G contains k edge-disjoint S-trees.*

Proof. Form $\hat{G}$ by adding to G a vertex v and any $\lceil 6.5k \rceil$ edges joining v to S. Let $\hat{S} = S \cup \{v\}$; note that $\hat{S}$ is $\lceil 6.5k \rceil$-edge-connected in $\hat{G}$. Partition $E(v)$ into $E_0, \ldots, E_k$ with $|E_0| \geq k$. Applying Theorem 5.1 to $\hat{G}$ and $\hat{S}$ instead of G and S yields subgraphs $H_0, \ldots, H_k$. By property (3) in Theorem 5.1, $H_1, \ldots, H_k$ contain the desired S-trees. □

Definition 5.2. *Minimal counterexample G_0.* If Theorem 5.1 is not true, then there is a graph G_0 with fewest edges such that S, v, λ, k and $E_0, \ldots, E_k$ satisfy the hypotheses of Theorem 5.1 (where λk is an integer) and yet no proper extension of $E_0, \ldots, E_k$ exists. Among such structures, choose one such that $\overline{S}$ is smallest, where $\overline{S} = V(G_0) - S$. Henceforth let G_0 be such a minimal counterexample. In the lemmas of this section, we obtain properties that G_0 must satisfy, eventually obtaining a contradiction. Minimality implies that G_0 is connected. Also, a λk-edge-connected set of size at least 2 cannot have a loop at a vertex of degree λk, so we may assume there is no loop at the fixed vertex v. □

Lemma 5.3. *In G_0, the set $\overline{S}$ of non-terminal vertices is independent.*

Proof. Let e be an edge with endpoints in $\overline{S}$. If S is λk-edge-connected in $G_0 - e$, then by the minimality of G_0 there exist $H_0, \ldots, H_k$ that properly extend $E_0, \ldots, E_k$ in $G_0 - e$. These subgraphs also properly extend $E_0, \ldots, E_k$ in G_0.

Hence S is not λk-edge-connected in $G_0 - e$. Let F be a subset of $E(G_0)$ with exactly λk edges (including e) such that S is not connected in $G_0 - F$. Exactly two components of $G_0 - F$ contain vertices of S, since S is λk-edge-connected in G_0. Let G_1 and G_2 be the graphs obtained by contracting one of these components to a single vertex, calling that vertex v_j in G_j. For $j \in \{1, 2\}$, let $S_j = (S \cap V(G_j) \cup \{v_j\}$; note that S_j is λk-edge-connected in G_j. By symmetry, we may assume that the special vertex v in S lies in $V(G_1)$.

Since the endpoints of e are in $\overline{S}$, the cut F does not isolate a vertex, so G_1 and G_2 are smaller than G_0. Hence there exist $H_0^1, \ldots, H_k^1$ that properly extend $E_0, \ldots, E_k$ in G_1. Let $E_i^2 = E(H_i^1) \cap F$ for $0 \le i \le k$. In G^2, we obtain $H_0^2, \ldots, H_k^2$ that properly extend $E_0^2, \ldots, E_k^2$. For $0 \le i \le k$, let H_i be the subgraph of G with $E(H_i) = E(H_i^1) \cup E(H_i^2)$. Now $H_0, \ldots, H_k$ properly extend $E_0, \ldots, E_k$ in G_0, a contradiction. □

For $x, y \in V(G)$, let $\kappa'(x, y; G)$ denote the *local edge-connectivity* of x and y in G, which is the minimum number of edges whose deletion leaves x and y in different components. Mader's Splitting Lemma is a powerful inductive tool involving local edge-connectivity.

Theorem 5.4 (Mader's Splitting Lemma [8])**.** *Let x be a non-cut-vertex of G. If x has degree at least 2 (except when $d_G(x) = 3$ and x has three distinct neighbors), then there is a shortcut $\hat{G}$ of G at x such that $\kappa'(u, v; G) = \kappa'(u, v; \hat{G})$ whenever $u, v \in V(G) - \{x\}$.*

To simplify our subsequent proofs, we need a slightly stronger version of Mader's Lemma that is less well known.

Theorem 5.5 (Mader's Splitting Lemma, variation)**.** *If $x \in V(G)$ and x is not incident to a cut-edge of G, then there is a shortcut $\hat{G}$ of G at x that preserves local edge-connectivity in $V(G) - \{x\}$ unless $d_G(x) = 3$ and x has three distinct neighbors.*

Proof. By Lemma 5.4, we may assume that x is a cut-vertex of G. Since x is not incident to a cut-edge, x has at least two neighbors in each component of $G-x$. Let $G_1,\ldots,G_t$ be the components of $G-x$. Let y and y' be neighbors of x in G_1, and let z and z' be neighbors of x in G_2. Form G' from G by the shortcut replacing yx and xz with yz. We show that $\kappa'_{G'}(u,v) \geq \kappa'_G(u,v)$ for $u,v \in V(G)$.

Suppose first that $u,v \in V(G_i) \cup \{x\}$. Any family of edge-disjoint u,v-paths in G lies in the subgraph induced by $V(G_i) \cup \{x\}$ and remains in G' unless it uses one of the shortcut edges. Hence we may assume $i=1$, by symmetry. In that case, the shortcut edge yx can be replaced by a path through the edge yz, a zz'-path in G_2, and the edge z_2x to obtain a family of the same size in G'.

Hence we may assume that u and v lie in different components of $G-x$. Let $\ell = \min\{\kappa'_G(u,x), \kappa'_G(v,x)\}$. We showed in the previous paragraph that no set of $\ell-1$ edges separates x from u or v in G'. Hence also no set of $\ell-1$ edges separates u from v in G'. Since u and v lie in different components of $G-x$, all u,v-paths in G pass through x, and hence $\kappa'_G(u,v)=\ell$, which completes the proof. □

Since Theorem 5.1 trivially holds for a graph that has only two vertices (both in S), the next structural property of G_0 allows us to assume henceforth that $|S| \geq 3$.

Lemma 5.6. *In G_0, every vertex of $\overline{S}$ has degree 3, with three distinct neighbors in S (and hence $|S| \geq 3$).*

Proof. Consider $x \in \overline{S}$. If x is incident to a cut-edge e, then S is contained within one component of $G-e$, since S is λk-edge-connected in G. In this case, we can apply minimality in the choice of G_0, restricting the graph to that component.

We may therefore assume that x is not incident to a cut-edge. Except when $d_{G_0}(x)=3$ and x has three distinct neighbors, Mader's Splitting Lemma now implies that S is λk-edge-connected in some shortcut of G_0 at x. By minimality in the choice of G_0, that shortcut of G_0 has a proper extension of $E_0,\ldots,E_k$, which implies that G_0 does also.

We may therefore assume that $d_{G_0}(x)=3$ and x has three distinct neighbors. By Lemma 5.3, those three distinct neighbors lie in S. □

Definition 5.7. *The modified set S' of terminals.* Within G_0, pick a vertex u_i from N_i for $1 \leq i \leq k$. These vertices need not be distinct and may lie in S. Let $U=\{u_1,\ldots,u_k\}$, $S'=S-\{v\}$, $N'_i = N_i - u_i - S'$ and $X=\bigcup_{i=1}^k N'_i$ (see Figure 2). Let M be the maximal bipartite subgraph of G_0 with partite sets X and S'. Note that $|S'| \geq 2$. □

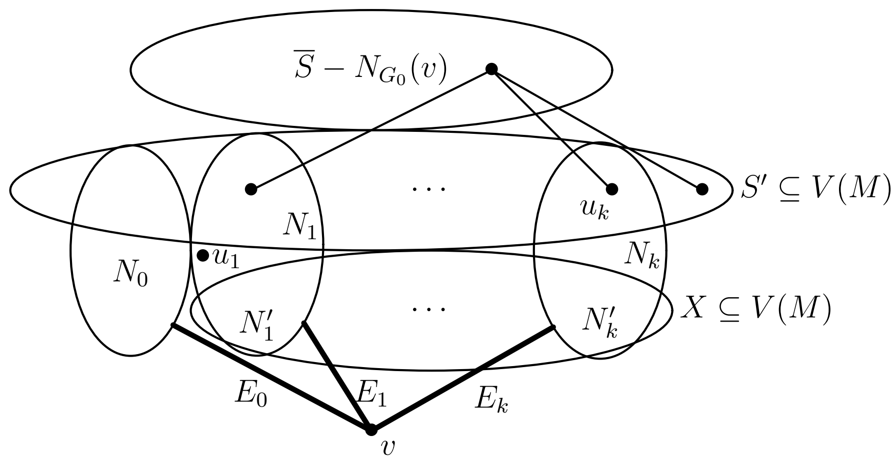

Figure 2: Vertices and vertex sets in G_0; let $G' = G_0 - v - X$

Lemma 5.8. *In G_0, there exists a subgraph M' of M such that:*
(1) $d_{M'}(x) = 1$ for all $x \in X$; and
(2) $d_{M'}(s) \geq \lfloor d_M(s)/2 \rfloor$ for all $s \in S'$.

Proof. By Definition 5.7, $X \subseteq \overline{S} \cap N_{G_0}(v)$. Hence every vertex in X has two distinct neighbors in M, by Lemma 5.6. By adding one vertex adjacent to all vertices of odd degree in M and following an Eulerian circuit in each component of the resulting graph, we obtain an orientation D of M (ignoring the edges added to M) in which every vertex $s \in S'$ has outdegree $\lfloor d_M(s)/2 \rfloor$ or $\lceil d_M(s)/2 \rceil$ and every vertex of M has indegree 1. The subgraph of M whose edges are those oriented from S' to X in D is the desired subgraph M'. □

Definition 5.9. *The derived graph G' and special parity function.* Given G_0 as in Definition 5.2, let $G' = G_0 - v - X$. Using S' as the set of terminals, where $S' = S - \{v\}$ as in Definition 5.7, we define a special S'-parity function g as follows

$$g(u) = \begin{cases} 0, & u \in (N_0 \cup U) - S'; \\ 1, & u \in \overline{S} - N_{G_0}(v); \\ \max\{k - d_{M'}(u) - |E(u) \cap E_0|, 0\}, & u \in S'. \end{cases}$$ □

We will prove that G' has a (k, g)-family for the terminal set S' and this S'-parity function g. Because the proof is lengthy, we first motivate it by using such a (k, g)-family to complete the proof of Theorem 5.1. Obtaining a proper extension of $E_0, \ldots, E_k$ contradicts the definition of G_0, thus forbidding counterexamples and proving Theorem 5.1.

Lemma 5.10. *If the graph G' derived from G_0 has a (k, g)-family for the S'-parity function g in Definition 5.9, then there is a proper extension of $E_0, \ldots, E_k$ in G_0.*

Proof. We will use a (k, g)-family in G' to extend $E_0, \ldots, E_k$ in G_0, adding edges to E_i to form H_i, thereby satisfying (1) in Theorem 5.1. For $1 \le i \le k$, we will add to E_i the edges of one S'-connector and additional edges needed to ensure (3) in Theorem 5.1. To extend E_0, we will use the oriented paths in the (k, g)-family, suitably adjusted.

In order to handle vertices of $U - S'$ (recall that $U = \{u_1, \ldots, u_k\}$), we first adjust the (k, g)-family in G'. We are given S'-connectors $H'_1, \ldots, H'_k$ and oriented paths $P_1, \ldots, P_{g(V(G'))}$. We may assume that $H'_1, \ldots, H'_k$ are minimal S'-connectors. Thus each path joining vertices of S' in H'_j is an edge or has length 2 with internal vertex in $\overline{S}$.

Minimality also implies that short-cutting the paths forming H'_j turns H'_j into a tree T'_j with vertex set S'. *Mark* an edge in T'_j with label i if it arises by short-cutting the two-edge path through u_i for some $u_i \in U - S'$. Since such a vertex u_i has degree 2 in G', and $H'_1, \ldots, H'_k$ are edge-disjoint, each label marks an edge in at most one tree. We will modify $T'_1, \ldots, T'_k$ so that each T'_j contains at most one marked edge.

If some such tree T has two marked edges, then let e be one of them. At most k edges are marked, so some tree T' in the list has none. Adding e to T' completes a unique cycle via a path that crosses from one component of $T - e$ to the other using an edge e' of T'. Replacing T and T' with $T - e + e'$ and $T' - e' + e$ yields a new set of trees in which fewer have more than one marked edge. The edge switch corresponds in G' to switching paths in the edge-disjoint S'-connectors.

Repeat the switching argument until no tree has more than one marked edge. Re-index the resulting S'-connectors so that each $u_i \in U - S'$ occurs in none of $H'_1, \ldots, H'_k$ other than H'_i. For $1 \le i \le k$, let $\hat{H}_i$ be the spanning subgraph of G_0 with edge set $E_i \cup E(H'_i) \cup B_i$, where B_i is the set of edges in $E(M) - E(M')$ incident to N'_i. Let $\hat{H}_0$ be the spanning subgraph of G_0 with edge set $E_0 \cup E(M') \cup \bigcup_{j=1}^{g(V(G))} E(P_j)$.

Since H'_i is an S'-connector in G', all of $S - \{v\}$ is connected in $\hat{H}_i - v$. If $x \in N'_i$, then x has two incident edges in M; one is in M' (by Lemma 5.8) and the other connects x to $S - \{v\}$ in $\hat{H}_i - v$. Now all of $(S - \{v\}) \cup N_i$ is connected in $\hat{H}_i - v$, except possibly u_i if $u_i \in U - S'$. In this case, u_i is not in M but is in G'. By the switching argument given above, if the two edges incident to u_i in G' are in $\bigcup_{j=1}^{k} H'_j$, then they are in H'_i, and we let $H_i = \hat{H}_i$. Otherwise, we add those two edges to $\hat{H}_i$ to form H_i, unless they form some path P_r in the g-family (note that $g(u_i) = 0$), in which case we add the edge leaving u_i in P_r to $\hat{H}_i$ to form H_i. In each case, u_i is now connected to S', and we have satisfied (3) in Theorem 5.1.

In forming H_i, we may have removed one edge of one path P_r from $\hat{H}_0$. Let H_0 be the subgraph of $\hat{H}_0$ that remains after all such edges have been removed. No edges of E_0 were removed, so $d_{H_0}(v) \geq k$, and we need only check that H_0 has enough edges at each $s \in S'$ to satisfy (2) in Theorem 5.1. There remain at least $g(s)$ edges from the paths in the g-family, since we removed only edges leaving vertices of $U - S'$. Adding $E(s) \cap E_0$ and the edges of M' yields $d_{H_0}(s) \geq g(s) + |E(s) \cap E_0| + d_{M'}(s) \geq k$. □

By Lemma 5.10, the next lemma completes the proof of Theorem 5.1 and hence also Theorem 1.5. This is where we use $\lambda \geq 6.5$. Although introducing the vertex set U complicates the construction in Lemma 5.10, it enables us to improve our result from $\lambda \geq 10$ to $\lambda \geq 6.5$ by reducing the requirement on $d_{H_0}(s)$ in (2) of Theorem 5.1 from $2k$ to k.

Lemma 5.11. *Given G_0, the derived graph G' has a (k, g)-family for the S'-parity function g in Definition 5.9.*

Proof. By Theorem 2.2, it suffices to prove that the SPC holds for G' and g. That is, $f_g(P) \geq 0$ for each S'-partition P of G'. Recall the definition:

$$f_g(P) = \sum_{A_i \in P} \delta_{G'}(A_i) - 2k(|P| - 1) - g(B_P) - 2g(T_P). \tag{4}$$

Our discussion of P and the sets B_P and T_P is always with respect to G'. It suffices to prove $f_g(P) \geq 0$ for a S'-partition P with special properties among those that minimize f_g.

By Lemma 5.6, every vertex of $V(G_0) - S$ has degree 3 in G_0, with three distinct neighbors in S. If $w \in A_i - S'$ for some block A_i in P, and w has no neighbor in A_i, then w has a neighbor in some block A_j other than A_i, and switching w from A_i to A_j produces an S'-partition P' of G' with $f_g(P') < f_g(P)$. Hence we may assume that every vertex of $V(G') - S'$ in a block of P has a neighbor in that block.

Next, the definition of g immediately yields $g(B_P) = n_o(B_P)$ (computed in G'). If $w \in B_P$, then $d_{G'}(w) \in \{2, 3\}$, and the neighbors of w are distinct vertices of S'. If $d_{G'}(w) = 2$, or if $d_{G'}(w) = 3$ and w has two neighbors in one block of P, then let P' be the S'-partition formed from P by moving w into a block containing at least half of $N_{G'}(w)$. Regardless of whether $d_{G'}(w)$ is 2 or 3, we obtain $f_g(P') \leq f_g(P)$. Iterating this operation yields P minimizing f_g such that every vertex in B_P has neighbors in three different blocks of P, and $g(B_P) = |B_P|$. Hence also v has no neighbor in B_P.

We can now exclude $|P| = 1$. If $|P| = 1$, then $|S'| \geq 2$ implies $T_P = \varnothing$. Since vertices of B_P must have neighbors in three blocks, also $B_P = \varnothing$. Hence $\delta(A_1) = 0$ and $f_g(P) = 0$.

To prove $f_g(P) \ge 0$ when $|P| > 1$, we need lower bounds on $\delta_{G'}(A_i)$. We obtain these using the λk-edge-connectedness of S in G_0. Vertices of X are not in G', but in G_0 they have exactly two neighbors in S'. For $x \in X$ and $j \in \{1, 2\}$, put $x \in X_j$ when $N(x) \cap V(G')$ intersects exactly j blocks in P; thus X_1 and X_2 partition X. Add each vertex of X_1 to the block of P containing its neighbors, forming $A'_1, \ldots, A'_{|P|}$ from $A_1, \ldots, A_{|P|}$; we have $\delta_{G'}(A_i) = \delta_{G_0}(A'_i) - \big|[A'_i, X_2 \cup \{v\}]\big|$. Since S is λk-edge-connected in G_0, its subset S' is also λk-edge-connected in G_0. Since $|P| > 1$, we thus have $\delta_{G_0}(A'_i) \ge \lambda k$ for $1 \le i \le |P|$. Since each vertex of X_2 is adjacent to v and two vertices of S', and v has no neighbor in B_P, in G_0 we have $\sum_{i=1}^{|P|} \big|[A'_i, X_2 \cup \{v\}]\big| = d_{G_0}(v) + |X_2|$. These computations yield

$$\begin{aligned}\sum_{A_i \in P} \delta_{G'}(A_i) &= \sum_{i=1}^{|P|} \Big(\delta_{G_0}(A'_i) - \big|[A'_i, X_2 \cup \{v\}]\big| \Big) \\ &\ge \lambda k\,|P| - d_{G_0}(v) - |X_2| \;=\; \lambda k(|P| - 1) - |X_2|. \end{aligned} \tag{5}$$

Also $3|B_P| = \delta_{G'}(B_P) \le \sum_{A_i \in P} \delta_{G'}(A_i)$, so $g(B_P) \le \frac{1}{3} \sum_{A_i \in P} \delta_{G'}(A_i)$. Using (5),

$$\sum_{A_i \in P} \delta_{G'}(A_i) - g(B_P) - 2k(|P| - 1) \ge \frac{2}{3}[(\lambda - 3)k(|P| - 1) - |X_2|]. \tag{6}$$

Now, to prove $f_g(P) \ge 0$, using the definition in (4) and applying (6), it suffices to prove

$$(\lambda - 3)k(|P| - 1) - |X_2| - 3g(T_P) \ge 0. \tag{7}$$

Our last preliminary computation bounds $|X_2|$. Since $X \subseteq \overline{S}$, vertices of X have no incident multi-edges. Hence $X \cap N_0 = \varnothing$, and we explicitly discarded $u_1, \ldots, u_k$ to form the sets comprising X. Hence E_0 and the k edges from v to U do not reach X. Since $d_{G_0}(v) = \lambda k$, we conclude

$$|X_2| \le |X| \le (\lambda - 2)k. \tag{8}$$

Let $T'_P = \{s \in T_P \colon g(s) > 0\}$; note that $g(T'_P) = g(T_P)$. We complete the proof by considering four cases in terms of $|P|$ and $|T'_P|$, showing in each case that $f_g(P) \ge 0$.

Case 1: $|P| = 2$ and $|T'_P| = 0$. Since $|P| < 3$, we have $B_P = \varnothing$. Using (5) and (8) instead of (7) yields $f_g(P) \ge \lambda k(|P| - 1) - (\lambda - 2)k - 2k(|P| - 1) = (\lambda - 2)k(|P| - 2) = 0$.

Case 2: $|T'_P| \le |P| - 2$. We may assume $|P| \ge 3$. Let L denote the left side of (7). Using $g(s) = k - d_{M'}(s) - |E_0 \cap E(s)|$ for $s \in T'_P$, we have

$$L \ge (\lambda - 3)k(|P| - 2) + (\lambda - 2)k - |X_2| - k - 3k\,|T'_P| + 3 \sum_{s \in T'_P} d_{M'}(s)$$

If $|P| \geq 4$ and $|T'_P| \leq |P| - 2$, then (8) and $\lambda \geq 6.5$ yield $L \geq (\lambda - 6)k(|P| - 2) - k \geq 0$. Hence we may assume $|P| = 3$. We obtain $L \geq (\lambda - 4)k \geq 0$ if $|T'_P| = 0$, so we may also assume $|T'_P| = 1$. Now let s be the one vertex of T'_P. The computation simplifies to

$$L \geq -0.5k + (\lambda - 2)k - |X_2| + 3d_{M'}(s).$$

Now $|X_2| \leq (\lambda - 2.5)k$ or $d_{M'}(s) \geq k/6$ suffices. If both fail, then $\big|[s, v]\big| \leq d_{G_0}(v) - |X_2| < 2.5k$ (since $d_{G_0}(v) = \lambda k$) and $\big|[s, X_2]\big| \leq d_M(s) \leq 2d_{M'}(s) + 1 < k/3 + 1$.

Now index the blocks of P so that $s \in A_1$. Focusing on A_1, we compute

$$\begin{aligned} f_g(P) &= \sum_{A_i \in P} \delta_{G'}(A_i) - 4k - |B_P| - 2g(s) \;\geq\; 2\big|[A_1, A_2 \cup A_3]\big| + 3|B_P| - 4k - |B_P| - 2k \\ &= 2\delta_{G'}(A_1) - 6k \;=\; 2\big(\delta_{G_0}(A_1) - \big|[s, X_2 \cup \{v\}]\big|\big) - 6k \\ &> 13k - 2(k/3 + 1 + 2.5k) - 6k \;>\; 0. \end{aligned}$$

Case 3: $|T'_P| = |P| - 1 \geq 1$. Each $x \in X_2$ has neighbors in S in two blocks of P; hence x has a neighbor in T'_P. Thus $\sum_{s \in T'_P} d_M(s) \geq |X_2|$. Also, $g(T'_P) \leq k\,|T'_P| - \sum_{s \in T'_P} d_{M'}(s)$. Starting again from L, the left side of (7), and using $\lambda k \geq 6k + 1$, we have

$$\begin{aligned} L &\geq (\lambda - 3)k(|P| - 1) - |X_2| - 3k(|P| - 1) + 3\sum_{s \in T'_P} d_{M'}(s) \\ &\geq (|P| - 1) + \sum_{s \in T'_P} d_{M'}(s) + \sum_{s \in T'_P} (d_M(s) - 1) - |X_2| \;\geq\; 0. \end{aligned}$$

Case 4: $|T'_P| = |P| \geq 2$. Here $T'_P = S'$, and each block of P contains just one vertex of S', so $X_1 = \varnothing$ and $X = X_2$. Also, $d_{M'}(T'_P) = d_{M'}(S') = d_{M'}(X) = |X|$. Hence $g(T'_P) = k\,|P| - |X| - \big|[v, S'] \cap E_0\big|$.

We need to strengthen the lower bound on $\sum_{A_i \in P} \delta_{G'}(A_i)$ and upper bound on $|B_P|$ used in (5). Let $W = \{w \in \overline{S}\colon\ vw \in E_0\}$. Note that $\big|[v, W]\big| = |W|$, since $W \subseteq \overline{S}$. If $w \in W \cap A_i$, then w is adjacent to the vertex of S' in A_i (by our initial reduction of P) and to a vertex of S' in another block A_j (by Lemma 5.6). Hence $\delta_{G'}(A_i) = \delta_{G'}(A_i - W)$. Since $\big|[S', X]\big| = 2|X|$, and $X \subseteq N(v)$, and S' is λk-edge-connected in G_0, we have

$$\begin{aligned} \sum_{A_i \in P} \delta_{G'}(A_i) &= \sum_{A_i \in P} \big(\delta_{G_0}(A_i - W) - \big|[A_i - W, X \cup \{v\}]\big|\big) \\ &\geq \lambda k\,|P| - d(v) - |X| + |W| \;=\; \lambda k(|P| - 1) - |X| + |W|. \end{aligned}$$

Each vertex of B_P supplies three of the edges leaving blocks of P, but not the edges leaving blocks of P to or from vertices of W; hence $3|B_P| \leq \left(\sum_{A_i \in P} \delta_{G'}(A_i)\right) - 2|W|$. Now

$$\begin{aligned} f_g(P) &= \sum_{A_i \in P} \delta_{G'}(A_i) - |B_P| - 2k(|P|-1) - 2g(T'_P) \\ &\geq \frac{2}{3}(\lambda k(|P|-1) - |X|) + \frac{4}{3}|W| - 2k(|P|-1) - 2k\,|P| + 2|X| + 2\big|[v,S'] \cap E_0\big| \\ &= \left(\frac{2}{3}\lambda - 4\right) k(|P|-1) + \frac{4}{3}|W| + 2\big|[v,S'] \cap E_0\big| + \frac{4}{3}|X| - 2k \\ &\geq \frac{1}{3}k(|P|-1) + \frac{4}{3}k - 2k. \end{aligned}$$

In the last step, we used $|W| + \big|[v,S'] \cap E_0\big| = |E_0| \geq k$, along with $\lambda \geq 6.5$ and $|X| \geq 0$. The final expression is nonnegative when $|P| \geq 3$.

This leaves the case $|T'_P| = |P| = 2$. As in Case 2, $B_P = \varnothing$, and we have

$$\begin{aligned} f_g(P) &= \sum_{A_i \in P} \delta_{G'}(A_i) - 2k - 2g(T'_P) \\ &\geq \lambda k - |X| + |W| - 2k - 4k + 2|X| + 2\big|[v,S'] \cap E_0\big| \;>\; 0. \end{aligned}$$

□

6 S-connector packing

To prove Theorem 1.6, we prove a theorem for S-connectors analogous to Theorem 5.1. Note that Theorem 6.1 immediately yields Theorem 1.6 in the way that Theorem 5.1 yields Theorem 1.5, by applying it to a graph obtained from the given graph by adding one vertex. The difference from Theorem 5.1 is that, because we seek connectors instead of trees in (3) and (4), the threshold we need in (2) is $2k$ instead of k. This leads to the later computations needing $\lambda \geq 10$ instead of $\lambda \geq 6.5$.

Theorem 6.1. *Fix $k \in \mathbb{N}$ and $\lambda k \in \mathbb{N}$ such that $\lambda \geq 10$. Consider $S \subseteq V(G)$ and $v \in S$ such that S is λk-edge-connected in G and $d_G(v) = \lambda k$. If $E_0, \ldots, E_k$ is a partition of $E(v)$ such that $|E_0| \geq 2k$, then there exist edge-disjoint subgraphs $H_0, \ldots, H_k$ such that*
(1) $E_i \subseteq E(H_i)$;
(2) $d_{H_0}(s) \geq 2k$ for any $s \in S$;
(3) For $i > 0$, H_i is an S-connector; and
(4) For $i > 0$, deleting from the family of paths forming H_i the paths that use edges of E_i leaves an $(S - v)$-connecting family.

The proof of Theorem 6.1 is similar to the proof of Theorem 5.1; we describe the differences without repeating the full argument.

As in Section 5, we consider a minimal counterexample G_0. The arguments of Lemmas 5.3 and 5.6 show that the non-terminal vertices in G_0 form an independent set in which every vertex has degree 3, with three distinct neighbors in S. This time we do not choose special vertices $u_1, \ldots, u_k$. With $S' = S - \{v\}$, $N_i' = N_i - S'$, and $X = \bigcup_{i=1}^{l} N_i'$, we let M be the maximal bipartite subgraph of G_0 with partite sets X and S'. The argument of Lemma 5.8 yields the subgraph M' such that $d_{M'}(x) = 1$ for $x \in X$ and $d_{M'}(s) \geq \lfloor d_M(s)/2 \rfloor$ for $s \in S$.

Again let $G' = G_0 - v - X$. This time we define a slightly different S'-parity function on G': there is no set U, and for $u \in S'$ we replace k with $2k$ in the definition.

$$g(u) = \begin{cases} 0, & u \in N_0 - S'; \\ 1, & u \in \overline{S} - N_{G_0}(v); \\ \max\{2k - d_{M'}(u) - |E(u) \cap E_0|, 0\}, & u \in S'. \end{cases} \tag{9}$$

We reduce the problem to showing that G' has a (k, g)-family for S' and this g, by proving as in Lemma 5.10 that $E_0, \ldots, E_k$ extend in G_0 as specified in Theorem 6.1 when G' has a (k, g)-family with g as in (9). This time the reduction is easier, since we have no chosen vertices $u_1, \ldots, u_k$ to complicate the construction.

Lemma 6.2. *If the graph G' derived from G_0 has a (k, g)-family for the S'-parity function g defined by (9), then $E_0, \ldots, E_k$ extend in G_0 as specified in Theorem 6.1.*

Proof. Given a (k, g)-family in G', let $H_1', \ldots, H_k'$ be the S'-connectors, and let $P_1, \ldots, P_{g(V(G'))}$ be the oriented paths. Constructing H_i by augmenting E_i yields (1) in Theorem 6.1.

Let H_0 be the spanning subgraph of G with edge set $E_0 \cup E(M') \cup \bigcup_{j=1}^{g(V(G))} E(P_j)$. For $1 \leq i \leq k$, let H_i be the spanning subgraph of G with edge set $E_i \cup E(H_i') \cup B_i$, where B_i is the set of edges in $E(M) - E(M')$ incident to N_i'.

For (3), note for $1 \leq i \leq k$ that $E_i \cup B_i$ is a nonempty set of paths that join v to vertices of S'. We do not require H_0 to be an S-connector.

For (4), when we delete the paths formed by $E_i \cup B_i$, we return to H_i', which is an S'-connector in G' and hence is an $(S - \{v\})$-connecting family in $G - v$.

For (2), we check that H_0 gains enough edges at each vertex of S'. For $s \in S'$, in H_0' there are at least $g(s)$ edges incident to s, provided explicitly by the paths in the (k, g)-family. Adding $E_0 \cap E(s)$ and the edges of M' yields $d_{H_0}(s) \geq g(s) + |E_0 \cap E(s)| + d_{M'}(s) \geq 2k$. Also $d_{H_0}(v) \geq 2k$, since $|E_0| \geq 2k$. □

Finally, we prove the analogue of Lemma 5.11.

Lemma 6.3. *Given G_0, the derived graph G' has a (k, g)-family for the S'-parity function g defined by (9).*

Proof. By Theorem 2.2, it suffices to prove that the SPC holds for G' and g. That is, $f_g(P) \geq 0$ for each S'-partition of G', where

$$f_g(P) = \sum_{A_i \in P} \delta(A_i) - 2k(|P| - 1) - g(B_P) - 2g(T_P).$$

As in Lemma 5.11, we may assume that every vertex of $\overline{S}$ has degree 3 in G_0, that every vertex outside S' in a block of P has a neighbor in that block, that every vertex in B_P has neighbors in three different blocks of P, and that $g(B_P) = |B_P|$. Similarly, vertices of X have exactly two neighbors in S'. Again let X_2 be the subset of X whose vertices having neighbors in distinct blocks of P. Arguing exactly as in Lemma 5.11 yields (5), (6), (7), (8), except that now we use $\big|[v, N_0]\big| = E_0 \geq 2k$ instead of $\big|[v, N_0 \cup U]\big| \geq 2k$, since there is no U and instead we increased the requirement on $|E_0|$ to $2k$.

There remain only the computations in the Cases. Again let $T'_P = \{s \in T_P \colon\ g(s) > 0\}$. The computations for $|P| = 1$ and Case 1 ($|T'_P| = |P| - 2 = 0$) are unchanged.

Case 2: $|T'_P| \leq |P| - 2$ and $|P| \geq 3$. Again let L be the left side of (7). Using (8) and $\lambda \geq 10$ and $g(T'_P) \leq 2k|T'_P|$,

$$L/k \geq (\lambda - 3)(|P| - 2) - 1 - 6|T'_P| \geq -1 + (|P| - 2) + 6(|P| - 2 - |T'_P|) \geq 0.$$

Case 3: $|T'_P| = |P| - 1 \geq 1$. With $g(T'_P) \leq 2k\,|T'_P| - \sum_{s \in T'_P} d_{M'}(s)$ and $\lambda k \geq 10k \geq 9k+1$, the computation becomes

$$\begin{aligned} L &\geq (\lambda - 3)k(|P| - 1) - |X_2| - 6k(|P| - 1) + 3\sum_{s \in T'_P} d_{M'}(s) \\ &\geq (|P| - 1) + \sum_{s \in T'_P} (d_{M'}(s) - 1) + \sum_{s \in T'_P} d_M(s) - |X_2| \quad \geq \quad 0. \end{aligned}$$

Case 4: $|T'_P| = |P| \geq 2$. As in Case 4 of Lemma 5.11, the computation starts with

$\sum_{A_i \in P} \delta_{G'}(A_i) \geq \lambda k(|P|-1) - |X| + |W|$ and $3|B_P| \leq \left(\sum_{A_i \in P} \delta_{G'}(A_i)\right) - 2|W|$. It ends with

$$\begin{aligned} f_g(P) &= \sum_{A_i \in P} \delta_{G'}(A_i) - |B_P| - 2k(|P|-1) - 2g(T'_P) \\ &\geq \frac{2}{3}(\lambda k(|P|-1) - |X|) + \frac{4}{3}|W| - 2k(|P|-1) - 4k\,|P| + 2|X| + 2\big|[v, S'] \cap E_0\big| \\ &= \left(\frac{2}{3}\lambda - 6\right) k(|P|-1) + \frac{4}{3}|W| + 2\big|[v, S'] \cap E_0\big| + \frac{4}{3}|X| - 4k \\ &\geq \frac{2}{3}k(|P|-1) + \frac{8}{3}k - 4k. \end{aligned}$$

In the last step, we used $|W| + \big|[v, S'] \cap E_0\big| \geq |E_0| \geq 2k$, along with $\lambda \geq 10$ and $|X| \geq 0$. The final expression is nonnegative when $|P| \geq 3$.

This leaves the case $|T'_P| = |P| = 2$. As in Case 2, $B_P = \varnothing$, and $\lambda \geq 10$ is enough to give

$$\begin{aligned} f_g(P) &= \sum_{A_i \in P} \delta_{G'}(A_i) - 2k - 2g(T'_P) \\ &\geq \lambda k - |X| + |W| - 2k - 8k + 2|X| + 2\big|[v, S'] \cap E_0\big| \quad \geq \quad 0. \end{aligned}$$

□

Acknowledgment

We thank Chandra Chekuri and Lap Chi Lau for helpful discussions about this problem. We also thank one anonymous referee for a very thorough reading of the paper, which led to substantial improvement of the exposition, and another for catching an error in Definition 5.8.

References

[1] J. Edmonds, Matroid partition. *Mathematics of the Decision Sciences, Part I (Seminar, Stanford, Calif., 1967)* (1968), 335–345.

[2] A. Frank, T. Király, and M. Kriesell, On decomposing a hypergraph into k connected subhypergraphs, *Discrete Appl. Math.* 131 (2003), 373–383.

[3] S. L. Hakimi, On the degrees of the vertices of a directed graph. *J. Franklin Inst.* 279 (1965), 290–308.

[4] M. Kriesell, Edge-disjoint trees containing some given vertices in a graph. *J. Combin. Theory Ser. B* 88 (2003), 53–65.

[5] M. Kriesell, Edge disjoint Steiner trees in graphs without large bridges. *J. Graph Theory* 62 (2009), 188–198.

[6] L. C. Lau, An approximate max-Steiner-tree-packing min-Steiner-cut theorem. *Combinatorica* 27 (2007), 71–90.

[7] C. St. J. A. Nash-Williams, Edge disjoint spanning trees of finite graphs, *J. London Math. Soc.* 36 (1961), 445–450.

[8] W. Mader, A reduction method for edge-connectivity in graphs. *Ann. Discrete Math.* 3 (1978), 145–164.

[9] W. T. Tutte, On the problem of decomposing a graph into n connected factors, *J. London Math. Soc.* 36 (1961), 221–230.