

# STAR-FACTORS OF TOURNAMENTS

**Guantao Chen**\*

*Georgia State University, Atlanta, GA 30303*

**Xiaoyun Lu**

*Chinese University of Hong Kong, Hong Kong*

**Douglas B. West**†

*University of Illinois, Urbana, IL 61801-2975*

**Abstract.** Let  $S_m$  denote the  $m$ -vertex simple digraph formed by  $m - 1$  edges with a common tail. Let  $f(m)$  denote the minimum  $n$  such that every  $n$ -vertex tournament has a spanning subgraph consisting of  $n/m$  disjoint copies of  $S_m$ . We prove that  $m \lg m - m \lg \lg m \leq f(m) \leq 4m^2 - 6m$  for sufficiently large  $m$ .

If  $D$  is an acyclic digraph of order  $n$ , then  $D$  occurs as a subgraph of every tournament with at least  $2^{n-1}$  vertices; this follows easily by induction on  $n$ . Special digraphs can be guaranteed to appear even when there are no extra vertices. A digraph of order  $n$  is *unavoidable* if it appears in every  $n$ -vertex tournament. A *claw* is a digraph obtained by identifying the sources of a set of edge-disjoint paths. The question of which  $n$ -vertex claws are unavoidable is studied in [5,6,10]. Further references and additional unavoidable directed trees appear in [7].

A claw formed using paths of length one is a *star*; the  $m$ -star  $S_m$  consists of  $m - 1$  edges with a common tail and  $m - 1$  distinct heads. We use  $kS_m$  to denote a disjoint union of  $k$  stars of order  $m$ . A 3-vertex tournament need not contain a spanning  $S_3$ , but every 6-vertex tournament contains a spanning  $2S_3$ . Some 8-vertex tournaments avoid  $2S_4$  (see Theorem 3). A copy of  $kS_m$  in a tournament of order  $km$  is an  $m$ -star-factor of the tournament.

If  $m$ -star-factors are unavoidable for some multiple of  $m$ , then inductively they are unavoidable for all larger multiples of  $m$ . Let  $f(m)$  be the least  $n$  such that every  $n$ -vertex tournament contains an  $m$ -star-factor (if such  $n$  exists). We prove that  $m \lg m - m \lg \lg m \leq f(m) \leq 4m^2 - 6m$  for sufficiently large  $m$ . The upper bound holds for all  $m$ . (Always  $\lg$  denotes  $\log_2$ .)

Reid [9] asked for the minimum  $n$  such that all  $n$ -vertex tournaments have  $k$  pairwise-disjoint transitive subtournaments of order  $m$ . For fixed  $m$ , Erdős proved the existence

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of  $g(m)$  such that vertices of tournaments with order  $g(m)$  can be partitioned into sets inducing transitive tournaments of order  $m$ . Since every transitive tournament has a spanning star,  $f(m) \leq g(m)$ , and our lower bound requires  $g(m) \geq m \lg m - m \lg \lg m$  (see our final remark for an upper bound on  $g(m)$ ). (Lonc and Truszczyński [4] observed the weaker conclusion that the vertices of a sufficiently large tournament can be partitioned into sets of size *at least*  $k$  that induce transitive subtournaments, but their result holds in a more general setting.)

Our lower bound depends on a bound for another problem. We say that a vertex of a tournament *dominates* all its successors, and a tournament is *k-dominated* (historically, “*satisfies property  $P_k$* ”) if for every  $k$ -set  $U$  of vertices there is a vertex outside  $U$  that dominates  $U$ . Erdős [2] proved that when  $n$  is sufficiently large, some  $n$ -vertex tournament is  $k$ -dominated, because then the expected number of undominated  $k$ -sets in the random tournament is less than 1. The expected number of undominated  $k$ -sets is  $\binom{n}{k}(1 - 2^{-k})^{n-k}$ . Since  $\binom{n}{k}(1 - x)^t < (ne/k)^k e^{-xt}$ , the expectation is less than 1 when  $n > 2^k k^2 (\ln 2)(1 + o(1))$ . (Erdős and Moser [3] obtained even stronger conclusions when  $n$  is above this threshold.)

Let  $h(k)$  be the minimum  $n$  such that some  $n$ -vertex tournament is  $k$ -dominated. The bounds are  $ck2^k \leq h(k) \leq 2^k k^2 (\ln 2)(1 + o(1))$ ; we will use the upper bound. (The lower bound is by Szekeres and described in [8]; Erdős had proved that  $h(k) \geq 2^{k+1} - 1$ . The upper bound is that of Erdős described above; see [1, p.5-6] for further discussion of this problem and these computations).

**THEOREM 1.** If  $n < m \lg m - m \lg \lg m$  (for sufficiently large  $m$ ), then some  $n$ -vertex tournament has no spanning  $m$ -star-factor.

**Proof:** We show first that  $kS_m$  is avoidable when  $km \geq h(k)$ . Since  $km \geq h(k)$ , some  $km$ -vertex tournament  $T$  is  $k$ -dominated. Let  $U$  be a set of  $k$  sources of stars in  $T$ . Since  $T$  is  $k$ -dominated, some vertex in  $T$  dominates  $U$ . Such a vertex belongs to none of these  $k$  stars. Thus  $T$  is not spanned by  $k$  stars of any sizes.

Thus it suffices to show that  $n \geq h(n/m)$  when  $n$  is a multiple of  $m$  such that  $n < m(\lg m - \lg \lg m)$ . Let  $k = n/m$ . Since  $h(k) \leq 2^k k^2 (\ln 2)(1 + o(1))$ , it suffices to show that  $n \geq 2^{(n/m)}(n/m)^2 (\ln 2)(1 + o(1))$ . This simplifies to  $2m \lg m \geq n + m(\lg n + O(1))$ , which is satisfied for  $n < m(\lg m - \lg \lg m)$  when  $m$  is sufficiently large. ■

**THEOREM 2.** If  $n > 4m^2 - 6m$  and  $n$  is a multiple of  $m$ , then every  $n$ -vertex tournament has an  $m$ -star-factor.

**Proof:** Consider an  $n$ -vertex tournament, and let  $x$  be a vertex of maximum out-degree. We construct an  $m$ -star-factor. In the subtournament induced by  $N^-(x)$ , we use as many disjoint  $m$ -stars as possible. Let  $A$  be the subset of  $N^-(x)$  not covered by these stars. We have  $|A| \leq 2m - 3$ , else within  $A$  we can find another  $m$ -star. If some  $m$ -star has source in  $A$  and leaves in  $N^+(x)$ , we use it. When no further  $m$ -stars of this type can be found, let  $A'$  be the remaining subset of  $N^-(x)$ , and let  $a = |A'|$ . Figure 1 illustrates these sets.

Let  $B$  be the remaining subset of  $N^+(x)$ , and let  $B'$  be the set of vertices in  $B$  that dominate  $A'$ . By the definition of  $A'$ , there are at most  $a(m - 2)$  edges from  $A'$  to  $B$ . At least  $|B| - a(m - 2)$  vertices of  $B$  are untouched by these edges, and hence  $|B'| \geq$

$|B| - a(m - 2)$ . Since  $d^+(x) \geq (n - 1)/2$  and at most  $2m - 3 - a$  stars were used with source in  $A$  and leaves in  $N^+(x)$ , we have  $|B| \geq (n - 1)/2 - (m - 1)(2m - 3 - a)$ . Hence  $|B'| \geq (n - 1)/2 - (m - 1)(2m - 3) + a$ .

Since  $n > 2m(2m - 3)$ , we have  $|B'| \geq 2m - 3$ . This means that the subtournament induced by  $B'$  has a vertex  $y$  with out-degree at least  $m - 2$ . Let  $R$  be a subset of  $B'$  consisting of  $y$  and  $m - 2$  successors of  $y$ . If  $a \geq m$ , we use a vertex of  $B'$  outside of  $R$  as the source of an  $m$ -star dominating  $m - 1$  vertices in  $A'$ . Since  $a \leq 2m - 3$ , fewer than  $m$  vertices of  $A'$  remain uncovered. If any remain uncovered, we use  $y$  as the source of an  $m$ -star that dominates the rest of  $A'$  (or all of  $A'$  if  $0 < a < m$ ) and as much of  $R$  as needed to complete the star.

We have now covered all vertices except  $x$  and a subset of  $B$ . We iteratively use  $m$ -stars from the subtournament induced by  $B$ . As long as at least  $2m - 2$  vertices remain in  $B$ , we can find another  $m$ -star. When fewer than  $2m - 2$  vertices remain, we use a star with source  $x$ . Now fewer than  $m - 1$  vertices remain, which must equal 0 since  $n$  is a multiple of  $m$ . ■

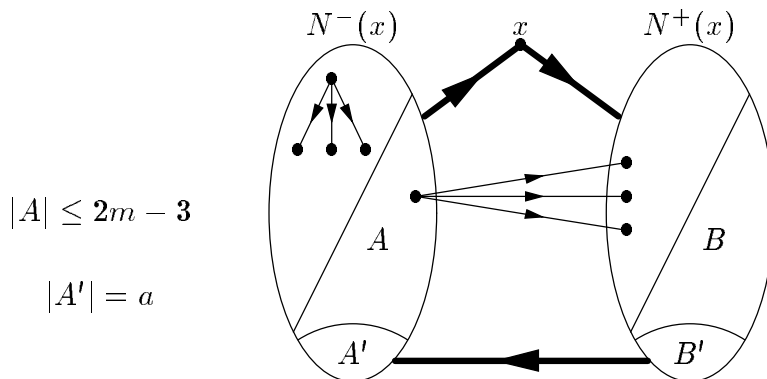


Figure 1. Structure of the  $m$ -star factor in Theorem 2.

For the interested reader, we discuss the first few values of  $f(m)$ .

**THEOREM 3.**  $f(2) = 2$ ,  $f(3) = 6$ , and  $f(4) \geq 12$ .

**Proof:** Trivially,  $f(2) = 2$ . The cyclic triple has no  $S_3$ , so  $f(3) \geq 6$ . Let  $T$  be an arbitrary tournament of order 6. Let  $a|bc$  denote a star centered at  $a$  with the remaining vertices as leaves, and let  $(abc)$  denote a cycle with edges  $ab, bc, ca$ . Every subtournament induced by at least four vertices contains  $S_3$ , so we begin with  $x|yz$  and the edge  $yz$ . Given any  $S_3$  in  $T$ , we are finished unless the remaining three vertices form a cyclic triple, so we also have  $(uvw)$ . If  $d^+(x) = 5$ , then we have a 3-star within  $T - x$  and another with center  $x$ . Thus we may assume the edge  $ux$ . Now  $u|xv \Rightarrow (yzw)$ ,  $w|uy \Rightarrow (xzv)$ , and  $v|xw \Rightarrow (yzu)$ . We now have  $2S_3$  formed by  $z|vw$  and  $u|xy$ .

For  $m = 4$ , we present 8-vertex tournaments having no  $2S_4$ . Our examples contain a particular 6-vertex tournament  $T_6$ . Construct the 3-cycles  $(x_1x_2x_3)$  and  $(y_3y_2y_1)$ . Add the

three edges of the form  $x_i y_i$  and all six edges of the form  $y_i x_j$  for  $i \neq j$ . We have  $d^+(x_i) = 2$  and  $d^+(y_i) = 3$ . Each copy of  $S_4$  in this tournament has  $y_i$  as its center, for some  $i$ , with leaves  $y_{i-1}, x_{i-1}, x_{i+1}$  (indices modulo 3). Let  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y_2, y_3\}$ .

Add a vertex  $u$  such that  $N^-(u) = X$  and  $N^+(u) = Y$ . The resulting 7-vertex tournament  $T_7$  is regular and does not contain  $S_4 + S_3$ . Each copy of  $S_4$  consists of a vertex and all its successors. Deleting the 4-star centered at  $u, x_i, y_i$  leaves the 3-cycles  $(x_1 x_2 x_3)$ ,  $(x_{i-1} y_{i-1} y_{i+1})$ ,  $(y_{i+1} x_i u)$ , respectively.

To construct an 8-vertex tournament avoiding  $2S_4$ , it suffices to add a sink to  $T_7$ . An 8-vertex tournament with a sink contains  $2S_4$  if and only if the 7-vertex tournament obtained by deleting the sink contains  $S_4 + S_3$ , which  $T_7$  does not. Another such tournament, with no sink, is obtained by adding to  $T_7$  a vertex  $v$  such that  $N^-(v) = X$  and  $N^+(v) = Y \cup \{u\}$ . In this tournament  $T_8$ , the vertices of  $Y \cup \{u\}$  have out-degree 3, and those of  $X \cup \{v\}$  have out-degree 4. Deleting the 4-star centered at  $u$  leaves a 4-vertex tournament with a sink. Deleting the 4-star centered at  $y_i$  leaves  $x_i, v$  with out-degree 2 and  $y_{i+1}, u$  with out-degree 1. Thus the centers of a copy of  $2S_4$  in  $T_8$  must both lie in  $X \cup \{v\}$ . Two centers from  $X$  cannot dominate all of  $Y$ , and a center from  $X$  along with  $v$  as a second center cannot dominate all of  $X$ . ■

One referee observed that  $f(3) = 6$  also follows immediately from the first sentence of [9]. For  $f(4)$ , Theorem 2 yields  $f(4) \leq 24$ . We convinced ourselves that  $f(4) = 12$  via a case analysis too tedious and painful to reproduce. We leave this open in the hope that someone will improve the general bound in Theorem 2.

Finally, we thank Zbigniew Lonc (Warsaw University of Technology) for providing a proof of a bound on  $g(m)$ , the existence of which was attributed to Erdős in [9] but appears never to have been published. First, let  $\theta(m)$  denote the minimum number of vertices that forces every tournament to have a transitive subtournament of order  $m$ . The bound  $\theta(m) \leq 2^{m-1}$  is usually attributed to Erdős and Moser and appears in Moon [8, p.15].

Lonc argued as follows that  $g(m) \leq m \lceil \theta((m-1)\theta(m))/m \rceil$ . Let  $T$  be a tournament of this order;  $T$  has a transitive subtournament  $A$  of order  $(m-1)\theta(m)$ . From  $T - A$  we delete transitive tournaments of order  $m$  until fewer than  $\theta(m)$  vertices remain outside  $A$ . For each remaining vertex  $v \in T - A$ , iteratively, we find  $m-1$  successors or  $m-1$  predecessors of  $v$  in  $A$ . Together with  $v$ , these form a transitive subtournament. When we process the last such vertex outside  $A$ , there remain at least  $2(m-1)$  vertices in  $A$ , so we can complete this phase. Finally, since  $A$  is transitive, the remaining vertices of  $A$  are partitioned into transitive subtournaments of order  $m$ .

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