Cycle Spectra of Hamiltonian Graphs

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Abstract

We prove that every Hamiltonian graph with n vertices and m edges has cycles with more than $\sqrt{p} - \frac{1}{2}\ln p - 1$ different lengths, where $p = m - n$. For general m and n, there exist such graphs having at most $2\left\lceil\sqrt{p+1}\right\rceil$ different cycle lengths.

Keywords: cycle, cycle spectrum, Hamiltonian graph, Hamiltonian cycle.

1 Introduction

The *cycle spectrum* of a graph G is the set of lengths of cycles in G. A cycle containing all vertices of a graph is a *spanning* or *Hamiltonian cycle*, and a graph having such a cycle is a *Hamiltonian graph*. An n-vertex graph is *pancyclic* if its cycle spectrum is $\{3, \ldots, n\}$. Our graphs have no loops or multiple edges. A graph is k-regular if every vertex has degree k (that is, k incident edges).

Interest in cycle spectra arose from Bondy's "Metaconjecture" (based on [3]) that sufficient conditions for the existence of Hamiltonian cycles usually also imply that a graph is pancyclic, with possibly a small family of exceptions. For example, Bondy [3] showed that the sufficient condition on n-vertex graphs due to Ore [16] (the degrees of any two nonadjacent vertices sum to at least n) implies also that G is pancyclic or is the complete bipartite

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graph $K_{\frac{n}{2},\frac{n}{2}}$. Schmeichel and Hakimi [13] showed that if a spanning cycle in an n-vertex graph G has consecutive vertices with degree-sum at least n, then G is pancyclic or bipartite or omits only $n-1$ from the cycle spectrum, the latter occurring only when the degree-sum is exactly n. Bauer and Schmeichel [1] used this to give unified proofs that the conditions for Hamiltonian cycles due to Bondy [4], Chvátal [5], and Fan [9] also imply that a graph is pancyclic, with small families of exceptions. Further results about the cycle spectrum under degree conditions on selected vertices in a spanning cycle appear in [10] and [14].

At the 1999 conference "Paul Erdős and His Mathematics", Jacobson and Lehel proposed the opposite question: *When sufficient conditions for spanning cycles are relaxed, how small can the cycle spectrum be if the graph is required to be Hamiltonian?* For example, consider regular graphs. Bondy's result [3] implies that $\lceil n/2\rceil$-regular graphs other than $K_{\frac{n}{2},\frac{n}{2}}$ are pancyclic. On the other hand, 2-regular Hamiltonian graphs have only one cycle length. For $3 \le k \le \lceil n/2\rceil - 1$, Jacobson and Lehel asked for the minimum size of the cycle spectrum of a k-regular n-vertex Hamiltonian graph, particularly when $k = 3$.

Let $s(G)$ denote the size of the cycle spectrum of a graph G. At the SIAM Meeting on Discrete Mathematics in 2002, Jacobson announced that he, Gould, and Pfender had proved $s(G) \ge c_k n^{1/2}$ for k-regular graphs with n vertices. Others later independently obtained similar bounds, without seeking to optimize c_k. For an upper bound, Jacobson and Lehel constructed the 3-regular example below with only $n/6+3$ distinct cycle lengths (when $n \equiv 0$ mod 6 and $n > 6$), and they generalized it to the upper bound $\frac{n}{2}\frac{k-2}{k} + k$ for k-regular graphs.

Example 1 When $k = 3$ and 6 divides n (with $n > 6$), take $n/6$ disjoint copies of $K_{3,3}$ in a cyclic order, with vertex sets $V_1, \ldots, V_{n/6}$. Remove one edge from each copy and replace it by an edge to the next copy to restore 3-regularity. A cycle of length different from 4 or 6 must visit each V_i, and in each V_i it uses 4 or 6 vertices. Hence the cycle lengths are 4, 6, and each even integer from $2n/3$ through n. For the generalization, use $K_{k,k}$ instead of $K_{3,3}$. $\square$

A related problem is the conjecture of Erdős [7] that $s(G) \ge \Omega\left(d^{\lfloor (g-1)/2\rfloor}\right)$ when G has girth g and average degree d. Erdős, Faudree, Rousseau, and Schelp [8] proved the conjecture for $g = 5$. Sudakov and Verstraëte [15] proved the full conjecture in a stronger form, obtaining $\frac{1}{8}\left(d^{\lfloor (g-1)/2\rfloor}\right)$ consecutive even integers in the cycle spectrum for graphs with fixed girth g and average degree $48(d+1)$. Gould, Haxell, and Scott [11] proved a similar result: for $c > 0$, there is a constant k_c such that for sufficiently large n, the cycle spectrum of every n-vertex graph G having minimum degree at least cn and longest even cycle length $2l$ contains all even integers from 4 up to $2l - k_c$ (see also [2]).

Prior arguments for lower bounds on $s(G)$ when G is regular and Hamiltonian used only the number of edges, not regularity. Suppose that G has n vertices and m edges. The coefficient c in a general lower bound of the form $s(G) \geq \sqrt{c(m-n)}$ cannot exceed 1, since $s(K_{\frac{n}{2},\frac{n}{2}}) = \sqrt{m-n+1}$. We give a construction for $m \leq n^2/4$ that is far from regular.

Example 2 For $t \leq n/2$, form a graph G by replacing one edge of $K_{t,t}$ with a path having $n-2t$ internal vertices; G has n vertices and m edges, where $m = t^2 - 2t + n \leq n^2/4$. The cycle spectrum of G consists of the $t-1$ even numbers in $\{4, \ldots, 2t\}$ and the $t-1$ numbers from $n-2t+4$ to n having the same parity as n. Thus $s(G) \leq 2(t-1) = 2\sqrt{m-n+1}$. Equality holds when $t \leq \lceil n/4 \rceil$, but when $\lceil n/4 \rceil < t \leq n/2$ and n is even the two sets of $t-1$ numbers overlap. They overlap more as m increases, becoming the same set when $m = n^2/4$, and indeed $s(K_{\frac{n}{2},\frac{n}{2}}) = \sqrt{m-n+1}$).

Deleting edges cannot enlarge the cycle spectrum. Hence in general we can let $t = \lceil \sqrt{m-n+1} \rceil + 1$, apply the construction above for n and t, and discard edges to wind up with m edges and $s(G) \leq 2\lceil \sqrt{m-n+1} \rceil$. □

Bondy [3] showed that every Hamiltonian graph with more than $n^2/4$ edges is pancyclic. Thus the lower bound on $s(G)$ jumps to $n-2$ when m exceeds $n^2/4$. At $m = n^2/4$, the size of the spectrum of $K_{n/2,n/2}$ is only $n/2-1$. For n-vertex Hamiltonian bipartite graphs (with $n > 6$), Entringer and Schmeichel [6] proved that $m > n^2/8$ suffices to make the graph *bipancyclic*, meaning that it has cycles of all $n/2-1$ even lengths.

In the construction of Example 2, the two segments overlap to yield bipancyclic graphs when m exceeds $n^2/16 + n/2$. The result of [6] implies that the construction is optimal among Hamiltonian bipartite graphs when m exceeds $n^2/8$, but whether this also holds for Hamiltonian non-bipartite graphs is unknown. It is also unknown whether there are non-bipancyclic constructions (bipartite or not) when $n^2/16 + n/2 < m \leq n^2/8$.

When $m < n^2/4$, the construction of Example 2 remains a candidate for a graph having the smallest cycle spectrum among Hamiltonian graphs with n vertices and m edges. We do know of one exception: when $(n, m) = (14, 21)$, the cycle spectrum of the Heawood graph (incidence graph of the projective plane of order 2) is smaller.

Our main result for the cycle spectra of n-vertex Hamiltonian graphs with m edges is that $s(G) > \sqrt{p} - \frac{1}{2}\ln p - 1$, where $p = m - n$.

2 The Lower Bound

A path with endpoints x and y is an *x,y-path*. A *chord* of a path (or cycle) P in a graph is an edge of the graph not in P whose endpoints are in P, and the *length* of the chord is the distance in P between its endpoints. In a path with vertices $v_1, \ldots, v_n$ in order, two chords $v_a v_c$ and $v_b v_d$ *overlap* if $a < b < c < d$.

Lemma 3 *If a graph G consists of an x, y-path P and h pairwise-overlapping chords of length l, then G contains x, y-paths having at least $h-1$ distinct lengths. Having only $h-1$ lengths requires l odd, $h \geq (l+3)/2$, and chords starting at h consecutive vertices along P.*

Proof. The claim is trivial for $h = 1$; assume $h \geq 2$. Let n be the length of P. Let $e_1, \ldots, e_h$ be the chords in order of appearance along P from x to y. Let d_i be the distance along P from the first endpoint of e_{i-1} to the first endpoint of e_i, for $2 \leq i \leq h$.

Let $P_{i,j}$ be the unique x, y-path using exactly two chords e_i and e_j, along with edges of P. Let p_j be the length of $P_{1,j}$, for $2 \leq j \leq h$. Note that $p_j = p_{j-1} - 2d_j$ for $3 \leq j \leq h$. The $h-1$ paths $P_{1,2}, \ldots, P_{1,h}$ have distinct lengths, which proves the first statement.

The length of $P_{1,2}$ is $n - 2d_2 + 2$. Thus the full path P provides an additional length unless $d_2 = 1$. If $d_j > 1$ for any larger j, then the length of $P_{2,j}$ is strictly between p_{j-1} and p_j. Hence an extra length arises unless the chords start at consecutive vertices along P.

In the remaining case, the $h-1$ lengths we have found are $n, n-2, \ldots, n-2h+4$. The length of any x, y-path that uses exactly one chord is $n - l + 1$. To avoid generating a new length, it must be that l is odd and $2h - 4 \geq l - 1$. □

Definition 4 Let G be a graph consisting of an n-cycle C plus q chords of length l, where $l < n/2$. Specify a forward direction along C. Let $C[u, v]$ denote the subpath of C traversed by moving forward from u to v along C. When uv is a chord of length l and $C[u, v]$ has length l, we say that u is its *start*, v is its *end*, and uv *covers* the edges and internal vertices of $C[u, v]$. For a chord e, let $F(e)$ consist of e and all chords covering the end of e.

Select a chord e_1 so that $|F(e_1)| \geq |F(e)|$ for every chord e. For $j > 1$, let e_j be the first chord encountered moving forward from e_{j-1} that does not overlap e_{j-1} or e_1; if no such chord exists, then stop and set $\alpha = j - 1$. Note that $F(e_i) \cap \{e_1, \ldots, e_\alpha\} = \{e_i\}$ for each i and that the sets $F(e_1), \ldots, F(e_\alpha)$ are pairwise disjoint. The selected edges $\{e_1, \ldots, e_\alpha\}$ form a *greedy chord system* for G (see Figure 1, which also includes notation used in Theorem 5). Given a greedy chord system beginning with e_1, let v_1 be the start of e_1, and let the vertices of C be $v_1, \ldots, v_n$ in forward order.

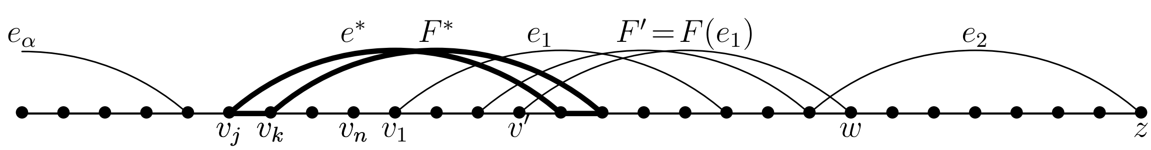

Figure 1: A greedy chord system

From a greedy chord system, we will build a large family of cycles with distinct lengths by using short cycles, long cycles, and cycles of intermediate lengths. The intermediate-length cycles are formed from the long cycles by replacing portions of C with chords.

Theorem 5 *Let G be a graph consisting of an n-cycle C plus q chords of length l, where $l < n/2$. The size $s(G)$ of the cycle spectrum of G is at least $(q-1)/2$ when l is even and at least $(q-1-\frac{q}{l})/2$ when l is odd.*

Proof. Consider a greedy chord system $e_1, \dots, e_\alpha$. Let $F' = F(e_1)$. Let w be the end of the last chord in F' (see Figure 1). Let F^* be the set of chords not in $\bigcup_{i=1}^{\alpha} F(e_i)$; since none of these chords overlaps e_α, each overlaps e_1. If $F^* \neq \varnothing$, then let e^* be the first chord of F^* following e_α (see Figure 1).

When $\alpha = 1$, we have $|F'| + |F^*| = q$. If also $F^* = \varnothing$, then $|F'| = q$. Otherwise, $F^* \subseteq F(e^*) - \{e_1\}$, so $|F^*| \leq |F(e^*)| - 1 \leq |F'| - 1$. Hence $|F'| - 1 \geq (q-1)/2$. Lemma 3 now yields v_1, w-paths of at least $(q-1)/2$ lengths that combine with $C[w, v_1]$ to form cycles of at least $(q-1)/2$ lengths. Hence we may assume $\alpha \geq 2$.

For $\alpha \geq 2$, we begin by using F^* to obtain at least $(|F^*|-1)/2$ short cycle lengths. We may assume $|F^*| \geq 2$. Define j by $e^* = v_j v_{j+l-n}$. Through each chord $v_k v_{k+l-n}$ in $F^* - \{e^*\}$, consider two cycles. One uses $v_k v_{k+l-n}$ and e^* and the two paths $C[v_j, v_k]$ and $C[v_{j+l-n}, v_{k+l-n}]$ that each have length $k-j$ (see Figure 1). The other uses $v_k v_{k+l-n}$ and e_1 and the two paths $C[v_k, v_1]$ and $C[v_{k+l-n}, v_{1+l}]$ that each have length $n-k+1$. The lengths of these cycles are $2(k-j+1)$ and $2(n-k+2)$; their minimum is at most $n-j+3$.

Taking the shorter for each k, we obtain $|F^*|-1$ cycles having length at most $n-j+3$, with each length occurring at most twice. This yields a set Q of $(|F^*|-1)/2$ values bounded by $n-j+3$. Since v_j is between the end of e_α and v_n, we have $j \geq 1+\alpha l$, and values in Q are bounded by $n-\alpha l+2$. Since $\alpha \geq 2$, these values are at most $n-\alpha(l-1)$.

With $\alpha \geq 2$, let z be the end of e_2 (see Figure 1), and say that a cycle in G is *long* if it contains $C[z, v_1]$ and has length at least $n-2(l-1)+1$. Let R be the set of lengths of long cycles, and let $\rho = |R|$.

From the long cycles in G, we construct shorter cycles. Since long cycles contain $C[z, v_1]$, they contain all edges of C covered by any of $e_3, \dots, e_\alpha$. These chords are pairwise non-overlapping and can replace parts of long cycles. Each such replacement yields ρ distinct lengths (within an interval of $2(l-1)$ values), shorter by $l-1$ than the previous set of lengths.

The set R and the $\alpha-2$ sets of size ρ produced by using $e_3, \dots, e_\alpha$ successively to reduce lengths together form $\alpha-1$ sets of size ρ. Since each set lies in an interval of length $2(l-1)$, each value appears in at most two of the sets. Also, the top part of R (values exceeding $n-(l-1)$) and the bottom part of the last translation (values at most $n-(\alpha-1)(l-1)$) appear only once. Let R' be the union of those two sets. Since every value in R is above $n-(l-1)$ or at most $n-(l-1)$, we have $|R'| = \rho$. Including also R', we now have α sets of size ρ, with each value appearing in at most two of them.

Hence the union contains at least $\alpha\rho/2$ cycle lengths, all at least $n-\alpha(l-1)+1$ (which exceeds $\max Q$). Thus $s(G) \geq (\alpha\rho + |F^*| - 1)/2$. It remains to study this quantity.

The greedy choice of e_1 yields $|F'| \geq \left\lceil \frac{q-|F^*|}{\alpha} \right\rceil$. To obtain a lower bound on $\alpha\rho$, we compare ρ to $|F'|$. Let G' be the induced subgraph of G consisting of $C[v_1, w]$ and the chords in F'. Since these chords are pairwise overlapping, Lemma 3 yields v_1, w-paths in G' with $|F'| - 1$ distinct lengths. Furthermore, there are at least $|F'|$ distinct lengths unless l is odd, $|F'| \geq (l+3)/2$, and the starts of the chords in F' are consecutive along C.

If $|F'| = 1$, then the greedy choice of e_1 implies that the chords are pairwise noncrossing and $s(G) = q + 1$. We may thus assume $|F'| > 1$ and $w \neq v_{l+1}$, so every v_1, w-path in G' has length at least 2. Adding $C[w, v_1]$ to v_1, w-paths of distinct lengths in G' creates cycles of distinct lengths in G. Since each such cycle contains $C[w, v_1]$, which has at least $n - 2l + 1$ edges, these cycles are long.

Thus when l is even, we have shown that $\rho \geq |F'|$. In this case

$$s(G) \geq \frac{\alpha\rho}{2} + \frac{|F^*| - 1}{2} \geq \frac{q - |F^*|}{2} + \frac{|F^*| - 1}{2} = \frac{q-1}{2}.$$

If l is odd, then $\frac{\alpha\rho}{2} \geq \left\lceil \frac{q-|F^*|}{\alpha} \right\rceil$ still holds if $|F'| > \left\lceil \frac{q-|F^*|}{\alpha} \right\rceil$, since $\rho \geq |F'| - 1$. Hence we may assume $|F'| \geq \left\lceil \frac{q-|F^*|}{\alpha} \right\rceil$. If $\rho \geq |F'|$ fails, then Lemma 3 implies that $|F'| \geq (l+3)/2$ and that the chords in F' are consecutive. Now R consists of the $|F'| - 1$ values from n through $n - 2|F'| + 4$ whose difference from n is even. We consider two cases, depending on whether e_2 overlaps some chord in F'.

Case 1: *e_2 overlaps no chord in F'.* Here e_2, like $e_3, \ldots, e_\alpha$, can be used to reduce cycle lengths by $l-1$. Since $|F'| \geq (l+3)/2$, the long cycle lengths include $n, n-2, \ldots, n-(l-1)$; there are $(l+1)/2$ of them. After using each of $e_2, \ldots, e_\alpha$ to reduce the lengths by $l-1$, we obtain all values with the same parity as n down to $n - \alpha(l-1)$. The smallest may equal $\max Q$. We keep $\frac{1}{2}\alpha(l-1)$ cycle lengths, each at least $n - \alpha(l-1) + 2$.

If $\alpha \geq q/l$, then $\frac{1}{2}\alpha(l-1) \geq \frac{1}{2}q(1 - \frac{1}{l}) \geq \frac{1}{2}(q - |F^*| - \frac{q}{l})$. If $\alpha < q/l$, then we use $l \geq |F'| = \left\lceil \frac{q-|F^*|}{\alpha} \right\rceil$ to compute

$$\frac{1}{2}\alpha(l-1) \geq \frac{1}{2}(|F'| - 1)\alpha \geq \frac{1}{2}(q - |F^*| - \alpha) > \frac{1}{2}\left(q - |F^*| - \frac{q}{l}\right).$$

Adding the $(|F^*| - 1)/2$ short lengths yields at least the desired number of lengths.

Case 2: *e_2 overlaps some chord in F'.* Since the chords in F' are consecutive, this case requires that e_2 starts just before the end of some chord e' in F'. Let v' be the start of e'. The cycle consisting of e_2 and e', the edge they both cover, and the path $C[z, v']$ (see Figure 1) has length $n - 2(l-1) + 2$; hence it is a long cycle. We obtain $\rho \geq |F'|$ unless this length already appears among those generated from Lemma 3, which requires $2|F'| - 4 \geq 2(l-1) - 2$, so $|F'| \geq l$. Since $|F'| \leq l$, equality holds.

As noted above, already $n, n-2, \ldots, n-2(l-2) \in R$. Lowering the bottom half of them by $l-1$ exactly $\alpha-2$ times yields $\frac{1}{2}\alpha(l-1)$ distinct cycle lengths. The least of them is $n-\alpha(l-1)+2$. This is exactly the situation we obtained in Case 1, so the same computation completes the proof. □

Theorem 6 *If G is an n-vertex Hamiltonian graph with m edges, then $s(G) > \sqrt{p}-\frac{1}{2}\ln p-1$, where $p = m-n$.*

Proof. Let C be a spanning cycle in G. Let L be the set of lengths of chords of C in G, and let $t = |L|$. For each $l \in L$, we obtain two lengths of cycles in G; they are $l+1$ and $n-l+1$ if $l < n/2$ (using one chord of length l), and they are $n/2+1$ and n if $l = n/2$. Hence $s(G) \geq 2t$, which suffices if $t \geq \frac{1}{2}\sqrt{p}$. We may therefore assume that $2t < \sqrt{p}$.

For $l \in L$, let q_l be the number of chords of length l. By Theorem 5, when $l < n/2$ there are at least $\frac{l-1}{2l}q_l - \frac{1}{2}$ lengths of cycles using only edges of C and chords of length l. The lower bound also holds when $l = n/2$, since then the chords are pairwise overlapping and Lemma 3 applies, and always $q_l - 1 > \frac{l-1}{2l}q_l - \frac{1}{2}$.

We may assume that $\frac{l-1}{2l}q_l - \frac{1}{2} \leq \sqrt{p}-\frac{1}{2}\ln p-1$ for odd $l \in L$, and $\frac{1}{2}q_l - \frac{1}{2} \leq \sqrt{p}-\frac{1}{2}\ln p-1$ for even $l \in L$. Thus $q_l \leq (\sqrt{p} - \frac{1}{2}\ln p - \frac{1}{2})c_l$, where $c_l = 2$ when l is even and $c_l = 2 + \frac{2}{l-1}$ when l is odd. We obtain a contradiction by showing that these bounds on q_l sum to less than p. In light of the form of c_l, it suffices to prove this when all values in L are odd. The bound is now the worst when L consists of the first t positive odd numbers. We compute

$$\begin{aligned}
p = \sum_{l\in L} q_l \leq \sum_{l\in L} \left(\sqrt{p} - \tfrac{1}{2}\ln p - \tfrac{1}{2}\right)\left(2 + \frac{2}{l-1}\right) \;&\leq\; \left(\sqrt{p} - \tfrac{1}{2}\ln p - \tfrac{1}{2}\right)\left[2t + \sum_{i=1}^{t}\frac{1}{i}\right] \\
< \left(\sqrt{p} - \tfrac{1}{2}\ln p - \tfrac{1}{2}\right)\left[\sqrt{p} + (1+\ln t)\right] \;&<\; \left(\sqrt{p} - \tfrac{1}{2}\ln p - \tfrac{1}{2}\right)\left[\sqrt{p} + \tfrac{1}{2}\ln p + (1-\ln 2)\right] \\
= p - \tfrac{1}{4}(\ln p)^2 - (\ln 2 - \tfrac{1}{2})\sqrt{p} - \tfrac{1}{4}(3-\ln 4)\ln p - \tfrac{1}{2}(1-\ln 2) \;&<\; p.
\end{aligned}$$

The contradiction completes the proof. □

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