Coloring, Sparseness, and Girth

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slides available on DBW preprint page

Joint work with

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Explicit: Lovász [1968], Nešetřil–Rödl [1979], Křiž [1989], Lubotzsky–Phillips–Sarnak [1988], Kostochka–Nešetřil [1999]

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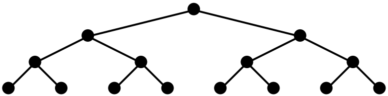

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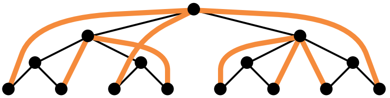

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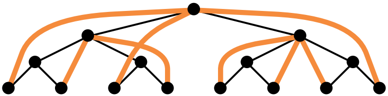

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Thm. For $d, r, g \in \mathbb{N}$, there is a bipartite (d, r, g)-graph.

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Nevertheless, (d, r, g)-graphs exist.

Edge-Coloring and f-Paths

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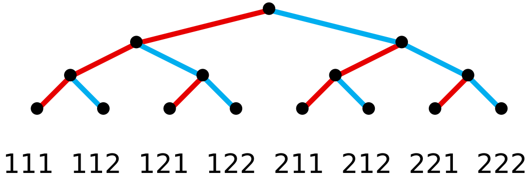

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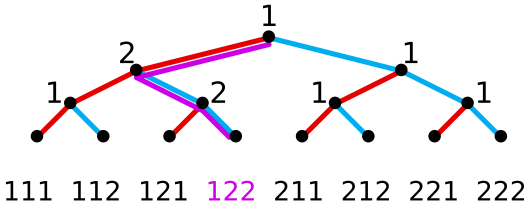

- Any $[d]$-coloring f of the internal vertices of T selects a unique f-path, where the color on each internal vertex agrees with that on the descending edge.

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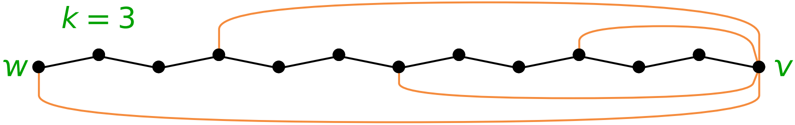

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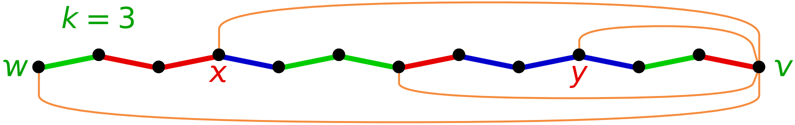

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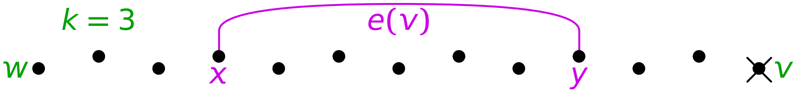

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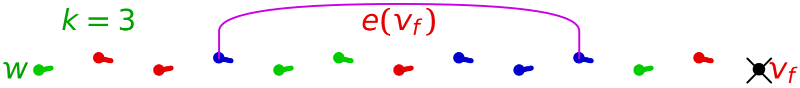

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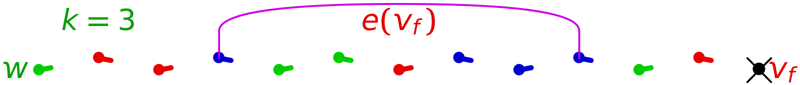

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Include xy as an edge in H; call it $e(v)$.

For any $[k]$-coloring f of U, the unique f-path in T ends at some leaf v_f. The edge $e(v_f)$ is monochromatic!

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Thm. For $k, g \in \mathbb{N}$, $\exists\, H$ with girth $\geq g$ and $\chi(H) > k$.

Pf. **(cont.)** We made H from a $(k, k+1, 2g)$-graph G.

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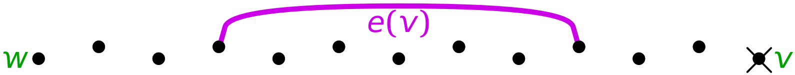
Cycles in H use only edges of the form $e(v)$ for $v \in L$.



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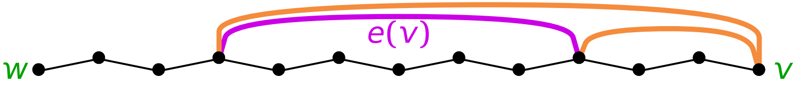

The edge $e(v) \in E(H)$ corresponds to two edges in G.

A cycle in H yields a trail twice as long in G.

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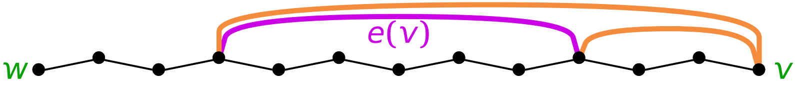

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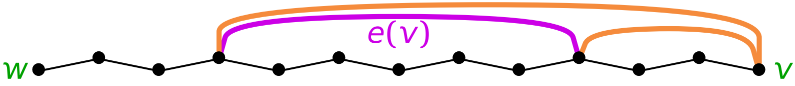

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Thm. For $k, g \in \mathbb{N}$ and $t \geq 2$, some t-uniform hypergraph H has girth $\geq g$ and $\chi(H) > k$.

Pf. Take $(k, (t-1)k+1, 2g)$-graph G; argue as before. ∎

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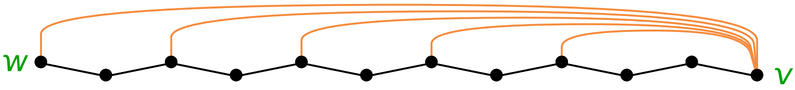
Lem. $m(d, r, 4) = 2r + 1$.



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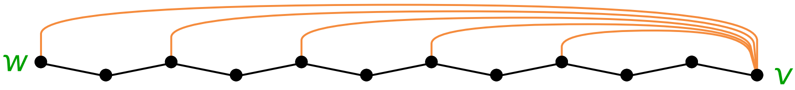

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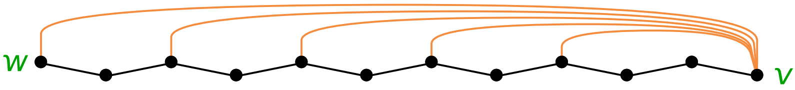

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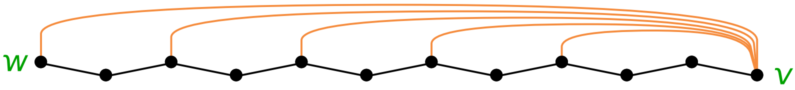

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Uses the prior statements for $(1, g)$ and (r, g) (for all d).

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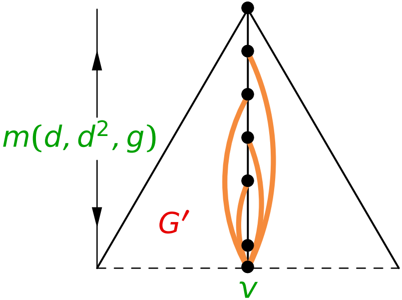

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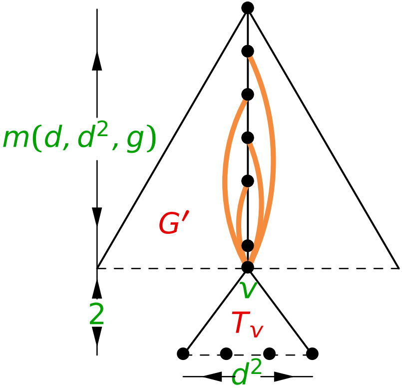

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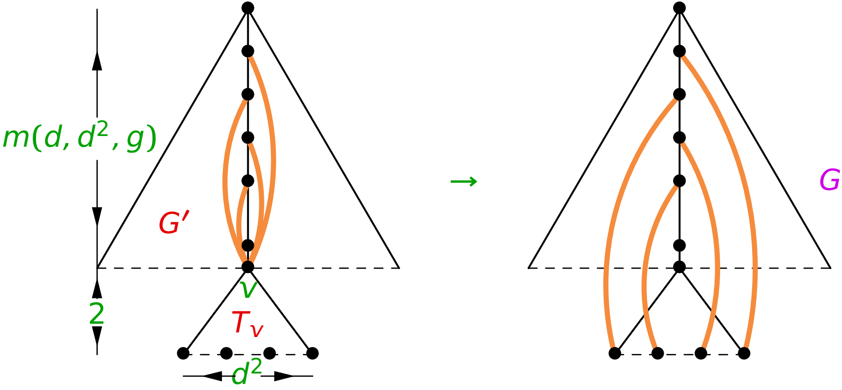

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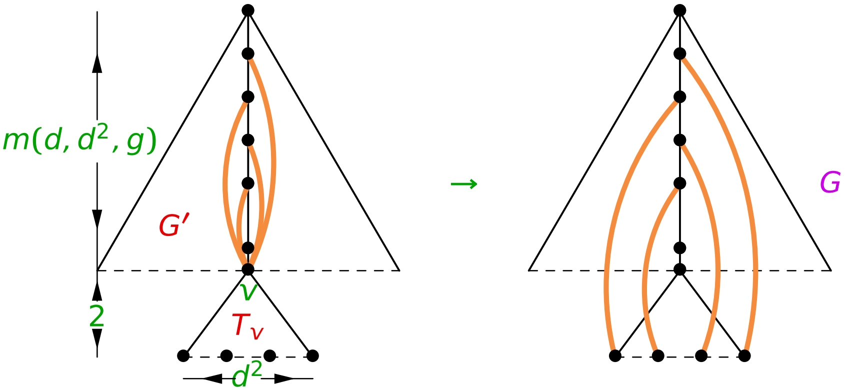

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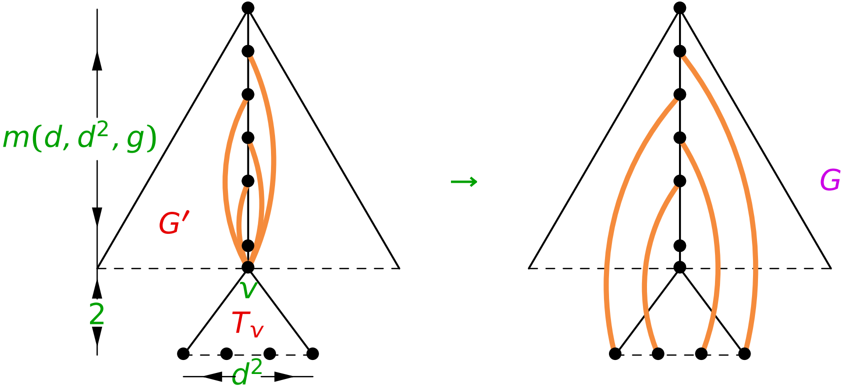

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If G' is bipartite, then also G is bipartite. ∎

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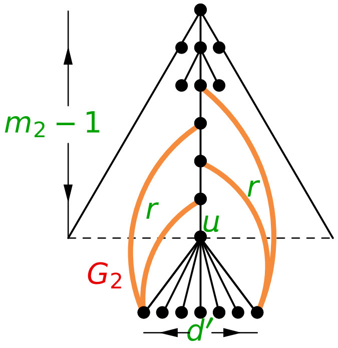

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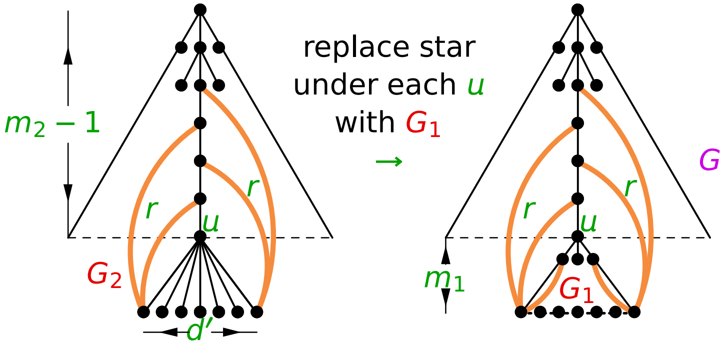

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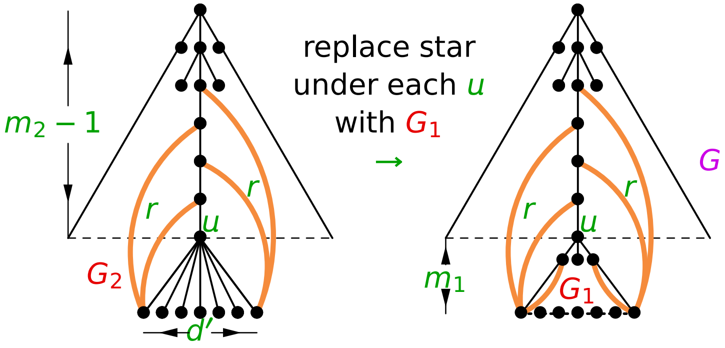
$m_1 = 2\left\lfloor \frac{m(d,1,g)}{2} \right\rfloor + 1$ and $m_2 = m(d^{m_1}, r, g)$.

Pf. Let G_1 be a $(d, 1, g)$-graph of height m_1.
Let H be a (d', r, g)-graph of height m_2, where $d' = d^{m_1}$.



Long-edge lengths up by $m_1 - 1$. Still G is bipartite.
Now $r+1$ augmenting edges. Girth maintained. ∎

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We prove that this result is sharp in a very strong sense.

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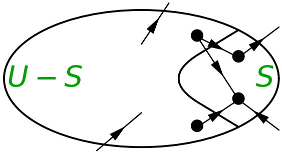

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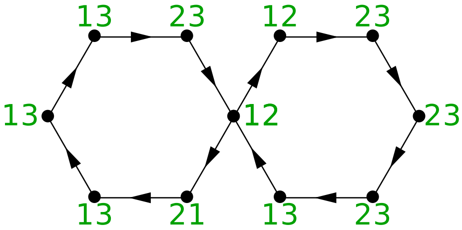
G_2 consists of two consistently oriented g-cycles sharing one vertex as the root.



Reduced (d, r, g)-graph

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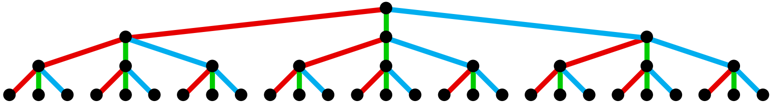

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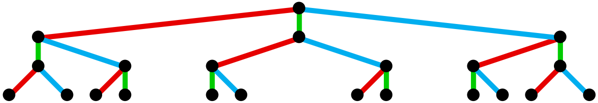

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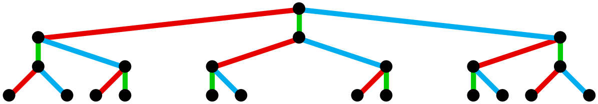

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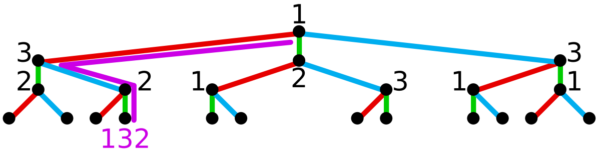

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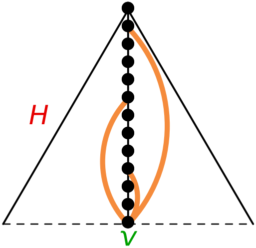
A proper $[d]$-coloring f of the subgraph of T induced by internal vertices still yields a unique f-path to a leaf.

Inductive Construction

For $k \geq 3$, given G_{k-1}, let $r = |V(G_{k-1})| - 1$.

Let H be a reduced $(k, r, 2g)$-graph with underlying tree T.



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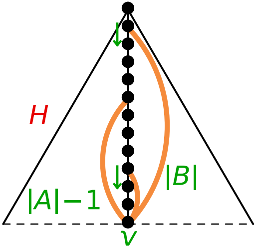

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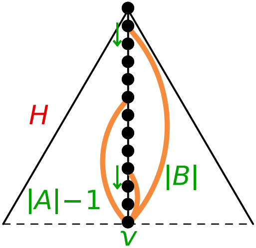

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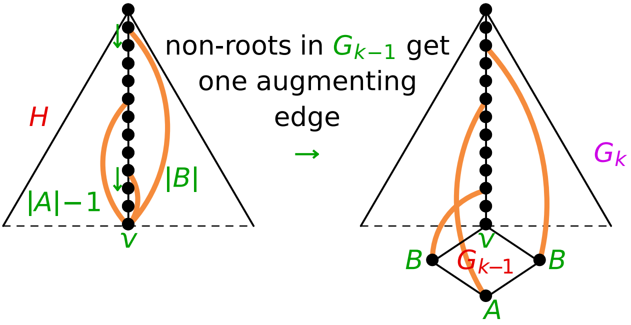
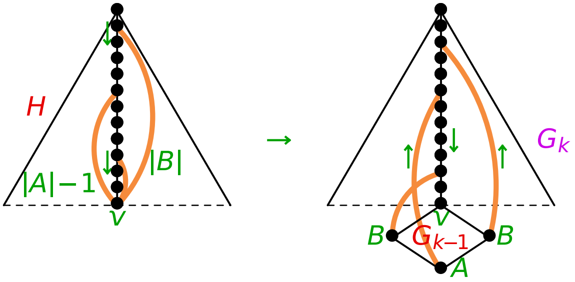
Form G_k by adding a copy of G_{k-1} rooted at each leaf v of T.



The Construction Succeeds



Orient: T away from the root, copies of G_{k-1} inductively,
augmenting edges oriented away from the copies of G_{k-1}.

Outdegree is $k-1$ except k at the root of T.

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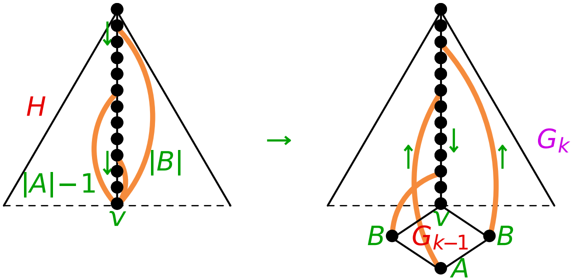

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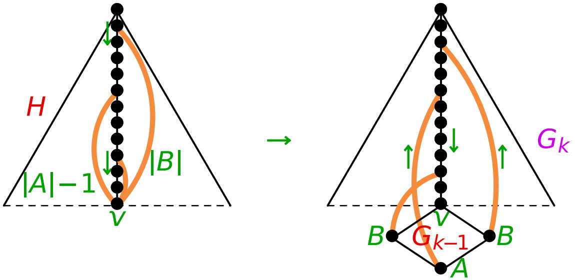

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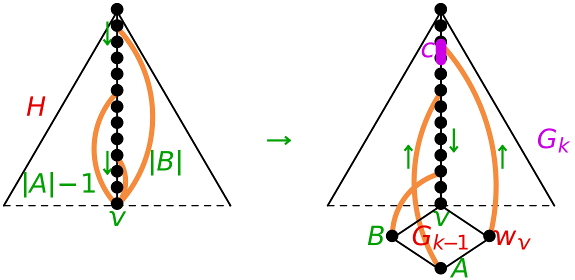

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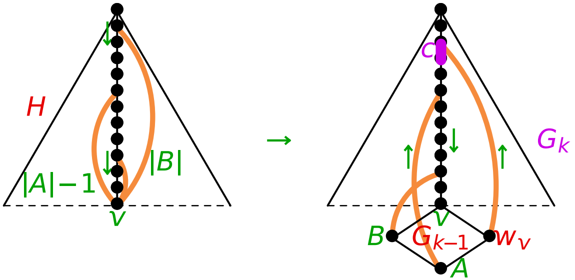

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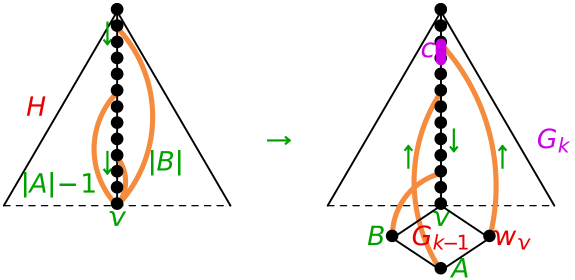

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In the copy of G_{k-1} for v, the color c added to each list is now forbidden, leaving only $L'(w)$ available at each w_v! ∎

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Thm. There is a non-k-choosable bipartite graph G with girth at least g and $\overline{d}(H) \le 2(k-1)$ for all $H \subsetneq G$, via a k-list assignment L such that $\left|\bigcup_{v \in V(G)} L(v)\right| = 2k-1$.

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Lem. $m(d, r, 4) = 2r + 1$.

Lem. $m(d, 1, g) \leq 2 + m(d, d^2, g - 2)$ for $g \geq 6$.

Lem. $m(d, r + 1, g) \leq m_1 + m_2 - 1$, where
$m_1 = 2 \left\lfloor \frac{m(d,1,g)}{2} \right\rfloor + 1$ and $m_2 = m(d^{m_1}, r, g)$.

Let $\text{tow}_{r,d}(x) = d^{\text{tow}_{r-1,d}(x)}$, where $\text{tow}_{0,d}(x) = x$.

$m(d, r + 1, g) \leq m(\text{tow}_{r,d}(m(d, 1, g)), 1, g)$.

Let $f = \text{tow}_{r,d}(m(d, 1, g))$.

$m(f, 1, g) \leq m(\text{tow}_{f^2,f}(m(f, 1, g - 2)), 1, g - 2)$.

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Alon: In general, one cannot do “much” better.