Minimum Degree and Dominating Paths

Douglas B. West

Departments of Mathematics
Zhejiang Normal University and
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

slides available on DBW preprint page

Joint work with
Ralph J. Faudree, Ronald J. Gould,
and Michael S. Jacobson

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What mindegree guarantees a (small) dominating path?

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Thm. thresholds for balanced spanning caterpillars ...

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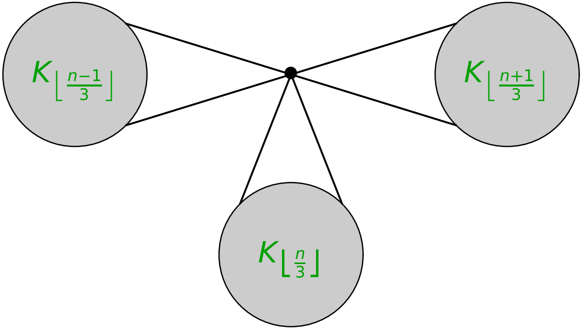

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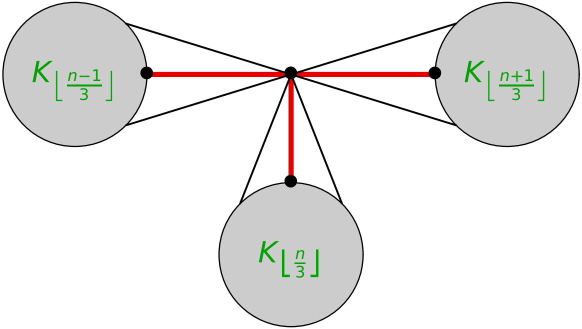

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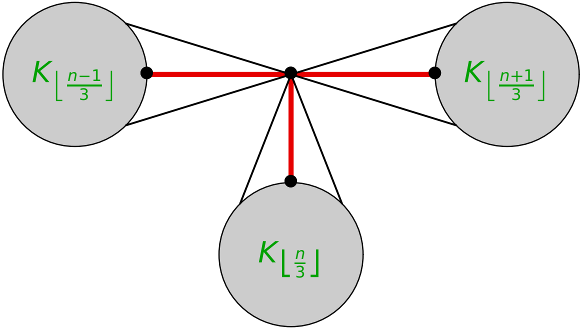

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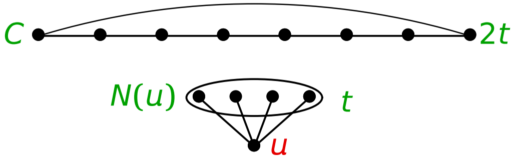

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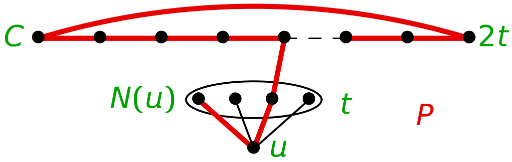

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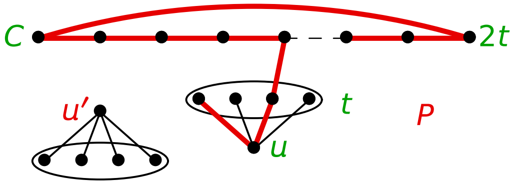

If $\exists\ u'$ undom. by $V(P)$, then $n \geq (2t+3)+(t+1) = 3t+4$.

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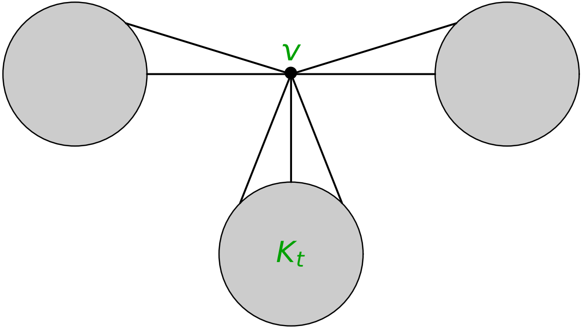

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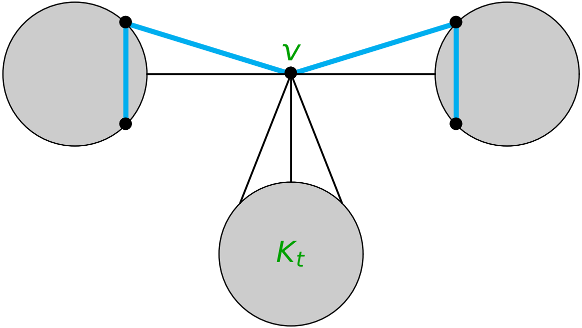

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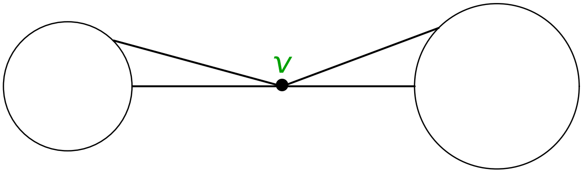

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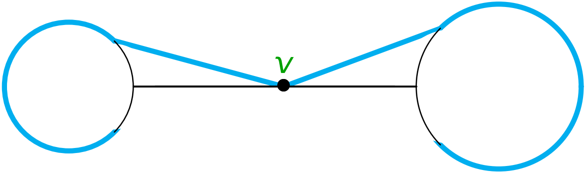

and hence long (dominating) cycles. ■

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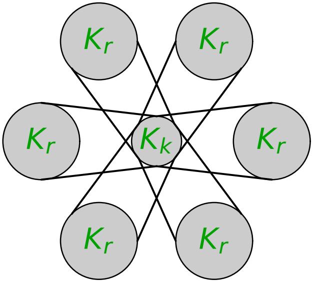

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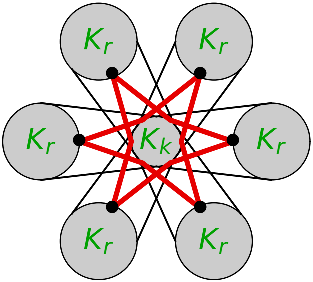

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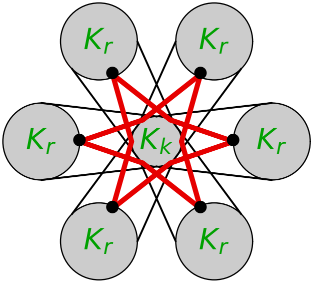

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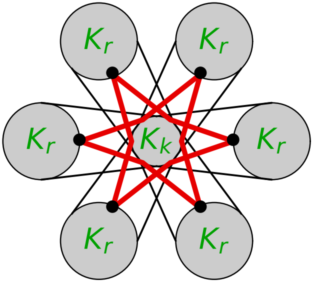

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Conj. If G is k-connected, then $\delta(G) > \frac{n-k}{k+2} - 1$ suffices.

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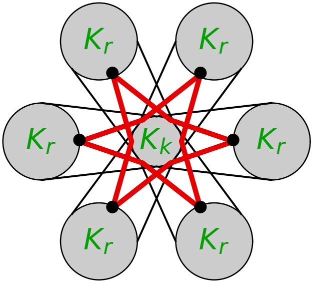

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(Broersma [1988])

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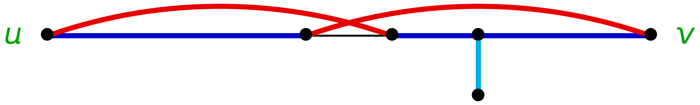

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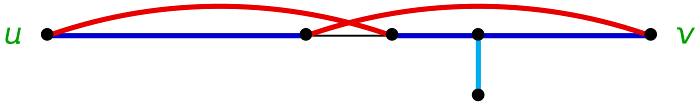

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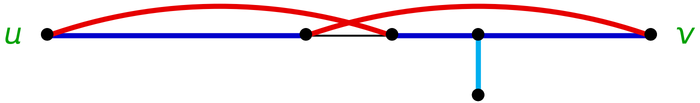

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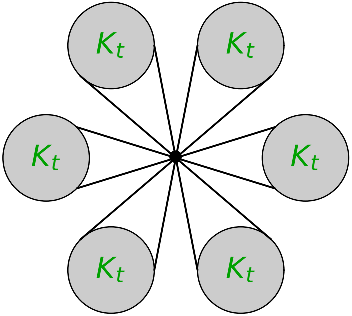
Hence $N(v)$, $N(u)^{-}$, and $\{v\}$ are pairwise disjoint, which requires at least $2\delta(G)+1$ vertices on P. ■

Sharpness

With minimum degree t, there may be no path with more than $2t+1$ vertices.

Ex. $K_1 \oplus rK_t$.



A Short Dominating Path

Thm. Let G be a connected n-vertex graph with $\delta(G) \geq \alpha n + \log_{\alpha/(1-\alpha)} n$, where $\alpha > 1/2$. For n suff'ly large, G has a dominating path with at most $\lceil \log_{\alpha/(1-\alpha)} n \rceil$ vertices, starting from any vertex x_1.

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Given $\langle x_1, \ldots, x_j \rangle$, let $T_j = N_G(x_j) - \{x_1, \ldots, x_{j-1}\}$, and let S_j be the subset of $V(G)$ undominated by $x_1, \ldots, x_j$.

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Claim: $\exists\, x_{j+1} \in T_j$ with $< \frac{1-\alpha}{\alpha}|S_j|$ nonneighbors in S_j.

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Inductively, $|S_j| \leq (\frac{1-a}{a})^j |S_0|$, so $S_r = \varnothing$. ■

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Thm. For $\frac{1}{3} < \alpha < 1$, there is a constant c_α such that if n is sufficiently large and $\delta(G) \geq \alpha n$, then G contains a dominating path with at most $c_\alpha \log_{1/(1-\alpha)} n$ vertices.

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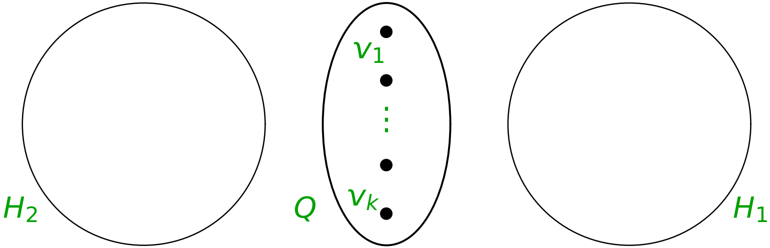

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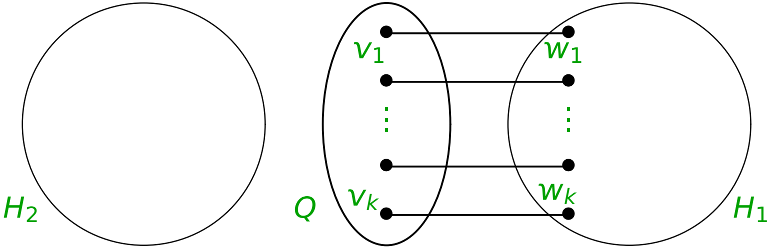

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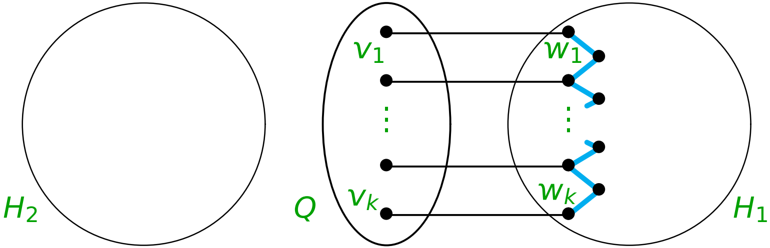

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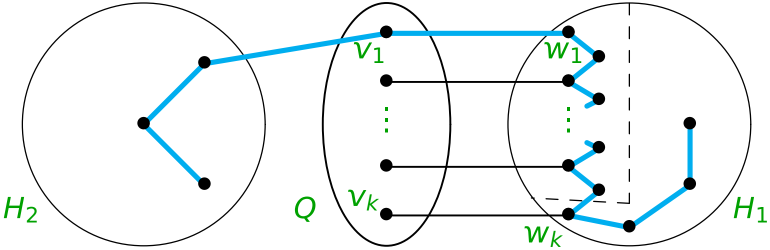

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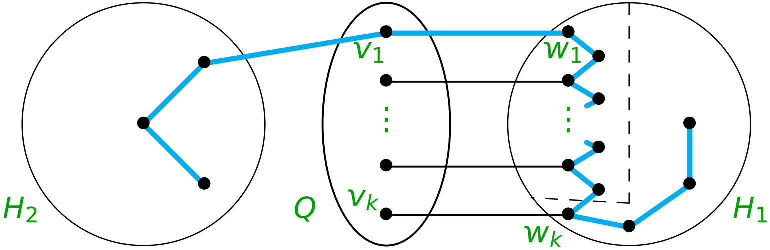

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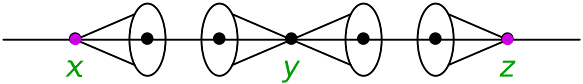

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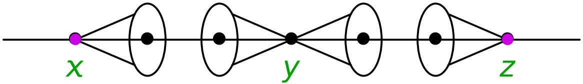

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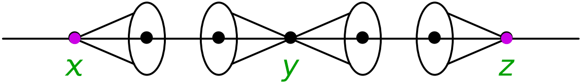

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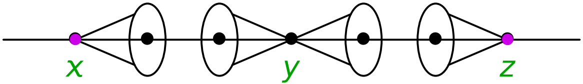

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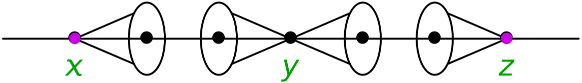

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Let c_α be the max of 5 and the constant from Case 1. ■

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The last inequality holds for large n since $c < 1$.

Now $d_G(y) < \delta(G) \Rightarrow$ no such $y \Rightarrow x_j$ exists. ∎

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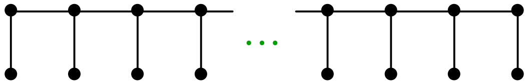

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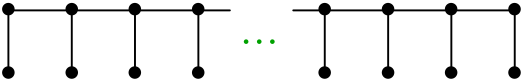

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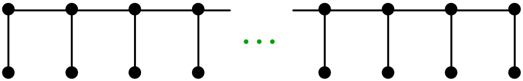

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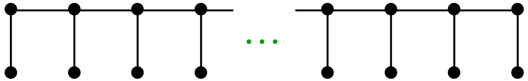

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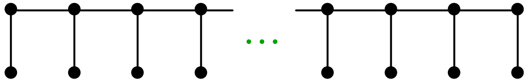

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p-packing = disjoint stars with $\leq p$ edges, center in Q, leaves in R. A star with p edges saturates its center.

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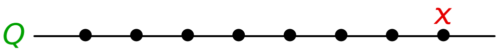

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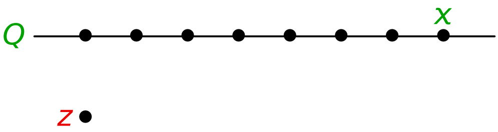

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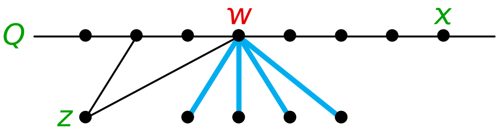

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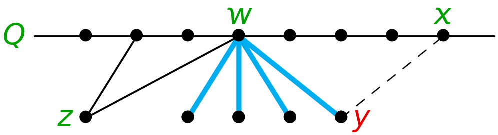

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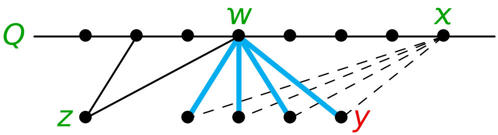

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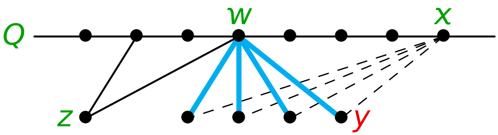

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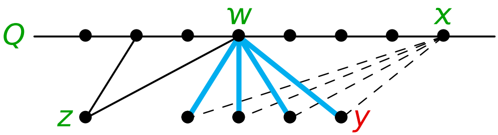

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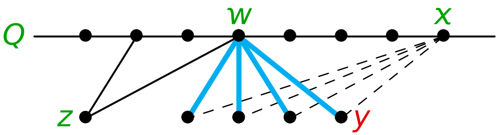

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With $|R| = \frac{pn}{p+1}$, this simplifies to $\delta(G) \le (1-\frac{p}{(p+1)^2})n-1$. ■

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