Bounds on the k-dimension of Products of Special Posets

Douglas B. West

Department of Mathematics
University of Illinois at Urbana-Champaign
west@math.uiuc.edu
http://www.math.uiuc.edu/~west/pubs/publink.html

Joint work with
Michael Baym
Department of Mathematics
Massachusetts Institute of Technology

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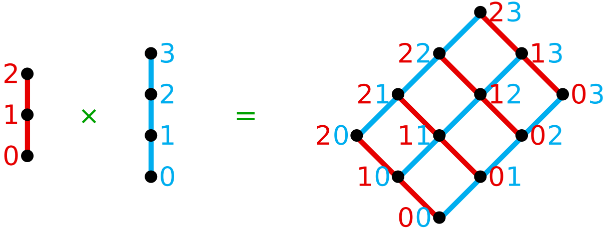
Ex. $P = \mathbf{3}$, $Q = \mathbf{4}$.



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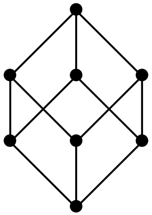

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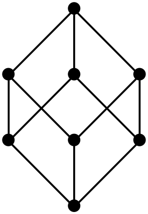

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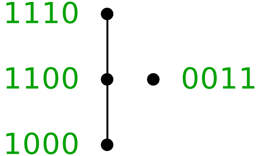

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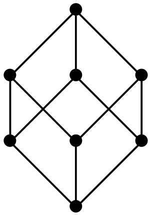

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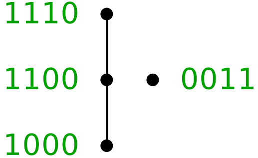

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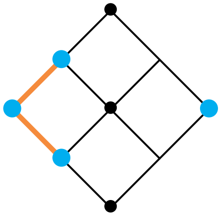

$\dim_3 P = 2$

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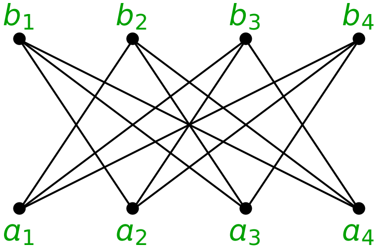

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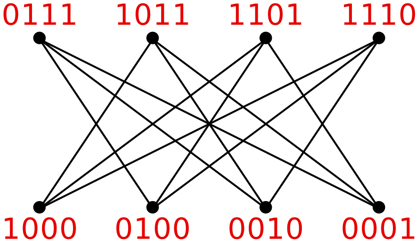

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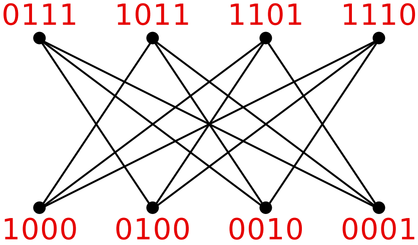

Lower bound: $\dim S_n \geq n$.

Each a_i must be above b_i on some extension.
Each extension has at most one such pair.

$$A - \{a_i\} < b_i < a_i < B - \{b_i\}$$

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Thm. (C. Lin [1991]) $\dim(S'_m \times S_n) = m + n - 2$, where S'_m is S_m plus a global max and global min.

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Thm. If P is a bounded poset and k is fixed, then
$2\dim_k mP - \dim_k(mP \times mP)$ is unbounded.

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Since $A_n \times A_n = A_{n^2}$,

$$\dim_2(A_n \times A_n) = 2\lg n + \frac{1}{2}\lg\lg n + O(\lg\lg\lg n).$$

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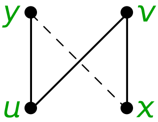

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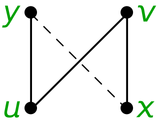

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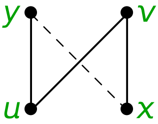

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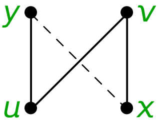

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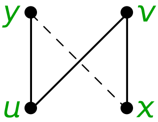

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- $\exists$ embedding in $\mathbf{k}^t$ $\Leftrightarrow$ $\exists$ k-extensions $f_1, \ldots, f_t$ such that $x \| y \Rightarrow f_i(x) > f_i(y)$ for some i.
- Suffices to handle (x, y) with x min'l and y max'l in R.

$S_m \times S_n$ - Critical Pairs and 2-extensions

Write $(a_{i,j}, b_{r,s})$ for $((a_i, a'_j), (b_r, b'_s))$ with $i, r \in [m]$ and $j, s \in [n]$. Note that $a_{i,j} < b_{r,s}$ if $i \neq r$ and $j \neq s$.

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Def. Critical pairs $(a_{i,j}, b_{r,s})$ are of three types:

vertical pairs - $i \neq r$ and $j = s$. $\exists$ $m^2n - mn$ of these.

horizontal pairs - $i = r$ and $j \neq s$. $\exists$ $mn^2 - mn$ of these.

straight pairs - $i = r$ and $j = s$. $\exists$ mn of these.

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- horizontal pairs - $i = r$ and $j \neq s$. $\exists\ mn^2 - mn$ of these.
- straight pairs - $i = r$ and $j = s$. $\exists\ mn$ of these.

Def. Types of 2-extensions ($m = 3$ and $n = 4$).

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

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Lem. Saturated 2-extensions are of types V, H, B, C. (Also for k-extensions with $k \geq 3$.)

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Lem. The # of critical pairs handled ($k = 3$ omitted):

	$k = 2$			$k \geq 4$		
	vertical	horiz.	str.	vertical	horiz.	str.
V	$m^2 - m$	0	m	$m^2 - m$	$2n - 2$	m
H	0	$n^2 - n$	n	$2m - 2$	$n^2 - n$	n
B	$m - 1$	$n - 1$	1	$2m - 2$	$2n - 2$	2
C	2	2	0			
tot	$m^2n - mn$	$mn^2 - mn$	mn	$m^2n - mn$	$mn^2 - mn$	mn

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For $k \geq 4$, $V + H + B < m+n-2 \Rightarrow V \leq n-2$ or $H \leq m-2$.
If $V = n - r$, then vertical $\Rightarrow$ $H + B \geq \frac{rm}{2} > m + r - 3$.

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If $x < y'$ and $x' < y$, then (x, y) and (x', y') form an "alternating cycle". ■

$\dim_2(S_m \times S_n)$, again

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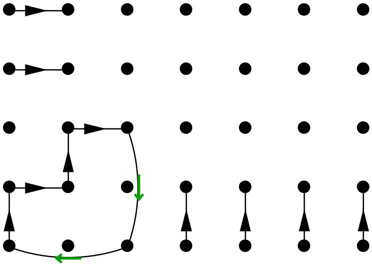

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Let $P = \prod_{i=1}^{t} S_{n_i}$, nontrivial, so $\dim_k P = \dim_k R$, where R consists of $\{$(all-min)$\} \cup \{$(all-max)$\}$.

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Idea: For $\prod_{i=1}^{t} S_t$, present a cycle C_t of length t^2 in the t-dimensional grid consisting of 2-conflicted arrows. Yields $\dim_2(\prod_{i=1}^{t} S_t) = t \cdot t$.

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For $\prod_{i=1}^{t} S_{n_i}$ with each $n_i \geq t$, add $n_i - t$ arrows for extra hyperplanes in the ith direction, copying arrows on C_t. Yields $\dim_2(\prod_{i=1}^{t} S_t) = \sum_{i=1}^{t} n_i$.

The t-lagging cycle

Def. The t-lagging cycle is the cycle C_t consisting of vectors $v_1, \dots, v_{t^2}$ in $[t]^t$ given by letting $v_1 = (1, \dots, 1)$ and, for $j \geq 1$, forming v_{j+1} from v_j by adding 1 (mod t) to each coordinate except the ith, where $i \equiv j \mod t$.

Ex. C_4, reading down columns:

$(1,1,1,1)$	$(4,4,4,4)$	$(3,3,3,3)$	$(2,2,2,2)$
$(1,2,2,2)$	$(4,1,1,1)$	$(3,4,4,4)$	$(2,3,3,3)$
$(2,2,3,3)$	$(1,1,2,2)$	$(4,4,1,1)$	$(3,3,4,4)$
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- Each arrow $v_j \to v_{j+1}$ represents $a_{v_j} \| b_{v_{j+1}}$.

It Works!

Lem. The arrows in C_t represent t^2 2-conflicted pairs.

Pf. Show that if a_{v_i} and b_{v_j} are incomparable, with $i+1 \neq j$, then $a_{v_{j-1}} < b_{v_{i+1}}$.

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 & & v_i & \rightarrow & v_{i+1} \\
q-2 \ldots q-1 & & q-1 \; \ldots \quad q & & q \quad \ldots q+1 \\
q-1 \ldots \quad q & & q \quad \ldots \; q+1 & & q+1 \ldots q+2 \\
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Now r is last index of q in v_{i+1} and s is first index of q in v_{j-1}. If overlap, then $s \leq r$. Now $r = s$, and $j = i+1$. ■