The Slow-Coloring Game

Douglas B. West

Departments of Mathematics
Zhejiang Normal University and
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

slides available on DBW preprint page

Joint work with
Thomas Mahoney and Gregory J. Puleo

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Def. This slow-coloring game ends when Painter has colored all vertices (properly!).
Lister seeks high total score (inefficient, slow coloring).
Painter seeks low total score.

Def. sum-color cost $\mathring{s}(G)$ - score in optimal play.

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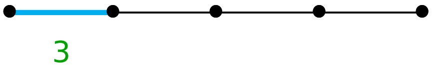

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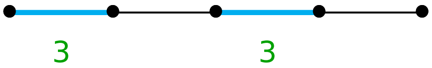

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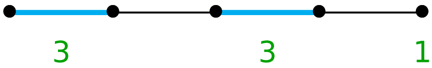

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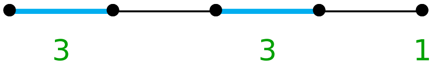

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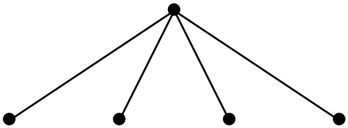

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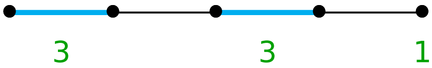

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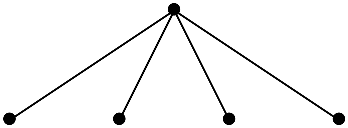

Lister marking all or just an edge gets only 6.

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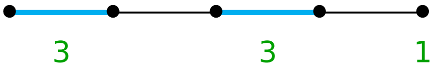

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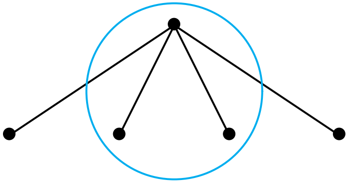

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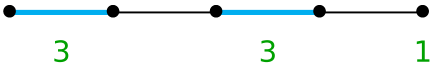

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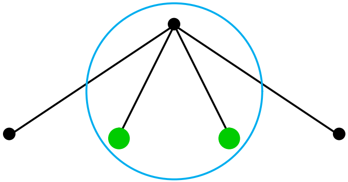

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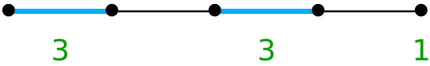

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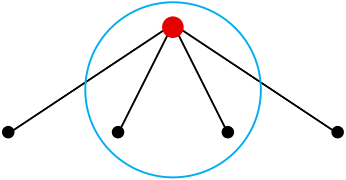

Score is $3 + \mathring{s}(P_3)$ or $3 + 4$.

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Equality also holds for paths, but not for stars.

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In f-painting, Lister need not name lists in advance.

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(If $G = \sum K_{n_i}$, then $\mathring{s}(G) = \chi_{sp}(G) = \sum \binom{n_i+1}{2}$.)

Bounds Involving Independence

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$\Sigma(G) = \sum_i i|V_i| = \sum_{i\ge1}\left|\bigcup_{j\ge i} V_j\right| \ge \sum_{k\ge0}(n - k\alpha(G))$. ■

Trees - Upper Bound

Thm. $\mathring{s}(T) \leq \lfloor 3n/2 \rfloor$ for every n-vertex tree T.

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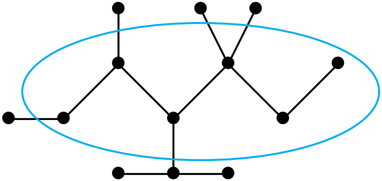
Pf. If Lister optimally marks M and Painter responds I, then inductively $\mathring{s}(T) \le |M| + \frac{3}{2}(n - |I|) - \frac{1}{2}o(T - I)$.



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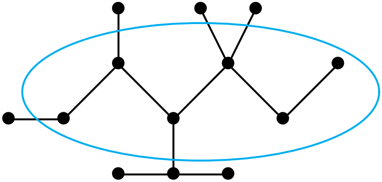

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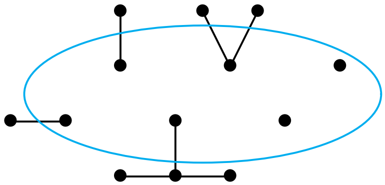

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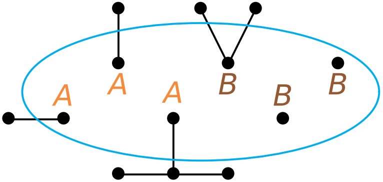

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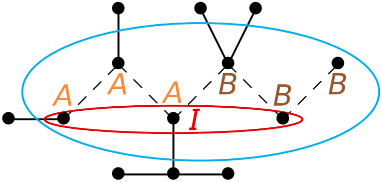

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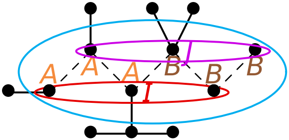

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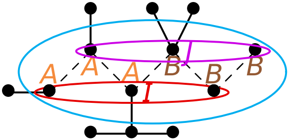

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Lem. When G is a graph and $\{A, B\}$ partitions $V(G)$,
$\mathring{s}(G[A]) + \mathring{s}(G[B]) \leq \mathring{s}(G) \leq \mathring{s}(G[A]) + \mathring{s}(G[B]) + |[A, B]|$.

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Def. A stem vertex has one non-leaf neighbor.

Computation for Forests

Thm. Let T be a forest. If v is a stem vertex of T and R is its set of leaf neighbors, with $r = |R|$, then

$$\mathring{s}(T) = \begin{cases} \mathring{s}(T - R - v) + r + 1 + u_r, & \text{if } r+1 \text{ is not triangular,} \\ \mathring{s}(T - R) + r + u_r, & \text{if } r+1 \text{ is triangular.} \end{cases}$$

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When $r + 1$ is triangular, the lower and upper bounds each consider possible optimal moves. ■

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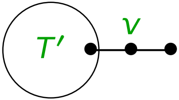

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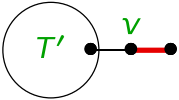

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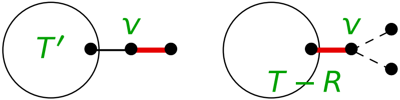

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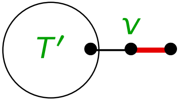

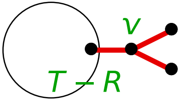

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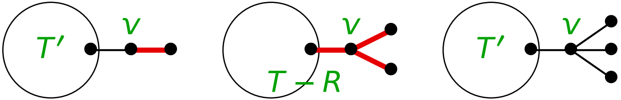

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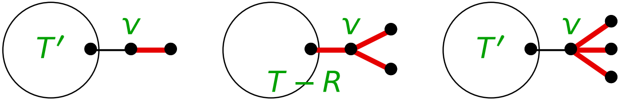

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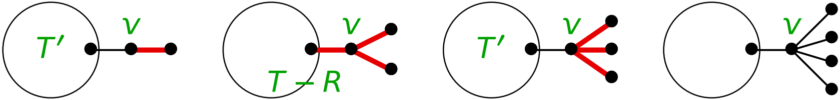

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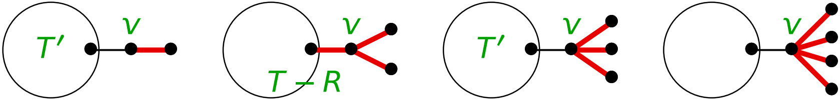

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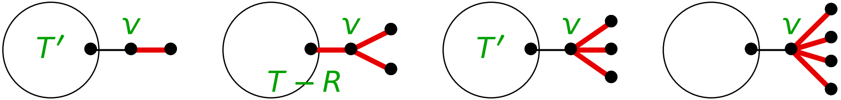

Minimum: Star unique when n is triangular.(?)

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The upper and lower bound proofs each take a page.

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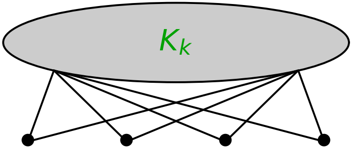

We know $\mathring{s}(P_n^k) \geq \left\lfloor \frac{n}{k+1} \right\rfloor \binom{k+2}{2} + \binom{r+1}{2}$, Maximum?

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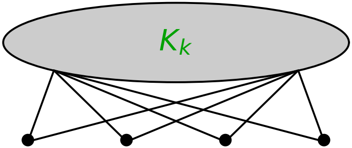

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Ques. Complexity?

Open Problems

Ques. What is $\mathring{s}(K_{r,s})$? Is $\mathring{s}(K_{r,r}) \sim 4r$ true?

Ques. What is $\max \mathring{s}(G)$ when G has n vertices and clique number k? Linear in kn? Max at Turán graph?

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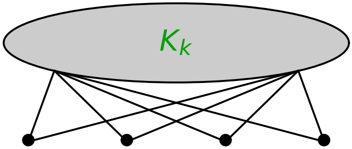

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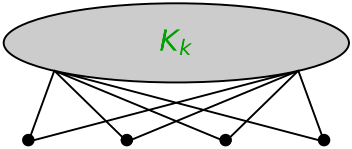

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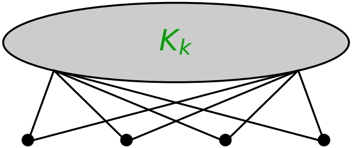

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