Online Sum-Paintability: The Slow-Coloring Game

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Abstract

The *slow-coloring game* is played by Lister and Painter on a graph G. On each round, Lister marks a nonempty subset M of the uncolored vertices, scoring $|M|$ points. Painter then gives a color to a subset of M that is independent in G. The game ends when all vertices are colored. Painter and Lister want to minimize and maximize the total score, respectively. The best score that each player can guarantee is the *sum-color cost* of G, written $\mathring{s}(G)$. The game is an online variant of online sum list coloring.

We prove $\frac{|V(G)|}{2\alpha(G)} + \frac{1}{2} \le \frac{\mathring{s}(G)}{|V(G)|} \le \max\left\{\frac{|V(H)|}{\alpha(H)} : H \subseteq G\right\}$, where $\alpha(G)$ is the independence number, and we study when equality holds in the bounds. We compute $\mathring{s}(G)$ for graphs with $\alpha(G) = 2$. Among n-vertex trees, we prove that $\mathring{s}$ is minimized by the star and maximized by the path. We also study $\mathring{s}(K_{r,s})$.

Keywords: slow-coloring game, sum choosability, sum paintability, Hall ratio, complete bipartite graph
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1 Introduction

A *proper coloring* of a graph G assigns each vertex in the vertex set $V(G)$ a color so that adjacent vertices have distinct colors. That is, the set of vertices assigned a given color must be an *independent set*, meaning a set of pairwise nonadjacent vertices. The chromatic number, written $\chi(G)$, is the least k such that G has a proper coloring using k colors.

To examine worst-case behavior of proper coloring when not all colors are available at all vertices, we study a coloring game played by Lister and Painter on a graph G. In the ith round, Lister marks a nonempty subset M of the uncolored vertices as eligible to receive color i, scoring $|M|$. Painter then gives color i to a subset of M that is independent in G.

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The game ends when all vertices are colored, producing a proper coloring. Painter's goal is to minimize the total score; Lister seeks to maximize it, thereby introducing a large total delay in the coloring process. We call this the *slow-coloring game*. The best score that each player can guarantee is the *sum-color cost* of G, written $\mathring{s}(G)$.

The slow-coloring game arose as an online variant of online sum list coloring. List coloring generalizes the classical model of graph coloring by introducing a *list assignment* L that assigns to each vertex v a set $L(v)$ of available colors. A graph G is *L-colorable* if it has a proper coloring ϕ such that $\phi(v) \in L(v)$ for every vertex v. Given $f\colon V(G) \to \mathbb{N}$, a graph G is *f-choosable* if G is L-colorable whenever $|L(v)| \ge f(v)$ for every vertex v.

Introduced by Vizing [24] and by Erdős, Rubin, and Taylor [7], the *choice number* or *choosability* $\chi_\ell(G)$ is the least k such that G is f-choosable whenever $f(v) \ge k$ for all $v \in V(G)$. Since one option for the lists is to be identical, always $\chi_\ell(G) \ge \chi(G)$. There are already hundreds of papers on aspects of choosability.

Instead of minimizing the threshold list size, we may seek the least sum (or average) of list sizes. Introduced by Isaak [12, 13] and studied in [1, 10, 11], the *sum-choosability* of a graph G, denoted $\chi_{\mathrm{SC}}(G)$, is the minimum of $\sum f(v)$ over all f such that G is f-choosable.

Using integer colors, a list assignment can be viewed as a schedule, presenting $\{v\colon i \in L(v)\}$ on round i as the set M_i of candidates to receive color i. In slow-coloring, Lister can model any list assignment, but Lister has other options and can change strategy in response to Painter's moves. Such flexibility for Lister leads to the *f-painting game*, introduced independently by Schauz [20] and by Zhu [26]. As in the slow-coloring game, Lister marks a set M and Painter colors an independent subset of M. Instead of fixed lists, we have a size $f(v)$ for each vertex v as the number of times Painter can allow v to be marked. Lister wins the game by marking some vertex v more than $f(v)$ times; Painter wins by coloring all the vertices before that happens. The graph is *f-paintable* if Painter has a winning strategy.

The *paint number* or *paintability* is the least k such that G is f-paintable whenever $f(v) \ge k$ for all $v \in V(G)$. The *sum paintability* of a graph G, introduced by Carraher, Mahoney, Puleo, and West [5] and written $\chi_{\mathrm{SP}}(G)$, is the minimum of $\sum f(v)$ over all f such that G is f-paintable. Finding f so that G is f-paintable corresponds to giving supplies to each vertex; $f(v)$ tokens are allocated to v, and one is used each time v is marked. In sum-paintability, Painter seeks to minimize the total number of tokens allocated.

The slow-coloring game differs from sum-paintability in that Painter need not allocate tokens in advance. Painter allocates tokens to vertices in response to the marked set. Since Painter can choose independent sets as dictated by an optimal strategy for sum-paintability, always $\mathring{s}(G) \le \chi_{\mathrm{SP}}(G)$. Note also that $\chi_{\mathrm{SC}}(G) \le \chi_{\mathrm{SP}}(G)$, since Painter can win the f-painting game in which Lister's moves are specified by L if and only if G is L-colorable (lists are truncated when vertices are colored). The inequality $\mathring{s}(G) \ge \chi_{\mathrm{SC}}(G)$ seems natural, but we will see in Example 1.10 that it is not always true.

Unlike sum-paintability, sum-color cost is given by an easily described (but hard to compute) recursive formula. The key point is that prior choices do not affect Painter's optimal strategy for coloring subsets of marked sets on the uncolored subgraph. Thus we can view colored vertices as having been "deleted" from the graph.

Proposition 1.1. $$\mathring{s}(G) = \max_{\varnothing \neq M \subseteq V(G)} \left(|M| + \min_{\textit{independent } I \subseteq M} \mathring{s}(G - I) \right).$$

Proof. In response to the initial marked set M, Painter minimizes the additional score over colored subsets $I \subseteq M$ such that I is independent in G. Lister chooses M to maximize the resulting total score. □

In studying optimal strategies for Lister and Painter, simple observations reduce the set of moves that need to be considered.

Observation 1.2. *On any graph, there are optimal strategies for Lister and Painter such that Lister always marks a set M inducing a connected subgraph, and Painter always colors a maximal independent subset of M.*

Proof. A move in which Lister marks a disconnected set M can be replaced with successive moves marking the vertex sets of the components of the subgraph induced by M. Also, coloring extra vertices at no extra cost cannot hurt Painter. □

Another easy observation sometimes yields a useful lower bound.

Observation 1.3. *If G_1 and G_2 are disjoint subgraphs of G, then $\mathring{s}(G) \geq \mathring{s}(G_1) + \mathring{s}(G_2)$.*

Proof. Lister can play an optimal strategy on G_1 while ignoring the rest and then do the same on G_2, achieving the score $\mathring{s}(G_1) + \mathring{s}(G_2)$. □

Observation 1.4. *If G is a subgraph of H, then $\mathring{s}(G) \leq \mathring{s}(H)$.*

Proof. On G, Painter can use a strategy that is optimal for the supergraph H. □

Proposition 1.5. $\mathring{s}(G) \leq \chi(G)\,|V(G)|$ *for every graph G.*

Proof. If G is k-colorable, then every marked set M contains an independent set of size at least $|M|/k$. By always coloring a largest marked independent set, Painter can guarantee that the contribution to the score is at most k times the number of vertices colored. By summing over all rounds, this strategy yields $\mathring{s}(G) \leq \chi(G)\,|V(G)|$. □

In Section 2, we analyse this Painter strategy more carefully to improve the upper bound $\chi(G)\,|V(G)|$ and also prove an easy lower bound, both in terms of independent sets. Let $\alpha(G)$ denote the maximum size of an independent set in a graph G.

Theorem 1.6. *The following are sharp bounds on* $\mathring{s}(G)$*:*

$$\left(\frac{|V(G)|}{2\alpha(G)}+\frac{1}{2}\right)|V(G)| \le \mathring{s}(G) \le \max\left\{\frac{|V(H)|}{\alpha(H)} \colon H \subseteq G\right\}|V(G)|.$$

Equality holds in the upper bound if and only if G *has no edges. Among complete multipartite graphs that are regular, equality holds in the lower bound if and only if* $\chi(G)=1$ *or* $\alpha(G)\le 2$.

Let $\rho(G)=\max\left\{\frac{|V(H)|}{\alpha(H)}\colon H\subseteq G\right\}$. Trivially, $\rho(G)\le\chi(G)$, so Theorem 1.6 improves Proposition 1.5. The quantity $\rho(G)$ has been called the *Hall ratio* of G, defined in [14] and explored further in [6, 15, 21, 22] (in fact, $\rho(G)$ is also bounded by the fractional chromatic number).

By well-known results on $\alpha(G)$ and $\chi(G)$ for random graphs (see Section 2), Theorem 1.6 implies that with high probability $\frac{\mathring{s}(G)}{|V(G)|}$ is within a constant multiple of $\chi(G)$. Although the upper bounds in Theorem 1.6 and Proposition 1.5 hold with equality only for edgeless graphs, they may be asymptotically best possible for k-colorable graphs.

In particular, the Turán graph $T_{n,k}$ is the complete k-partite graph with n vertices in which the sizes of parts differ by at most 1. When k divides n, all parts have size n/k, and $\alpha(T_{n,k})=n/k$. Theorem 1.6 yields $(k+1)n/2\le\mathring{s}(T_{n,k})\le kn$. Upon hearing this problem, Hehui Wu proved that the upper bound is asymptotically sharp; see Section 5.

Theorem 1.7 (Wu [25])**.** *When* k *is fixed,* $\mathring{s}(T_{n,k})$ *is asymptotic to the upper bound* kn.

In Section 2, we compute $\mathring{s}(G)$ for all G with $\alpha(G)=2$, and this aids in studying sharpness of the lower bound in Theorem 1.6. A *matching* is a set of pairwise disjoint edges.

Theorem 1.8. *If* $\alpha(G)=2$*, then* $\mathring{s}(G)=\binom{n-q+1}{2}+\binom{q+1}{2}$*, where* q *is the maximum size of a matching in the complement of* G.

We have noted that Painter can follow a winning strategy in an f-painting game to achieve $\mathring{s}(G)\le\chi_{\mathrm{SP}}(G)$. In Section 3, we characterize equality.

Theorem 1.9. $\mathring{s}(G)=\chi_{\mathrm{SP}}(G)$ *if and only if every component of* G *is complete.*

Examples give independence of the upper bounds $\mathring{s}(G)\le|V(G)|\,\rho(G)$ and $\mathring{s}(G)\le\chi_{\mathrm{SP}}(G)$.

Example 1.10. Note that $\mathring{s}(K_r)=\binom{r+1}{2}$; Painter can only color one vertex in each round, so on this graph it is optimal for Lister to always mark all uncolored vertices.

When G is the disjoint union of the complete graph K_r and an independent set of $n-r$ vertices, we have $\rho(G)=r$, but $\mathring{s}(G)=\chi_{\mathrm{SP}}(G)=n+\binom{r}{2}$, so $|V(G)|\,\rho(G)$ is much bigger.

When G is bipartite with at least one edge, $\rho(G)=2$. When $G=K_{2,r}$, we have $|V(G)|\,\rho(G)=2r+4$, but $\chi_{\mathrm{SP}}(G)\approx 2r+2\sqrt{r}$ (see [5]), so $\chi_{\mathrm{SP}}(G)$ is bigger. More generally, Füredi and Kantor [8] proved for $a\ge 3$ and $r>50a^2\log a$ that $\chi_{\mathrm{SC}}(K_{a,r})\ge 2r+.068a\sqrt{r\log a}$, so $\chi_{\mathrm{SP}}(K_{a,r})$ is at least as large, while $|V(K_{a,r})|\,\rho(K_{a,r})$ is only $2r+2a$. Hence in particular also $\chi_{\mathrm{SC}}(K_{a,r}) > \mathring{s}(K_{a,r})$ when $r > 50a^2\log a$ and a is sufficiently large, by Theorem 1.6.

On these complete bipartite examples, χ_{SP} is larger than $|V(G)|\,\rho(G)$, but the two bounds are asymptotically equal. Another bipartite example gives asymptotic ratio 5/4. Let $G = P_k \square K_2$, where $\square$ denotes the cartesian product (G is the 2-by-k "grid"). Note that G can be constructed from K_2 by successively adding ears of length 3. A lemma in [5] shows that each such addition increases the sum-paintability by 5, so $\chi_{\mathrm{SP}}(G) = 5k-2$. However, $|V(G)|\,\rho(G) = 4k$.

In Section 4, we prove sharp bounds on the sum-color cost of n-vertex trees.

Theorem 1.11. *Among n-vertex trees, the value of $\mathring{s}$ is minimized by the star and maximized by the path. Furthermore, with $u_t = \lfloor(-1+\sqrt{8t+1})/2\rfloor$ and T being an n-vertex tree,*

$$n+\sqrt{2n} \;\approx\; n+u_{n-1} \;=\; \mathring{s}(K_{1,n-1}) \;\le\; \mathring{s}(T) \;\le\; \mathring{s}(P_n) \;=\; \lfloor 3n/2 \rfloor .$$

In a subsequent paper, Puleo and West [19] provide a linear-time algorithm to compute $\mathring{s}$ on trees, via an inductive formula. The formula yields characterizations of the extremal n-vertex trees. The star is the unique n-vertex tree minimizing $\mathring{s}$ when $n-1$ and $n-2$ are not of the form $\binom{k}{2}$; otherwise a few trees like the star also achieve the minimum. The n-vertex trees T with $\mathring{s}(T) = \lfloor 3n/2 \rfloor$ are those having a spanning acyclic subgraph F in which every vertex has degree 1 or 3, except for one vertex in F having degree 0 or 6 when n is odd.

We do not know the complexity of computing $\mathring{s}(G)$ in general or on larger families than trees. It is not obvious that the decision problem is in NP.

We conjecture that Theorem 1.11 generalizes to k-trees. A *k-tree* is a graph obtained from K_k by iteratively adding a vertex whose neighborhood is a k-clique in the existing graph. The *join* $G \oplus H$ of graphs G and H is obtained from the disjoint union $G+H$ by making each vertex in G adjacent to each vertex in H. The *rth power* of G is the graph G^r with vertex set $V(G)$ where vertices are adjacent if and only the distance between them in G is at most r. The graphs $K_k \oplus \overline{K}_{n-k}$ and P_n^k are k-tree analogues of n-vertex stars and paths. Our argument to compute $\mathring{s}(K_{1,n-1})$ allows us more generally to compute $\mathring{s}(K_k \oplus \overline{K}_{n-k})$.

Theorem 1.12. *Let $u_t = \left\lfloor \frac{-1+\sqrt{8t+1}}{2} \right\rfloor$. For $r,s \in \mathbb{N}$,*

$$\mathring{s}(\overline{K}_r \oplus K_s) = r + \binom{s+1}{2} + s u_r .$$

Conjecture 1.13. *For $k \in \mathbb{N}$ and any k-tree T with n vertices,*

$$\mathring{s}(K_k \oplus \overline{K}_{n-k}) \;\le\; \mathring{s}(T) \;\le\; \mathring{s}(P_n^k).$$

An easy lower bound for $\mathring{s}(P_n^k)$ follows from Observation 1.3. Since P_n^k contains $\lfloor \frac{n}{k+1} \rfloor$ disjoint copies of K_{k+1}, we have $\mathring{s}(P_n^k) \geq \mathring{s}(\lfloor \frac{n}{k+1} \rfloor K_{k+1}) + \mathring{s}(K_r)$, where $r \equiv n \mod (k+1)$, and we conjecture that equality holds. This formula reduces to the correct answer for $k = 1$.

In Section 5, we study the slow-coloring game on complete bipartite graphs.

Theorem 1.14. *Let $u_t = \left\lfloor \frac{-1+\sqrt{8t+1}}{2} \right\rfloor$. For $r \geq s > 0$,*

$$r + \frac{5s-3}{2} + u_{r-s} \leq \mathring{s}(K_{r,s}) \leq r + s + 2\sqrt{rs}.$$

Note that this upper bound is $\chi(G)\,|V(G)|$ when $r = s$. The lower bound was later improved by Wu [25] to $r + s + u_r u_s$, which is very close to the upper bound.

In a further subsequent paper, Gutowski et al. [9] study the bounds on $\mathring{s}(G)$ for families of graphs containing forests. A graph is *d-degenerate* if every subgraph has a vertex of degree at most d. Inductively, $\chi(G) \leq 1 + d$ when G is d-degenerate, so Proposition 1.5 yields $\mathring{s}(G) \leq (1+d)\,|V(G)|$ when G is d-degenerate, and the disjoint union of copies of K_{d+1} show that the value can be as high as $(1 + \frac{d}{2})\,|V(G)|$. In [9], the bound $\mathring{s}(G) \leq (1 + \frac{3d}{4})\,|V(G)|$ is proved for d-degenerate graphs (improving to $(\frac{3}{4} + \frac{3d}{4})\,|V(G)|$ when d is odd).

A graph is *outerplanar* if it embeds in the plane with all vertices lying on a single face. It is an elementary exercise that outerplanar graphs are 2-degenerate, so the result mentioned above implies $\mathring{s}(G) \leq 2.5n$ when G is an n-vertex outerplanar graph. In [9], the bound is improved to $7n/3$. Disjoint copies of K_3 show that the value can be as large as $2n$, which is conjectured optimal.

The famous Four Color Theorem states $\chi(G) \leq 4$ when G embeds in the plane, so $\mathring{s}(G) \leq 4n$ for n-vertex planar graphs. In [9], the bound is improved to $3.9857n$, where the coefficient more precisely is $(13 + 4\sqrt{3})/5$. In fact, this bound holds for all acyclically 5-colorable graphs, which includes all planar graphs. Disjoint copies of K_4 show that the value can be as large as $2.5n$ on planar graphs; this may be optimal.

2 General Bounds

The lower bound we prove in Theorem 2.2 is stronger than that in Theorem 1.6, as explained in Corollary 2.3. It uses the "chromatic sum" of a graph, defined by Kubicka [16] (see [17] for a survey).

Definition 2.1. The *chromatic sum* of a graph G, written $\Sigma(G)$, is the minimum of $\sum_{v \in V(G)} c(v)$ over all proper colorings c that color the vertices using positive integers.

The chromatic sum is the outcome of the slow-coloring game when Lister follows the strategy of always marking the entire remaining graph. Let $G[M]$ denote the subgraph of G induced by a vertex subset M.

Theorem 2.2. *For every n-vertex graph G,*

$$\Sigma(G) \le \mathring{s}(G) \le n\rho(G).$$

Equality holds in the lower bound when all components of G are complete. Equality holds in the upper bound if and only if G has no edges.

Proof. First consider the lower bound. Given that Lister always marks all remaining vertices, let V_i be the set colored by Painter on round i. The vertices in V_i are marked i times. Thus the total cost is at least $\Sigma(G)$. Equality holds for a disjoint union of complete graphs, because Painter always colors one vertex in each component having a marked vertex.

For the upper bound, let $r = \rho(G)$. Given any marked set M, the *greedy strategy* for Painter colors a largest independent set in $G[M]$. The definition of $\rho(G)$ yields $\alpha(G[M]) \ge |M|/r$. For any game played against this strategy, let $m_1, \dots, m_t$ be the sizes of the marked sets in the successive rounds. In round i Painter colors at least m_i/r vertices, so $\sum_{i=1}^{t} \frac{m_i}{r} \le n$. Multiplying by r shows that Lister scores at most nr.

Equality for $G = \overline{K}_n$ is trivial. Conversely, if equality holds in the upper bound, then exactly m_i/r vertices must be colored in round i. In particular, in the last round, all m_i marked vertices are colored, requiring $m_i/r = m_i$, and hence $r = 1$. For any graph G having an edge, $\rho(G) \ge 2$, so equality holds only for edgeless graphs. □

Corollary 2.3. $\Sigma(G) \ge \frac{n}{2}(1 + \frac{n}{\alpha(G)})$ *when G has n vertices, and hence* $\mathring{s}(G) \ge \frac{n^2}{2\alpha(G)} + \frac{n}{2}$.

Proof. We use induction on n; the claim is trivial for $n = 0$. For $n > 0$, let I be the set of vertices receiving color 1 in a proper coloring of G with minimum sum, and let $a = \alpha(G)$. Note that $\Sigma(G) = |I| + (n - |I|) + \Sigma(G - I)$. Using the induction hypothesis, $\Sigma(G) \ge n + \frac{1}{2}(n - |I|)(1 + \frac{n-|I|}{\alpha(G-I)})$. Minimizing the numerator and maximizing the denominator, we have $\Sigma(G) \ge n + \frac{1}{2}(n - a)(1 + \frac{n-a}{a}) = \frac{n}{2}(1 + \frac{n}{a})$. □

The *binomial random graph model* (see [4]) is the probability space $\mathbb{G}(n,p)$ generating graphs with vertex set $\{1, \dots, n\}$ by letting vertex pairs be edges with probability p, independently. An event occurs *with high probability* if its probability in $\mathbb{G}(n,p)$ tends to 1 as $n \to \infty$.

Corollary 2.4. *For fixed $p \in (0,1)$, there is a positive constant c such that for G sampled from $\mathbb{G}(n,p)$, with high probability $c\chi(G) \le \frac{\mathring{s}(G)}{n} \le \chi(G)$.*

Proof. The upper bound always holds, by Proposition 1.5. For the lower bound, it suffices by Corollary 2.3 to obtain a constant c' such that $c'\chi(G) \le \frac{n}{2\alpha(G)} + \frac{1}{2}$ with high probability. By well-known results on the concentration of the clique number and the chromatic number in $\mathbb{G}(n,p)$ [2, 3, 4], there are positive constants c_1 and c_2 (depending on p) such that for any positive ϵ, with high probability $\chi(G)$ is within a fraction $1 + \epsilon$ of $c_1 \frac{n}{\log n}$ and $\alpha(G)$ is within a fraction $1 + \epsilon$ of $c_2 \log n$. The result follows. □

We next determine $\mathring{s}(G)$ when $\alpha(G) \le 2$, showing that the lower bound from Theorem 2.2 holds with equality in that case. Let $\overline{G}$ denote the complement of a graph G.

Lemma 2.5. *If $\alpha(G) \le 2$, then $\Sigma(G) = \binom{n-q+1}{2} + \binom{q+1}{2}$, where q is the maximum size of a matching in $\overline{G}$.*

Proof. When $\alpha(G) \le 2$, all independent sets have size at most 2. To minimize the sum of the colors, the color classes of size 2 should be given the lowest colors, and the largest possible number of disjoint classes of size 2 should be used. This largest number is q, and the remaining vertices must have distinct colors. Thus q vertices receive colors 1 through q, and the remaining vertices receive colors 1 through $n-q$. The result follows. $\square$

Lemma 2.6. *If $G_{n,q}$ is the n-vertex complete multipartite graph with q parts of size 2 and $n-2q$ parts of size 1, then $\mathring{s}(G_{n,q}) = \binom{n-q+1}{2} + \binom{q+1}{2}$.*

Proof. The lower bound follows from Theorem 2.2 and Lemma 2.5. For the upper bound, let $f(n,q) = \binom{n-q+1}{2} + \binom{q+1}{2}$. We prove $\mathring{s}(G_{n,q}) \le f(n,q)$ by induction on n. The claim is trivial for $n \le 1$.

For $n > 1$, let s be the total score of the game. Consider the first round. If Lister marks some two nonadjacent vertices, then Painter colors such a pair, yielding $s \le n + \mathring{s}(G_{n-2,q-1}) = (n-q) + q + f(n-2, q-1) = f(n,q)$. If Lister marks at most one vertex from each part, including a part of size 1, then Painter colors such a part, yielding $s \le n - q + \mathring{s}(G_{n-1,q}) = n - q + f(n-1,q) = f(n,q)$. If Lister marks only single vertices from parts of size 2, then $s \le q + \mathring{s}(G_{n-1,q-1}) = q + f(n-1,q-1) = f(n,q)$. $\square$

Theorem 2.7. *If G is an n-vertex graph with $\alpha(G) \le 2$, and q is the maximum size of a matching in $\overline{G}$, then $\mathring{s}(G) = \Sigma(G) = \binom{n-q+1}{2} + \binom{q+1}{2}$. Thus there is a cubic-time algorithm to determine $\mathring{s}(G)$ and $\Sigma(G)$ in the class of graphs with independence number at most 2.*

Proof. Let M be a maximum matching in $\overline{G}$. Let H be the supergraph of G with vertex set $V(G)$ such that M is the set of edges in $\overline{H}$. The graph H is a complete multipartite graph with q parts of size 2. By Lemma 2.5, Theorem 2.2, Observation 1.4, and Lemma 2.6,

$$\binom{n-q+1}{2} + \binom{q+1}{2} = \Sigma(G) \le \mathring{s}(G) \le \mathring{s}(H) = \binom{n-q+1}{2} + \binom{q+1}{2}.$$

Thus equality holds throughout.

By examining all 3-sets of vertices, it can be tested whether $\alpha(G) \le 2$. There is an algorithm to find the maximum size of a matching in $\overline{G}$ that runs in time $O(n^{2.5})$ ([18]; see [23] for a proof). Thus $\Sigma(G)$ and $\mathring{s}(G)$ can be computed in cubic time when $\alpha(G) \le 2$. $\square$

Using Theorem 2.7 and the the formula in Theorem 1.12 for $\mathring{s}(\overline{K_r} \oplus K_s)$, which we will prove in Section 4, we can determine whether some graphs achieve the weaker lower bound in Corollary 2.3.

Theorem 2.8. *If G is an n-vertex complete multipartite graph that is regular, then $\mathring{s}(G) = \frac{n^2}{2\alpha(G)} + \frac{n}{2}$ if and only if $\chi(G) = 1$ or $\alpha(G) \le 2$.*

Proof. Let $t = \chi(G)$ and $r = \alpha(G)$; note that G is a complete t-partite graph in which all parts have size r, and $n = rt$. Corollary 2.3 yields the desired lower bound $\mathring{s}(G) \ge \frac{rt}{2}(1+t)$, which we write as $r\binom{t+1}{2}$ (Lister guarantees this much by Observation 1.3, using a covering of $V(G)$ by r disjoint t-cliques). When r or t equals 1, the graph is complete or empty, and we have seen that equality holds in these cases. When $r = 2$, the graph is $G_{n,n/2}$ of Lemma 2.6 with $t = q = n/2$, which yields $\mathring{s}(G) = 2\binom{t+1}{2}$.

When $r > 2$ and $t > 1$, a strategy for Lister establishes a stronger lower bound. In each of the first $t-1$ rounds, Lister marks $r-1$ vertices from each part having no colored vertices. By Observation 1.2, Painter responds optimally by coloring all the marked vertices in one part, reducing it to one uncolored vertex. The remaining marked parts still have no colored vertices. After $t-1$ rounds, Lister has scored $\sum_{i=2}^{t} i(r-1)$, which equals $(r-1)\binom{t+1}{2} - (r-1)$, and the subgraph induced by the uncolored vertices is $\overline{K}_r \oplus K_{t-1}$. In Theorem 4.3 we will compute $\mathring{s}(\overline{K}_r \oplus K_{t-1})$; the formula yields $\mathring{s}(\overline{K}_r \oplus K_{t-1}) \ge r + \binom{t}{2} + (t-1)(\sqrt{2r}-1)$. Summing the contributions from the first $t-1$ rounds and the remaining graph $\overline{K}_r \oplus K_{t-1}$ yields $\mathring{s}(G) \ge r\binom{t+1}{2} + (t-1)(\sqrt{2r}-2)$. This lower bound exceeds $r\binom{t+1}{2}$ when $r > 2$ and $t \ge 2$. □

3 All graphs G such that $\mathring{s}(G) = \chi_{SP}(G)$

Here we prove Theorem 1.9: Equality holds in $\mathring{s}(G) \le \chi_{SP}(G)$ if and only if every component of G is complete. Sufficiency is immediate, since $\mathring{s}(K_n) = \chi_{SP}(K_n) = \binom{n+1}{2}$. For necessity, we show that equality holds only when $\mathring{s}(G) = |V(G)| + |E(G)|$ and that this latter equality holds only when every component is complete.

As noted in [5], $|V(G)| + |E(G)|$ is an easy upper bound on $\chi_{SP}(G)$, proved by Painter playing greedily with respect to some vertex ordering. A graph is *sp-greedy* [5] when equality holds in that bound. Our first step is to show that $\mathring{s}(G) = \chi_{SP}(G)$ only when G is sp-greedy.

In the f-painting game, Painter must immediately color any vertex with no remaining tokens. Therefore, when Lister marks a vertex v having exactly one token, Lister should also mark all neighbors of v. Zhu formalized this observation. Let $N(v)$ denote the set of neighbors of v.

Proposition 3.1 (Zhu [26])**.** *If $f(v) = 1$ for a token assignment f on a graph G, then G is f-paintable if and only if $G - v$ is f'-paintable, where*

$$f'(w) = \begin{cases} f(w) - 1, & \text{if } w \in N(v), \\ f(w), & \text{otherwise.} \end{cases}$$

Lemma 3.2. *Given $\mathring{s}(G) = \chi_{SP}(G)$, let f be an assignment of $\chi_{SP}(G)$ tokens under which G is f-paintable. Let Lister play optimally in the slow-coloring game (guaranteeing to score at least $\mathring{s}(G)$). If Painter interprets Lister as playing in the f-painting game and responds using an optimal strategy there, then the set colored by Painter always consists of vertices that began the round with one token.*

Proof. Under the given strategies, let $g(v)$ be the number of times that a vertex v is marked. By Lister's strategy, $\mathring{s}(G) = \sum_{v\in G} g(v)$. On the other hand, $\mathring{s}(G) = \chi_{SP}(G) = \sum_{v\in G} f(v)$. Since Painter wins, $g(v) \le f(v)$ for all v, so $g(v) = f(v)$ for all v. Since v has $f(v) - g(v) + 1$ tokens at the beginning of the round in which it is colored, the claim follows. □

Let $d(v)$ denote the degree of a vertex v in a graph G.

Lemma 3.3. *Let G be a graph, and let f be an assignment of $\chi_{SP}(G)$ tokens such that G is f-paintable. If $G - v$ is sp-greedy for some vertex v such that $f(v) = 1$, then G is sp-greedy.*

Proof. By Proposition 3.1, the graph $G - v$ is f'-paintable, where $f'(w) = f(w) - 1$ for $w \in N(v)$ and $f'(w) = f(w)$ otherwise. Since also $G - v$ is sp-greedy,

$$\sum_{w\in V(G)} f(w) - f(v) - d(v) \;=\; \sum_{w\in V(G-v)} f'(w) \;\ge\; |V(G)| - 1 + |E(G)| - d(v).$$

Thus $\sum_{w\in V(G)} f(w) \ge |V(G)| + |E(G)| + f(v) - 1$. Since $f(v) = 1$, G is sp-greedy. □

Let $N[v]$ denote the *closed neighborhood* of a vertex v, meaning $N[v] = N(v) \cup \{v\}$.

Lemma 3.4. *If G is not sp-greedy, then $\mathring{s}(G) < \chi_{SP}(G)$.*

Proof. Let G be a counterexample with fewest vertices, so G is not sp-greedy, but $\mathring{s}(G) = \chi_{SP}(G)$. Let f be an assignment of $\chi_{SP}(G)$ tokens such that G is f-paintable. Let Lister play an optimal strategy in the slow-coloring game on G. Since $\mathring{s}(G) = \chi_{SP}(G)$, an optimal strategy for Painter would be to follow an optimal strategy $\mathcal{S}$ in the f-painting game on G.

Let M be the set marked by Lister on the first move. Since $\mathcal{S}$ would be optimal for Painter, by Lemma 3.2 there exists $v \in M$ such that $f(v) = 1$. Let $G' = G - v$, and let $M' = M - N[v]$. Let I' be Painter's response to M' in an optimal strategy $\mathcal{S}'$ for the slow-coloring game on G'. The set $I' \cup \{v\}$ is independent. Instead of using $\mathcal{S}$, Painter responds to M on G by coloring $I' \cup \{v\}$ and then continues play according to $\mathcal{S}'$.

By Lemma 3.3, G' is not sp-greedy. Since G is a minimal counterexample, $\mathring{s}(G') < \chi_{SP}(G')$. On the other hand, Proposition 3.1 implies $\chi_{SP}(G') \le \chi_{SP}(G) - (d(v) + 1)$. Thus $d(v) + 1 + \mathring{s}(G') < \chi_{SP}(G)$.

In the game on G, the total scored by Lister is at most $1 + d(v) + \mathring{s}(G')$, since $\mathring{s}(G')$ counts everything scored in the game except $M - M'$ in the first round. Since $d(v) + 1 + \mathring{s}(G') < \chi_{SP}(G)$, this contradicts the hypothesis that Lister can score at least $\chi_{SP}(G)$. □

Corollary 3.5. *If* $\mathring{s}(G) = \chi_{\mathrm{SP}}(G)$*, then* $\mathring{s}(G) = |V(G)| + |E(G)|$.

The following theorem completes the proof of Theorem 1.9.

Theorem 3.6. $\mathring{s}(G) = |V(G)| + |E(G)|$ *if and only if every component of* G *is complete.*

Proof. It suffices to show $\mathring{s}(G) < |V(G)| + |E(G)|$ when G is connected and not complete. Let M be the set marked by Lister on an optimal first move in the slow-coloring game on G.

Case 1: $M = V(G)$. Since G is connected and not complete, G has a vertex v with nonadjacent neighbors w and w'. Let $G' = G - \{w, w'\}$, and let $M' = V(G) - (N[w] \cup N[w'])$; note that $M' \subsetneq V(G')$. Let I' be Painter's response to M' in an optimal strategy $\mathcal{S}'$ on G'.

The set $I' \cup \{w, w'\}$ is independent. In response to M on G, Painter colors $I' \cup \{w, w'\}$ and continues play according to $\mathcal{S}'$. In the game on G, the total scored by Lister is at most $2 + |N(w) \cup N(w')| + \mathring{s}(G')$, since $\mathring{s}(G')$ counts everything scored in the game except $|N[w] \cup N[w']|$ on the first round. Since $\mathring{s}(G') \le |V(G')| + |E(G')|$,

$$\mathring{s}(G) \le 2 + |N(w) \cup N(w')| + |V(G')| + |E(G')| = |V(G)| + |E(G')| + |N(w) \cup N(w')|.$$

Since $v \in N(w) \cap N(w')$, we have $|E(G')| \le |E(G)| - (|N(w) \cup N(w')| + 1)$. Hence $\mathring{s}(G) \le |V(G)| + |E(G)| - 1$.

Case 2: $\varnothing \ne M \subsetneq V(G)$. Since G is connected, G has an edge vw with $w \in M$ and $v \notin M$. Let $G' = G - w$ and $M' = M - N[w]$. Let I' be Painter's response to M' in an optimal strategy $\mathcal{S}'$ on G'. In response to M on G, Painter colors $I' \cup \{w\}$ and continues play according to $\mathcal{S}'$. Let $M_0 = N(w) \cap M$. Adding the part of the score in the first round that is not counted in the game on G', we have

$$\mathring{s}(G) \le |M_0| + 1 + \mathring{s}(G') \le |M_0| + 1 + |V(G')| + |E(G')| = |V(G)| + (|M_0| + |E(G')|).$$

Since $v \notin M$, we have $|E(G')| \le |E(G)| - |M_0| - 1$. Hence $\mathring{s}(G) \le |V(G)| + |E(G)| - 1$. □

4 Bounds for n-Vertex Trees

It is easy for Lister to score $\lfloor 3n/2 \rfloor$ on the n-vertex path P_n. Lister first marks all n vertices. Since $\alpha(P_n) = \lceil n/2 \rceil$, Lister can score $\lfloor n/2 \rfloor$ more by marking all vertices that remain after Painter colors an independent set. Indeed, $\mathring{s}(G) \ge 2n - \alpha(G)$ in any graph by Lister marking all uncolored vertices for two rounds. We thus can prove $\mathring{s}(T) \le \mathring{s}(P_n)$ for each n-vertex tree T by proving $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$. There are several ways to prove this fact; the efficient phrasing we present here was suggested by Xuding Zhu.

Theorem 4.1. *If* T *is an* n*-vertex tree, then* $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$.

Proof. We use induction on n; the statement holds by inspection for small n. Let M be the set marked in an optimal first move for Lister. It suffices to prove that M contains an independent set I such that $\mathring{s}(T-I) \le \frac{3}{2}n - |M|$. By Observation 1.2, we may assume that $T[M]$ is connected. Our Painter strategy colors X or Y, where $T[M]$ is a bipartite graph with parts X and Y. It thus suffices to prove $\mathring{s}(T-X) + \mathring{s}(T-Y) \le 3n - 2|M|$.

Let $o(H)$ denote the number of components with odd order in a graph H. Summing the inductive bound over all components of $T-I$ yields $\mathring{s}(T-I) \le \frac{3}{2}(n-|I|) - \frac{1}{2}o(T-I)$. We apply this computation to both $T-X$ and $T-Y$.

Let $T' = T - E(T[M])$. Each component of T' contains exactly one vertex of M. Consider a vertex $v \in X$, contained in the component R of T'. If R is odd, then R is counted in $o(T-Y)$. If R is even, then R contributes at least 1 to $o(T-X)$. The symmetric statement holds for $v \in Y$. Thus $o(T-X) + o(T-Y) \ge |M|$. We compute

$$\begin{aligned}\mathring{s}(T-X) + \mathring{s}(T-Y) &\le \frac{3}{2}(n-|X|) - \frac{1}{2}o(T-X) + \frac{3}{2}(n-|Y|) - \frac{1}{2}o(T-Y)\\ &\le 3n - \frac{3}{2}|M| - \frac{1}{2}|M| = 3n - 2|M|. \end{aligned}$$ □

Next we determine $\mathring{s}(K_{1,n-1})$ and prove $\mathring{s}(T) \ge \mathring{s}(K_{1,n-1})$ for every n-vertex tree T. More generally, we compute $\mathring{s}(\overline{K}_r \oplus K_s)$. For $k, r \in \mathbb{N} \cup \{0\}$, let $t_k = \binom{k+1}{2}$ and $u_r = \max\{k \colon t_k \le r\}$. Note that $u_r = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor$. The numbers of the form $\binom{k+1}{2}$ are the *triangular numbers.* Before computing $\mathring{s}(\overline{K}_r \oplus K_s)$, we need a technical lemma about u_r.

Lemma 4.2. *$u_{r-u_r} = u_r$ when $r+1$ is a triangular number, and otherwise $u_{r-u_r} = u_r - 1$.*

Proof. If $u_r = k$, then $t_k \le r < t_{k+1}$. Also $t_{k+1} - t_k = k+1$. Thus $r - k = t_k$ if $r+1 = t_{k+1}$, yielding $u_{r-u_r} = u_r$. However, $t_{k-1} \le r-k < t_k$ if $t_k \le r \le t_{k+1} - 2$, yielding $u_{r-u_r} = u_r - 1$. □

Theorem 4.3. *For nonnegative integers r and s,*

$$\mathring{s}(\overline{K}_r \oplus K_s) = r + \binom{s+1}{2} + su_r.$$

Proof. We use induction on $r+s$. Let $f(r,s) = r + \binom{s+1}{2} + su_r$. When r or s is 0, the claim clearly holds. For $rs > 0$, let $G = \overline{K}_r \oplus K_s$. Also let $[r] = \{1, \dots, r\}$. Let R and S denote the sets of vertices with degree s and degree $r+s-1$, respectively.

If Lister marks no vertex of S, then Painter colors all marked vertices. By the induction hypothesis, this easily gives the desired upper bound on the final score. Hence we may assume that Lister marks some vertex of S. Since Painter can color at most one vertex of S in response, Lister should mark all of S plus perhaps some of R. Painter responds by coloring one vertex of S or all marked vertices of R. Applying the recurrence of Proposition 1.1 and the induction hypothesis,

$$\mathring{s}(\overline{K}_r \oplus K_s) = \max_{k\in[r]}\left((k+s) + \min\{\mathring{s}(\overline{K}_{r-k} \oplus K_s), \mathring{s}(\overline{K}_r \oplus K_{s-1})\}\right) = \max_{k\in[r]} g(k),$$

where

$$g(k) = k + s + \min\{f(r-k,s), f(r,s-1)\}.$$

By the induction hypothesis, $g(k)$ is the best result Painter can obtain when Lister marks S and k vertices of R on the first round. We compute

$$\begin{aligned} g(u_r) &= u_r + s + \min\{r - u_r + \binom{s+1}{2} + su_{r-u_r}, r + \binom{s}{2} + (s-1)u_r\} \\ &= \min\{r + \binom{s+1}{2} + s(u_{r-u_r}+1), r + \binom{s+1}{2} + su_r\} \\ &= \min\{f(r,s) + s(1 + u_{r-u_r} - u_r), f(r,s)\}. \end{aligned}$$

By Lemma 4.2, $u_r - u_{r-u_r} \in \{0,1\}$, so $g(u_r) = f(r,s)$. Furthermore, if Lister marks u_r vertices in R and all of S, then coloring a vertex of S is an optimal response for Painter.

We seek $\max_k g(k)$. Note that $g(0) = s + f(r,s-1)$, since $f(r,s) > f(r,s-1)$. If $f(r-k,s) \geq f(r,s-1)$, then $g(k) = g(0) + k$, so in this range we maximize k. Since also $f(r-k,s)$ decreases as k increases, and $f(r-k,s)+k$ is nonincreasing, for k larger than this range g is nonincreasing. Hence $g(k)$ is maximized by the largest k such that $f(r-k,s) \geq f(r,s-1)$. To show that this is u_r, it suffices to show $f(r-u_r,s) \geq f(r,s-1)$ and $f(r-(u_r+1),s) < f(r,s-1)$.

When $r+1$ is not a triangular number, Lemma 4.2 yields $f(r-u_r,s) = f(r,s-1)$. Since $f(r-(u_r+1),s) < f(r-u_r,s)$, the desired value of k is u_r.

When $r+1$ is a triangular number, Lemma 4.2 yields $f(r-u_r,s) = f(r,s-1)+s$. Since r itself then is not a triangular number, Lemma 4.2 and $u_{r-1} = u_r$ yield $f(r-(u_r+1),s) = f(r-1,s-1) < f(r,s-1)$. Again the desired value is u_r. □

Setting $s = 1$, we have $\mathring{s}(K_{1,n-1}) = n + u_{n-1} = n + \left\lfloor \frac{-1+\sqrt{8n-7}}{2} \right\rfloor$.

Theorem 4.4. *If T is an n-vertex tree, then $\mathring{s}(T) \geq \mathring{s}(K_{1,n-1}) = n + v_n$, where $v_n = u_{n-1}$.*

Proof. We use induction on n. Since $\mathring{s}(P_n) = \lfloor 3n/2 \rfloor$ and always $\lfloor n/2 \rfloor \geq v_n$, the claim holds when T is a path. Hence also the claim holds for $n \leq 4$.

The main idea is that Lister can play separately on disjoint induced subgraphs, yielding $\mathring{s}(T) \geq \mathring{s}(T_1) + \mathring{s}(T_2)$ when T_1 and T_2 are the components obtained by deleting an edge of T. If $n_i = |V(T_i)|$, then $\mathring{s}(T) \geq n + v_{n_1} + v_{n_2}$. It therefore suffices to find an edge e such that $v_{n_1} + v_{n_2} \geq v_n$. We may assume $n_1 \leq n_2$.

When $n \geq 5$, we have $v_n \leq 1 + v_{n-3}$. If T has an edge whose deletion leaves a component with two or three vertices, then $v_{n_1} = 1$, and $v_{n_1} + v_{n_2} \geq 1 + v_{n-3} \geq v_n$, as desired. If T is not a star, then T has an edge not incident to a leaf, and when $n \le 7$ every edge not incident to a leaf has this property that $n_1 \le 3$.

In the remaining case, $n \ge 8$ and $n_1, n_2 \ge 4$. When $n \ge 8$, we have $v_n \le 1 + v_{n-4}$. Let $h(x) = \frac{-1+\sqrt{8x-7}}{2}$; note that $v_n = \lfloor h(n) \rfloor$. We need $v_{n_1} + v_{n_2} \ge v_n$.

Let $p = 4$ and $q = n - 4$. Since h is concave, $h(p+x) - h(p) \ge h(q) - h(q-x)$ when $0 \le x \le n/2 - 4$, which yields $h(p+x) + h(q-x) \ge h(p) + h(q)$. If $a + b \ge c + d$, then $\lfloor a \rfloor + \lfloor b \rfloor \ge \lfloor a+b \rfloor - 1 \ge \lfloor c+d \rfloor - 1 \ge \lfloor c \rfloor + \lfloor d \rfloor - 1$. Applying this with $(a,b,c,d) = (h(n_1), h(n_2), h(p), h(q))$, and using $v_4 = 2$, we obtain $v_{n_1} + v_{n_2} \ge v_4 + v_{n-4} - 1 = 1 + v_{n-4} \ge v_n$, as desired. ☐

5 Bounds for Complete Bipartite Graphs

Upper and lower bounds on $\mathring{s}(K_{r,s})$ arise from strategies for Painter and Lister, respectively. Our upper bound is very close to Wu's later lower bound. We also present our earlier lower bound, omitting the proof; both are summarized in Theorem 1.14.

Remark 5.1. *Properties of optimal strategies.* In light of symmetry, we can describe a move by Lister in the slow-coloring game on $K_{r,s}$ as a pair (j,i), marking j vertices in the part X of size r and i in the part Y of size s. An optimal response by Painter colors all marked vertices in one part, and the cost under optimal play will be $j + i + \min\{\mathring{s}(K_{r-j,s}), \mathring{s}(K_{r,s-i})\}$.

Since Lister can restrict play to an induced subgraph, $\mathring{s}(K_{r,s})$ increases with r and with s (in fact strictly, by Observation 1.3). If an optimal response by Painter for the move (j,i) is to color in X, and $j' > j$, then

$$\mathring{s}(K_{r-j',s}) < \mathring{s}(K_{r-j,s}) \le \mathring{s}(K_{r,s-i}),$$

and hence it is also optimal to color in X in response to (j', i). Therefore, given i, there is a threshold J such that in response to (j,i), Painter should color the j vertices in X when $j \ge J$ and should color the i vertices in Y when $j < J$. As a result, we specify a Painter strategy by specifying J as a function of i when playing on $K_{r,s}$.

Theorem 5.2. *$\mathring{s}(K_{r,s}) \le r + s + 2\sqrt{rs}$.*

Proof. Let $f(r,s) = r + s + 2\sqrt{rs}$. We prove $\mathring{s}(K_{r,s}) \le f(r,s)$ by induction on $r+s$, with basis $s = 0$, where $\mathring{s}(K_{r,s}) = r = f(r,0)$. For $r + s > 0$, without loss of generality we may assume $r \ge s$. Let X and Y be the parts of the bipartition, with $|X| = r$ and $|Y| = s$.

As explained in Remark 5.1, we specify J as a function of i so that in response to Lister's first move (j,i), Painter colors the j marked vertices in the part of size r if $j \ge J$ and otherwise colors the i marked vertices in the part of size s. When $i = 0$, the threshold is 0. Painter colors all the marked vertices, and the induction hypothesis yields the desired bound. Hence we may assume $i > 0$.

Below the threshold, the remaining game is on $K_{r,s-i}$, independent of j. Hence in this case Lister should maximize j to gain the most initial cost, and $i+J+\mathring{s}(K_{r,s-i})$ is an upper bound on the total cost. Above the threshold, by the induction hypothesis the value $i+j+f(r-j,s)$ is an upper bound on the total cost, given that Painter's strategy colors the j vertices in Y. The value of the bound is $i+r+s+2\sqrt{(r-j)s}$. As a continuous function of j, this value is strictly decreasing. We consider only values at least J, so $i+J+f(r-J,s)$ is an upper bound on the total cost when Lister's play against Painter's strategy is in this range. This statement does not require J to be an integer.

Using the induction hypothesis, we compute

$$\begin{aligned}\mathring{s}(K_{r,s}) &\le \max_i \left[i+J+\max\{f(r,s-i),f(r-J,s)\}\right] \\ &= r+s+\max_i \left[\max\{J+2\sqrt{r(s-i)}, i+2\sqrt{(r-J)s}\}\right].\end{aligned}$$

It then suffices to prove the following inequalities for $1 \le i \le s$ under an appropriate threshold function J:

$$J+2\sqrt{r(s-i)} \le i+2\sqrt{(r-J)s} \le 2\sqrt{rs}.$$

Define the threshold function for Painter by $J = i\sqrt{r/s}$. The inequality on the right is equivalent to $2 \ge \frac{i}{\sqrt{rs}-\sqrt{rs-sJ}} = \frac{i(\sqrt{rs}+\sqrt{rs-sJ})}{sJ}$. Thus it suffices to have $\frac{2i\sqrt{rs}}{sJ} \le 2$, as is given.

The inequality on the left is equivalent to

$$J-i \le 2(\sqrt{rs-sJ}-\sqrt{rs-ir}) = 2\frac{ir-sJ}{\sqrt{rs-sJ}+\sqrt{rs-ir}}.$$

Thus it suffices to prove $J-i \le \frac{ir-sJ}{\sqrt{rs}}$, which simplifies to $J \le \frac{ir+i\sqrt{rs}}{s+\sqrt{rs}}$. Since $J = i\sqrt{r/s}$, it suffices to prove $\frac{i\sqrt{rs}}{s} \le \frac{ir+i\sqrt{rs}}{s+\sqrt{rs}}$. Cross-multiplying shows that the two sides are equal. □

When $r=s$, Theorem 5.2 reduces to the trivial upper bound $\mathring{s}(K_{r,r}) \le 4r$. The bound is not sharp, since equality in $\mathring{s}(G) \le \chi(G)|V(G)|$ holds only when G has no edges, but it is asymptotically sharp, as we will note shortly.

In the original version of this paper, we proved a lower bound on $\mathring{s}(K_{r,s})$.

Theorem 5.3. *If $r \ge s \ge 1$, then $\mathring{s}(K_{r,s} \ge r + \frac{5s-3}{2} + u_{r-s}$.*

This lower bound generalizes the exact answer for stars with non-triangular number of vertices. We gave an inductive proof using the Lister strategy that marks one vertex in the small part and u_{r-s} vertices in the large part when $r \ge s+2$, two vertices in each part when $r = s+1$, and one vertex in each part when $r=s$.

We now omit the details of this proof because Wu has proved a better lower bound that generalizes to all complete multipartite graphs.

Theorem 5.4 (Wu [25])**.** *If G is a complete k-partite graph with r_i vertices in the ith part, then $\mathring{s}(G) \ge \sum_{1\le i\le k} r_i + \sum_{1\le i<j\le k} u_{r_i}u_{r_j}$.*

Wu's inductive proof is a simple one-line computation based on the observation in Lemma 4.2 that $u_{r-u_r} \geq u_r - 1$ and the Lister strategy of marking u_{r_i} vertices in the ith part, for all i. Painter responds by coloring the vertices in one part, say the jth part. Letting G' denote the graph obtained by deleting these vertices, the induction hypothesis yields $\mathring{s}(G) \geq \sum_i u_{r_i} + (\sum r_i) - u_{r_j} + \sum_{i<i'} u_{r_i} u_{r_{i'}} - \sum_{i \neq j} u_{r_i}$, as desired.

Since $u_r = \left\lfloor \frac{-1+\sqrt{8r+1}}{2} \right\rfloor$, this lower bound is quite close to our upper bound in Theorem 5.2. Wu also slightly improved our upper bound, replacing $\sqrt{2r}\sqrt{2s}$ with $\sqrt{2r-1}\sqrt{2s-1}$. This improvement reduces to the exact value for most stars and generalizes to complete k-partite graphs.

References

[1] A. Berliner, U. Bostelmann, R. A. Brualdi, and L. Deaett, *Sum list coloring graphs*, Graphs Combin. **22** (2006), no. 2, 173–183. MR 2231989 (2007c:05070)

[2] B. Bollobás, *The chromatic number of random graphs*, Combinatorica **8** (1988), no. 1, 49–55. MR 951992 (89i:05244)

[3] B. Bollobás and P. Erdős, *Cliques in random graphs*, Math. Proc. Cambridge Philos. Soc. **80** (1976), no. 3, 419–427. MR 0498256 (58 #16408)

[4] B. Bollobás, *Random graphs*, second ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001. MR 1864966 (2002j:05132)

[5] J. M. Carraher, T. Mahoney, G. J. Puleo, and D. B. West, *Sum-paintability of generalized theta-graphs*, Graphs and Combinatorics, **31** (2015), no. 5, 1325–1334. MR 3386011

[6] M. Cropper, A. Gyárfás, and J. Lehel, *Hall ratio of the Mycielski graphs*, Discrete Math. **306** (2006), no. 16, 1988–1990. MR 2251581 (2007b:05153)

[7] P. Erdős, A. L. Rubin, and H. Taylor, *Choosability in graphs*, Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979) (Winnipeg, Man.), Congress. Numer., XXVI, Utilitas Math., 1980, pp. 125–157. MR 593902 (82f:05038)

[8] Z. Füredi and I. Kantor, *List colorings with distinct list sizes, the case of complete bipartite graphs.* In *Proc. European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2009)*, Electron. Notes Discrete Math. 34 (Elsevier, 2009), 323–327.

[9] G. Gutowski, T. Krawczyk, D. B. West, M. Zajac, and X. Zhu, The slow-coloring game on outerplanar, planar, and k-degenerate graphs, preprint.

[10] B. Heinold, *Sum choice numbers of some graphs*, Discrete Math. **309** (2009), no. 8, 2166–2173. MR 2510342 (2010m:05278)

[11] B. Heinold, *The sum choice number of $P_3 \square P_n$*, Discrete Appl. Math. **160** (2012), no. 7-8, 1126–1136. MR 2901131

[12] G. Isaak, *Sum list coloring $2 \times n$ arrays*, Electron. J. Combin. **9** (2002), no. 1, Note 8, 7. MR 1928792 (2003j:05046)

[13] G. Isaak, *Sum list coloring block graphs*, Graphs Combin. **20** (2004), no. 4, 499–506. MR 2108395 (2005h:05077)

[14] P. D. Johnson, Jr., *The Hall-condition number of a graph*, Ars Combin. **37** (1994), 183–190. MR 1282556 (95d:05052)

[15] P. D. Johnson, Jr., *The fractional chromatic number, the Hall ratio, and the lexicographic product*, Discrete Math. **309** (2009), no. 14, 4746–4749. MR 2533136 (2010i:05121)

[16] E. M. Kubicka, *The chromatic sum and efficient tree algorithms*, ProQuest LLC, Ann Arbor, MI, 1989, Thesis (Ph.D.)–Western Michigan University. MR 2637573

[17] E. Kubicka, *The chromatic sum of a graph: history and recent developments*, Int. J. Math. Math. Sci. **29–32** (2004), 1563–1573. MR 2085077

[18] S. Micali and V. V. Vazirani, An $O(\sqrt{|V|} \cdot |E|)$ algorithm for finding maximum matching in general graphs", Proc. 21st IEEE Symp. Foundations of Computer Science (1980), pp. 17–27.

[19] G. J. Puleo and D. B. West, Online sum-paintability: Slow-coloring of trees (submitted).

[20] U. Schauz, *Mr. Paint and Mrs. Correct*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 77, 18. MR 2515754 (2010i:91064)

[21] G. Simonyi, *Asymptotic values of the Hall-ratio for graph powers*, Discrete Math. **306** (2006), no. 19-20, 2593–2601. MR 2261921 (2007i:05094)

[22] Á. Tóth, *On the ultimate lexicographic Hall-ratio*, Discrete Math. **309** (2009), no. 12, 3992–3997.

[23] V. V. Vazirani, *A theory of alternating paths and blossoms for proving correctness of the $O(\sqrt{V}E)$ general graph maximum matching algorithm*, Combinatorica 14 (1994), no. 1, 71–109. MR 1273202

[24] V. G. Vizing, *Coloring the vertices of a graph in prescribed colors*, Diskret. Analiz (1976), no. 29 Metody Diskret. Anal. v Teorii Kodov i Shem, 3–10, 101. MR 0498216 (58 #16371)

[25] H. Wu, personal communication, presented at International Workshop on Graph Theory, Ewha Woman's University, Seoul, Korea, January 5, 2018.

[26] X. Zhu, *On-line list colouring of graphs*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 127, 16. MR 2558264 (2011a:05123)